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A study on a Cosmological Constant only universe

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Introduction

In the study of cosmology there are a few basilar equations and concepts that we will need to know in order to begin our study.
The first assumption that we make,on which everything will be founded on, is that we consider the Universe (on a very large scale) to be Homogeneous (the Universe looks the same at each point) and Isotropic (the Universe looks the same in all directions); this assumption is called the *cosmological principle*.
Let's take two arbitrary points in the universe with mass m and M (M>m) respectively. Thanks to Newton we can write the gravitational force that attracts the two bodies and so we can derive their potential energy from it by multiplying per the distance r between the two. So we obtain: $U = -F \cdot r = -\frac{G \cdot M \cdot m}{r^2} \cdot r = -\frac{G \cdot 4 \cdot \pi\rho \cdot r^3 \cdot m}{3 \cdot r} = -\frac{G \cdot 4 \cdot \pi \cdot r^2 \cdot m}{3}$, where we outlined a sphere of radius r and density ρ with the center in the more massive body. At this point we can introduce the total energy, given by $E = U + T$ where $T = \frac{1}{2} \cdot m \cdot \dot{r}^2$ is the kinetic energy. The universe is not a static frame, for this reason we need coordinates that may express the expansion or collapse of this environment; so we adopt the *comoving coordinates*, which allow us to maintain fixed the points but to extend or restrict the space without loosing the position: $\overrightarrow{r} = a(t)\overrightarrow{x}$, where a(t) is called the *scale factor of the universe*. If we substitute these coordinates in the Energy equation (and multiply per $\frac{2}{m \cdot a^2 \cdot x^2}$) we obtain the Friedman equation: $\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8 \cdot \pi \cdot G}{3} \cdot (\rho_M(t) + \rho_R(t) + \rho_\Lambda) - \frac{k \cdot c^2}{a^2(t)}$, where $\rho_M(t)$ is the density of the matter(we won't consider dark matter) in the universe, $\rho_R(t)$ is the density of the radiation (relativistic matter like photons, neutrinos..) in the universe, ρ_Λ is the Vacuum density (was introduced by Einstein as a mathematical tool and today is used to represent the Dark energy density) and the curvature constant $k = \frac{2 \cdot E}{m \cdot x^2}$. In truth the Friedmann equation is obtained from general relativity but the result is the same as derived from Newtonian physics.

Abstract

We are going to study a universe where there is no matter nor radiation, thus the main aspect of our analysis will be the Cosmological Constant Λ expressed through its density ρ_Λ (and its density parameter Ω_Λ). We also assume the Hubble constant ($H_0 = H(t_0) = \dfrac{\dot{a}(t_0)}{a(t_0)}$) to be positive.
In the first chapter we will introduce the Friedmann equation for such an universe and we will study the correlation between the density parameter $\Omega_{\Lambda 0}$ and the curvature constant k. At the end we will introduce the study of how $\Omega_{\Lambda 0}$ influences the possible regimes of the universe.
In the second chapter we will see the inter-correlation between $\Omega_{\Lambda 0}$ and a universe with or without Big Bang giving at the end an overall description of the attitude of $\Omega_{\Lambda 0}$.

Chapter 1

A Cosmological Constant only universe

Say we live in a universe where there is no matter ($\rho_M = 0$) nor radiation ($\rho_R = 0$). As we know the general form of the Friedman equation is

$$\frac{\dot{a}^2(t)}{a^2(t)} = H^2(t) = \frac{8 \cdot \pi \cdot G}{3} \cdot (\rho_M(t) + \rho_R(t) + \rho_\Lambda) - \frac{k \cdot c^2}{a^2(t)}$$

where $\frac{\dot{a}(t)}{a(t)} = H(t)$ is called the *Hubble parameter* and $\frac{\dot{a}(t_0)}{a(t_0)} = H(t_0)$ is called the *Hubble constant* (t_0 =present age of the universe=now). The Friedman equation in the former hypothesis it reduces to:

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8 \cdot \pi \cdot G}{3} \cdot \rho_\Lambda - \frac{k \cdot c^2}{a^2(t)}.$$

Then we define the *vacuum density parameter* $\Omega_\Lambda(t)$ and the *curvature density parameter* $\Omega_k(t)$ as:

$$\Omega_\Lambda(t) = \frac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3 \cdot H^2(t)} = \frac{\rho_\Lambda}{\rho_c(t)}$$

$$\Omega_k(t) = -\frac{k \cdot c^2}{a^2(t) \cdot H^2(t)}$$

where $\rho_c(t) = \frac{3 \cdot H^2(t)}{8 \cdot \pi \cdot G}$ is called the *critical density*. We are going to calculate these parameters for $t = t_0$, (we define $a(t_0) = 1$) then we find:

$$\Omega_\Lambda(t_0) = \Omega_{\Lambda 0} = \frac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3 \cdot H_0^2} = \frac{\rho_\Lambda}{\rho_c(t_0)}$$

$$\Omega_K(t_0) = \Omega_{K0} = -\frac{k \cdot c^2}{H_0^2}$$

where H_0^2 stands for $H^2(t_0)$ (let's observe that $H^2(t_0) = \dot{a}^2(t_0) = \dot{a}_0^2$).
At this point we can rewrite the Friedman equation using the density parameters evaluated at the time t_0; we find the following unit-less equation:

$$1 = \Omega_{\Lambda 0} + \Omega_{K0}.$$

Which gives us the following relation:

$$\Omega_{K0} = 1 - \Omega_{\Lambda 0}.$$

If we multiply the right hand side of the Friedman equation per $\frac{\dot{a}_0^2}{\dot{a}_0^2}$ then we obtain:

$$\frac{\dot{a}^2(t)}{a^2(t)} = H^2(t) = \Omega_{\Lambda 0} \cdot \dot{a}_0^2 + \Omega_{k0} \cdot \frac{\dot{a}_0^2}{a^2(t)}.$$

And if we substitute the value of Ω_{k0} we find:

$$H^2(t) = \Omega_{\Lambda 0} \cdot \dot{a}_0^2 + (1 - \Omega_{\Lambda 0}) \cdot \frac{\dot{a}_0^2}{a^2(t)} = \Omega_{\Lambda 0} \cdot H_0^2 + (1 - \Omega_{\Lambda 0}) \cdot \frac{H_0^2}{a^2(t)} \ .$$

As we know ρ_Λ doesn't depend on time, while we may have observed that $\Omega_\Lambda(t)$ does depend on time. This strange thing is due to the fact that in order to find the density parameter for the cosmological constant we divide the term $\frac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3}$ by $H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)}$, which depends on t.
We can also find a constant value for $\Omega_\Lambda(t)$ simply applying the initial condition $\Omega_\Lambda(t) = \Omega_{\Lambda 0}$.
We can find such a constant value by solving the following equation for $a(t)$:

$$\frac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3} \cdot \frac{1}{\frac{\dot{a}^2(t_0)}{a^2(t_0)(= 1)}} = \frac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3} \cdot \frac{1}{\frac{\dot{a}^2(t)}{a^2(t)}}$$

which solution is $a(t) = e^{|\dot{a}(t_0)| \cdot (t - t_0)}$. Then if we replace this value of $a(t)$ in the previous equation we find:

$$\frac{1}{\dot{a}^2(t_0)} = \frac{1}{\frac{\dot{a}^2(t_0) \cdot e^{2 \cdot |\dot{a}(t_0)| \cdot (t - t_0)}}{e^{2 \cdot |\dot{a}(t_0)| \cdot (t - t_0)}}} \ ,$$

which match!
We still didn't take into consideration the curvature constant k. The meaning of this constant is that it interprets our measurements and therefore it says which is the geometry of the universe:

- $k = 0$ The universe that we are taking into consideration has an Euclidean geometry (where the Euclidean postulates are all valids). The V postulate says that if two straight lines are parallel in one point then they will be parallel at any point, due to this axiom we have that such an universe must be infinite. Such a universe is usually called *flat universe*.

- $k > 0$ The universe that we are taking into consideration can be described using spherical coordinates. The surface traced by these coordinates is finite, then the dimension of this kind of universe must be finite. Such a universe is usually called *closed universe*.

- $k < 0$ The universe that we are taking into consideration has a hyperbolic geometry. This kind of universe must have a infinite dimension. Such a universe is usually called *open universe*.

The Friedman equation puts into close relationship our cosmological constant Λ (and in particular $\Omega_\Lambda(t)$) with the curvature k. We can analyse in which extent the value of $\Omega_\Lambda(t_0)$ is going to affect the geometry of the universe that we are studying.
We might use a half extended version of the Friedman equation for density parameters evaluated in t_0:

$$1 = \Omega_{\Lambda 0} - \frac{k \cdot c^2}{H_0^2} \ .$$

It is useful to observe that the sign of the second term on the right side of the equation depends entirely upon the sign of k.
Say that this universe is a flat universe ($k = 0$), then we will find that $\Omega_{\Lambda 0} = 1$.
If we suppose this universe to be closed ($k > 0$), then we will find that $\Omega_{\Lambda 0} = 1 + \frac{k \cdot c^2}{H_0^2} = 1 + \text{positive constant} > 1$.
Finally we may conjecture this universe to be a open universe ($k < 0$), then we will find that $\Omega_{\Lambda 0} = 1 + \frac{k \cdot c^2}{H_0^2} = 1 + \text{negative constant} < 1$.
As far as the previous study comprehends all the possible values of $\Omega_{\Lambda 0}$ we can reverse the implications and we find that:

If $\Omega_{\Lambda 0} = 1$ then we find a flat universe ($k = 0$).

If $\Omega_{\Lambda 0} < 1$ then we find a open universe ($k < 0$).

If $\Omega_{\Lambda 0} > 1$ then we find a closed universe ($k > 0$).

Still we don't know in which term and condition the universe that we are studying may expand, collapse or switch direction.
To begin our study we may easily find which is the value of $a(t)$ at which the universe stop in order to change direction..if it does.
To the Friedman equation we impose $\dot{a}(t_{halt}) = 0$ which is the condition for the halt point (the point where the universe stop and change velocity direction), so we find that
$a^2(t_{halt}) = \frac{\Omega_{\Lambda 0} - 1}{\Omega_{\Lambda 0}}$. It is important to observe that there can't be any turning point (halt point) in the direction of the expansion (or collapse) of the universe if $0 \leq \Omega_{\Lambda 0} < 1$ (in this case we would have $\Omega_{\Lambda 0} - 1 < 0$) because it would assign a negative value to $a^2(t_{halt})$.
Now we can face the problem for how the value of $\Omega_{\Lambda 0}$ may influence the direction of the speed of the expansion of the universe.
By multiplying the Friedman equation per $a^2(t)$ we find:

$$\dot{a}^2(t) = \frac{8 \cdot \pi \cdot G}{3} \cdot \rho_\Lambda \cdot a^2(t) - k \cdot c^2$$

If we derive it per t we will find:

$$2 \cdot \dot{a}(t) \cdot \ddot{a}(t) = \frac{8 \cdot \pi \cdot G}{3} \cdot \rho_\Lambda \cdot 2 \cdot a(t) \cdot \dot{a}(t) \cdot \frac{\dot{a}^2(t_0)}{\dot{a}^2(t_0)}$$

If we decide to study only the moment before and after the switching point then we can divide per $\dot{a}^2(t)$ and we find:

$$\ddot{a}(t) = a(t) \cdot \dot{a}^2(t_0) \cdot \Omega_{\Lambda 0}$$

As far as $a \cdot \dot{a}^2(t_0)$ is always positive we find that the sign of $\Omega_{\Lambda 0}$ directly determines the sign of the acceleration, and while $\Omega_{\Lambda 0}$ is a constant we have that the acceleration depends on time.
Then we find the following interrelation:

If $\Omega_{\Lambda 0} > 0$ then $\ddot{a}(t) > 0$.

If $\Omega_{\Lambda 0} < 0$ then $\ddot{a}(t) < 0$.

At this point we may ask ourselves which kind of regimes the universe may assume. One of our general assumption is that the Hubble constant H_0 is positive, which means that $\dot{a}(t_0) > 0$, since that we understand that it is impossible to have a universe where the velocity is negative and the acceleration too because when it would cut the $a(t)$ axis it would have a negative slope.

This fact restricts the possible regimes to only three of them because the acceleration doesn't change sign: an ever expanding universe, a collapsing-then-expanding universe and an expanding-then-collapsing universe.

For the first two regimes we have $sign(\ddot{a}(t)) = sign(\Omega_{\Lambda 0}) > 0$ (then $\Omega_{\Lambda 0} > 0$) and in particular for the second one we have also the condition that $\Omega_{\Lambda 0} > 1$ (otherwise there can't be any switching point), while for the last one we have $sign(\ddot{a}(t)) = sign(\Omega_{\Lambda 0}) < 0$ (then $\Omega_{\Lambda 0} < 0$).

Chapter 2

A Big Bang/non Big Bang universe

We are now going to analyse under which condition the De Sitter (the Dutch physicist who first theorized the Cosmological Constant only universe) universe have a Big Bang or not.
As we saw in the previous chapter we can rewrite the Friedmann equation as:

$$\dot{a}^2(t) = H_0^2 \cdot (a^2(t) \cdot \Omega_{\Lambda 0} - \Omega_{\Lambda 0} + 1)$$

If we want to study the universe assuming that it has the Big Bang then we can solve the Friedmann equation requiring the condition that a(0)=0. If we consider $\Omega_{\Lambda 0} > 0$ then we find the following solution:

$$\begin{aligned} a(t) = & \frac{\Omega_{\Lambda 0} - 1}{2 \cdot \sqrt{\Omega_{\Lambda 0} - \Omega_{\Lambda 0}^2}} \cdot (e^{\sqrt{H_0^2} \cdot \sqrt{\Omega_{\Lambda 0}} \cdot t} - e^{-\sqrt{H_0^2} \cdot \sqrt{\Omega_{\Lambda 0}} \cdot t}) = \\ & \frac{1 - \Omega_{\Lambda 0}}{2 \cdot \sqrt{\Omega_{\Lambda 0} - \Omega_{\Lambda 0}^2}} \cdot sinh(\sqrt{H_0^2} \cdot \sqrt{\Omega_{\Lambda 0}} \cdot t) \end{aligned}$$

The solution gives some existence condition to the Vacuum density: $0 < \Omega_{\Lambda 0} < 1$. Furthermore we may observe that both the velocity and the acceleration are positive. This gives us an ever-expanding universe. We can plot an example with the following values: $\Omega_{\Lambda 0} = 0.7$, $H_0^2 = 0.5$.

On the other hand if we require $\Omega_{\Lambda 0} < 0$ then we will find:

$$a(t) = \frac{\Omega_{\Lambda 0} + 1}{2 \cdot \sqrt{-\Omega_{\Lambda 0} - \Omega_{\Lambda 0}^2}} \cdot (e^{i \cdot \sqrt{H_0^2} \cdot \sqrt{\Omega_{\Lambda 0}} \cdot t} - e^{-i \cdot \sqrt{H_0^2} \cdot \sqrt{\Omega_{\Lambda 0}} \cdot t})$$

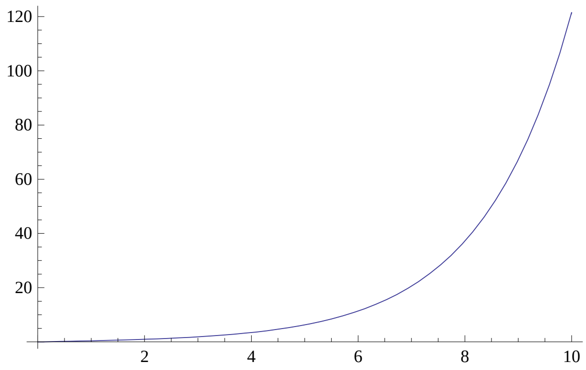

Figure 2.1: Expanding universe ($\Omega_{\Lambda 0} = 0.7$, $H_0^2 = 0.5$).

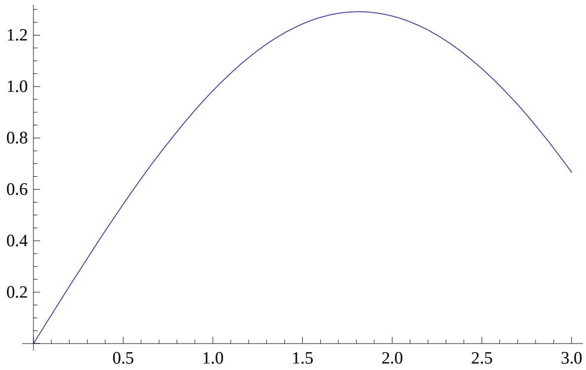

Figure 2.2: Expanding then collapsing universe ($\Omega_{\Lambda 0} = -1.5$, $H_0^2 = 0.5$).

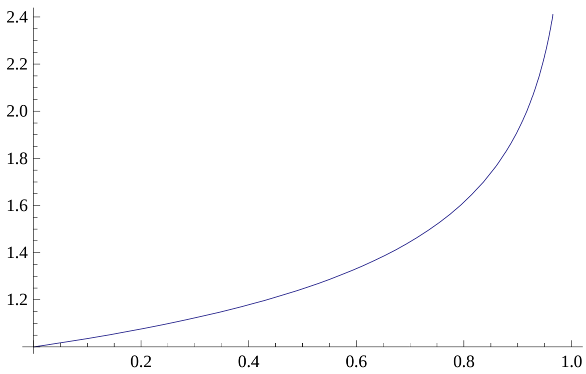

Figure 2.3: Time of the universe depending on the values of a positive $\Omega_{\Lambda 0}$

There are no existence condition for the Vacuum density except that: $\Omega_{\Lambda 0} < 0$. This kind of solution gives us an expanding-then-collapsing universe. As an example we may choose: $\Omega_{\Lambda 0} = -1.5$, $H_0^2 = 0.5$.

Now we can think about the age that a De Sitter universe with Big Bang might have.
We start from our Friedman equation:

$$H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = H_0^2 \cdot (\Omega_{\Lambda 0} + \frac{1 - \Omega_{\Lambda 0}}{a^2(t)})$$

If we apply a square root to the equation and then multiply both sides per $a(t)$ we will find:

$$\dot{a}(t) = \frac{da}{dt} = a(t) \cdot H_0 \cdot \sqrt{\Omega_{\Lambda 0} + \frac{1 - \Omega_{\Lambda 0}}{a^2(t)}}$$

from which we obtain:

$$\frac{dt}{da} = \dot{t}(a) = \frac{1}{a \cdot H_0 \cdot \sqrt{\Omega_{\Lambda 0} + \frac{1 - \Omega_{\Lambda 0}}{a^2}}}$$

From this equation, if $0 < \Omega_{\Lambda 0} < 1$ we find that:

$$\text{age of the universe} = t(1) = \frac{-log((1 - \Omega_{\Lambda 0}) \cdot \Omega_{\Lambda 0}) + 2 \cdot log(\sqrt{\Omega_{\Lambda 0}} + \Omega_{\Lambda 0})}{2 \cdot \Omega_{\Lambda 0} \cdot \sqrt{\Omega_{\Lambda 0}}}$$

While if $-1 < \Omega_{\Lambda 0} < 0$ we find:

$$t(1) = \frac{arctg(\sqrt{\Omega_{\Lambda 0}})}{H_0^2 \cdot \sqrt{\Omega_{\Lambda 0}}}$$

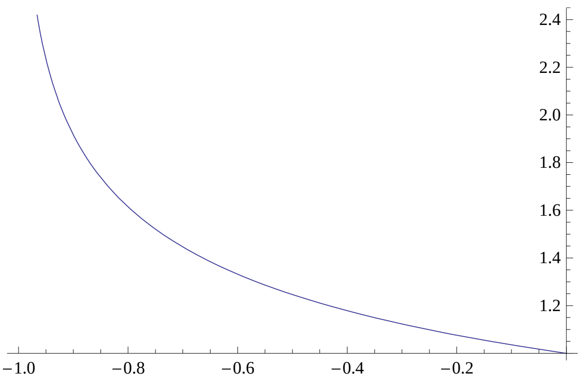

Figure 2.4: Time of the universe depending on the values of a negative $\Omega_{\Lambda 0}$

Now we can consider the case in which there is no Big Bang, which means that $a(0) \neq 0$; as far as we have the defining condition for the scale factor $a(t) \geq 0$ then we can simply say that for any positive real constant $c > 0$ $a(0) = c$.
Now we can solve the problem as we did in the case with the Big Bang but changing the initial condition!
Thus for $\Omega_{\Lambda 0} > 0$ we find:

$$a(t) = \frac{e^{-\sqrt{H_0^2 \cdot \Omega_{\Lambda 0}} \cdot t}}{2(c \cdot \Omega_{\Lambda 0} + \sqrt{\Omega_{\Lambda 0} - \Omega_{\Lambda 0}^2 + c^2 \cdot \Omega_{\Lambda 0}^2})} \cdot (\Omega_{\Lambda 0} - 1 + e^{2\sqrt{H_0^2 \cdot \Omega_{\Lambda 0}} \cdot t} - \Omega_{\Lambda 0} \cdot e^{2\sqrt{H_0^2 \cdot \Omega_{\Lambda 0}} \cdot t} + 2c^2 \cdot \Omega_{\Lambda 0} \cdot e^{2\sqrt{H_0^2 \cdot \Omega_{\Lambda 0}} \cdot t} + 2c\Omega_{\Lambda 0} \cdot e^{2\sqrt{H_0^2 \cdot \Omega_{\Lambda 0}} \cdot t} \sqrt{\Omega_{\Lambda 0} - \Omega_{\Lambda 0}^2 + c^2 \cdot \Omega_{\Lambda 0}^2})$$

And we find an ever expanding universe for any positive $\Omega_{\Lambda 0}$), with the condition that $c^2 \geq \dfrac{\Omega_{\Lambda 0} - 1}{\Omega_{\Lambda 0}}$.

On the other side for $\Omega_{\Lambda 0} < 0$ we find the same solution with the condition that $c^2 \leq \dfrac{\Omega_{\Lambda 0} - 1}{\Omega_{\Lambda 0}}$.

As far as we don't have any Big Bang, to give a measure of the time we have to distinguish two cases. The first one is the case in which the universe is ever expanding; in such a case we could decide to calculate the time since the universe had a certain value for the velocity; for example we could decide to calculate how long time has passed since $t(\dot{a} = d)$ where $d > 0$ is a real number. In the case for a collapsing-then-expanding universe we could use the distance since the universe changed direction, which means that we can start measuring the time since t_{halt}, where $\dot{a}(t_{halt}) = 0$.
Until now we have studied the behaviour of the universe depending on the value of $\Omega_{\Lambda 0}$. But we may ask ourselves which is the behaviour of $\Omega_\Lambda(t)$. We

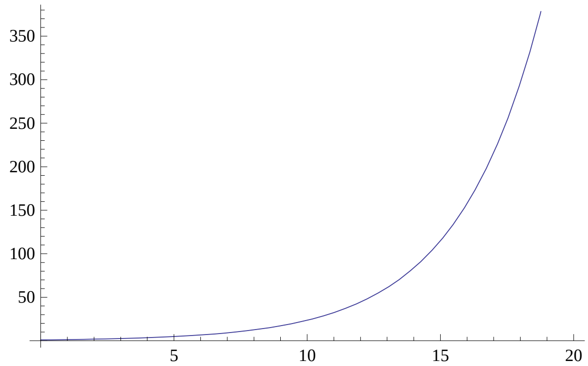

Figure 2.5: Expanding universe without Big Bang ($\Omega_{\Lambda 0} = 1$, $H_0^2 = 0.1$ and $c = 1$).

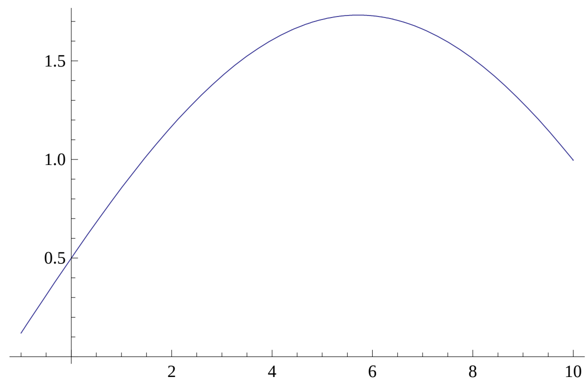

Figure 2.6: Expanding then collapsing universe without Big Bang ($\Omega_{\Lambda 0} = -0.5$, $H_0^2 = 0.1$ and $c = 0.5$).

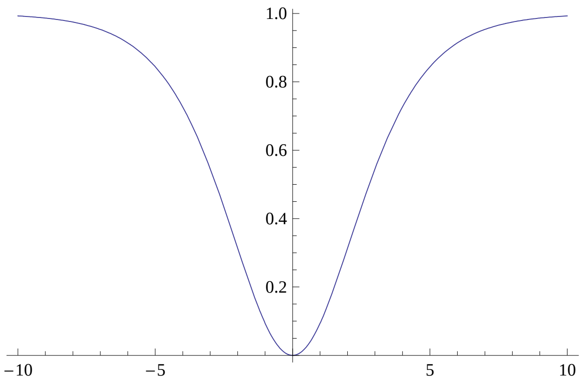

Figure 2.7: Values for $\Omega_\Lambda(t)$ for an expanding universe with Big Bang ($\Omega_{\Lambda 0} = 0.1$ and $H_0^2 = 1$).

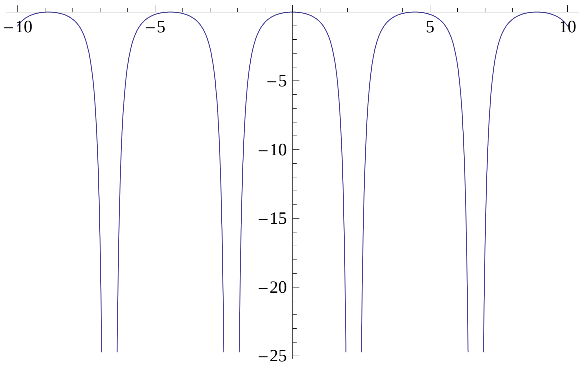

Figure 2.8: Values for $\Omega_\Lambda(t)$ for an expanding then collapsing universe with Big Bang ($\Omega_{\Lambda 0} = -0.5$ and $H_0^2 = 1$).

know that $\Omega_\Lambda(t) = \dfrac{8 \cdot \pi \cdot G \cdot \rho_\Lambda}{3H^2(t)} \cdot \dfrac{H_0^2}{H_0^2} = \dfrac{\Omega_{\Lambda 0} \cdot H_0^2}{H^2(t)}$. But $H^2(t) = H_0^2(\Omega_{\Lambda 0} - \dfrac{\Omega_{\Lambda 0} - 1}{a^2(t)})$, then we obtain

$$\Omega_\Lambda(t) = \frac{\Omega_{\Lambda 0}}{\Omega_{\Lambda 0} - \dfrac{\Omega_{\Lambda 0} - 1}{a^2(t)}}$$

At this point we are able to study $\Omega_\Lambda(t)$ by plugging in different values of $\Omega_{\Lambda 0}$ and choosing time by time the different solutions found for $a(t)$ (both in the case for a Big Bang and a non-Big Bang universe). Let's see some particular cases: figures 2.7-2.10 .

Due to numerical errors especially the figures 2.9 and 2.7 don't fit very much the initial conditions, anyway they give good ideas of what is happen-

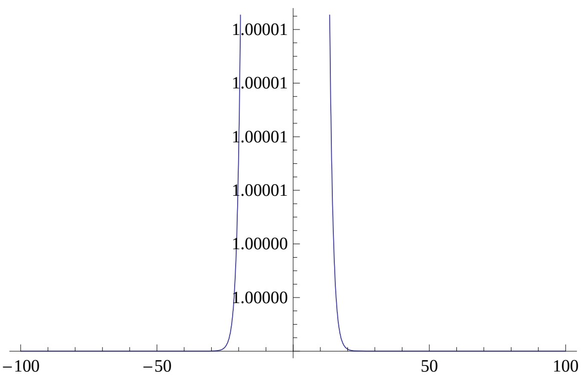

Figure 2.9: Values for $\Omega_\Lambda(t)$ for an expanding universe without Big Bang ($\Omega_{\Lambda 0} = 1.5$, $H_0^2 = 0.1$ and $c = 1$).

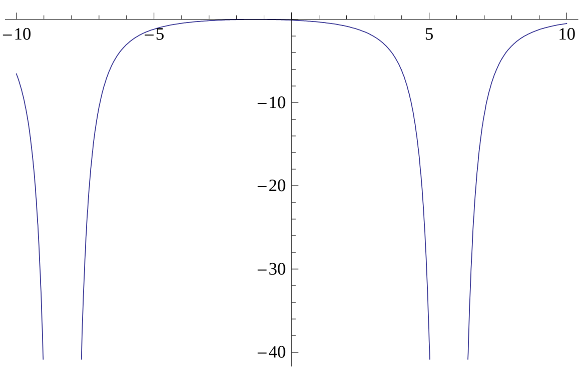

Figure 2.10: Values for $\Omega_\Lambda(t)$ for an expanding then collapsing universe without Big Bang ($\Omega_{\Lambda 0} = -0.5$, $H_0^2 = 0.1$ and $c = 0.5$).

ing.
What we see is that the value of $\Omega_\Lambda(t)$ maintains the same sign of the $\Omega_{\Lambda 0}$; moreover for the negative values $\Omega_\Lambda(t)$ tents to infinity while for the positive values it tents to stabilize with small positive values.

Bibliography

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