Università degli Studi di Pisa

Facoltà di Scienze Matematiche, Fisiche e Naturali
Corso di Laurea in Matematica

Khinchin theorem in Teichmüller dynamics

20 luglio 2012

Tesi di Laurea Magistrale

Candidato
Antonio De Capua
decapua@mail.dm.unipi.it

Relatore
Prof. Stefano Marmi
Scuola Normale Superiore

Controrelatore
Dott. Carlo Carminati
Università di Pisa

Anno Accademico 2011/2012

Table of Contents

Last edit on 16th July 2012

0 Introduction

0.1 Killing two birds with one stone

Two are the main objects of interest of this work, and they will be developed in parallel.

The first one is of geometrical nature, and is called *Teichmüller dynamics*. Teichmüller spaces, namely the spaces of the complex structures on a fixed surface up to isotopies, are not quite Riemannian manifolds, but a geodesic flow can nevertheless be defined. The interesting thing is that this flow can be visualised very simply if one considers it as a flow among *translation surfaces*, a special kind of flat Riemannian structures with a finite number of singularities. The flow is usually considered not on Teichmüller spaces but on a quotient, namely moduli spaces.

The second object of interest will be the *interval exchange maps*, which are much simpler to define: they are self-maps of an interval that divide it in sub-intervals and rearrange them preserving the orientation. *Rauzy dynamics* is a dynamics among interval exchange maps, obtained by considering the action of one of them on smaller and smaller portions of the original interval.

These two settings may seem totally unrelated to each other, but an identification is possible on two levels. If one considers a family of lines in a fixed direction on a translation surface, and an orthogonal segment, an interval exchange map is induced on the segment: one starts from any of its points and follows the corresponding line, until one gets back to the segment. On the other hand, a section for the Teichmüller flow can be defined, and the associated return map can be interpreted in terms of Rauzy dynamics.

The intensive study of Teichmüller/Rauzy dynamics began roughly in the 1970s-1980s with the pioneering works of Katok, Keane, Masur and Veech and is now an extremely active field of research with a growing number of contributors including several Field medalists (Kontsevich, McMullen and Yoccoz, just to name a few) as well as some younger researchers (Avila, Bufetov, Chaika, Matheus, Marchese, Ulcigrai. . .).

In this chapter we briefly introduce some preliminary notions from the theory of dynamical systems which will be used in this work; then we examine quickly two simpler kinds of objects which motivate the study of the entities listed above: namely flat tori, and rotations of the circumference. Chapter 1 is about the fundamentals of Teichmüller theory, and in Chapter 2 we give the basic notions needed to work with interval exchange maps.

After that, we begin talking about the results obtained in this twofold subject. Chapter 3 collects some 'historical' topics: ergodicity of the considered dynamics, and the properties of the Kontsevich-Zorich cocycle; then we treat rather in detail the generalisations and implications of the *Khinchin theorem*, investigated by Luca Marchese.

The Khinchin theorem, in its original version, is a theorem about a Diophantine condition for real numbers. Let us fix a generic $\vartheta \in [0, 1)$, and a decreasing sequence $\phi(n)$: we ask, how many times does it happen that $\{n\vartheta\} < \phi(n)$ (where $\{\cdot\}$ denotes the fractional part of a real number)? The answer is, finitely many if $\sum \phi(n) < +\infty$, infinitely many otherwise.

In Chapter 4 we will see that it is possible to define a condition regarding singularities of interval exchange maps, derived from the one considered by Khinchin. The number of solutions to this condition will be finite or infinite again according to $\sum \phi(n)$. The proof of this result provides an example of the most commonly employed method for studying i.e.m.s and related matters: "to plough in parameter space, and to harvest in phase space" (Adrien Douady).

In Chapter 5 the geometrical implications of this generalised Khinchin theorem are analysed. These ones include not only a mere geometrical restatement of the theorem found for interval exchange maps; but also interesting information about how the geodesics on a translation surface pass closer and closer to a fixed point; and a characterisation of how much the geodesic flow in moduli space gets «far towards infinity» and then gets back.

Large portions of this work are meant as an overview. For this reason, several proofs are only outlined; or also, deprived of the most technical parts, or totally omitted.

0.2 Small dictionary of asymptotic properties

This section collects some definitions and preliminary results related to the study of dynamical systems in general, before we describe the setting of our interest. The main source for this quick overview is [CM08], whose viewpoint is that: "Roughly speaking, the goal of the theory of dynamical systems is to understand *most* of the dynamics of *most* systems". It must not be forgotten that such abstract definitions may undergo modifications according to the context.

§ 0.2.A Discrete vs. continuous A dynamical system is a transformation of some *phase space* X, which is a set endowed with some structure e.g. a topology, a manifold structure, a measure etc.; the most typical distinction is between discrete and continuous dynamical systems.

A *discrete (deterministic) dynamical system* is given by a map $f : X \to X$ that «has a good behaviour with respect to some of the structures on X», that is it may be continuous, differentiable, measurable, etc. Fixed a point $x \in X$, its *orbit* is the sequence $(f^n(x))_{n\in\mathbb{N}}$ given by the images of x under iterates of f. When f is an *invertible* map, this is often called the *positive (half-)orbit*, as one may also consider the biinfinite sequence $(f^n(x))_{n\in\mathbb{Z}}$.

A *continuous dynamical system* is defined on a phase space X with a topological (typically a manifold, or orbifold) structure. Such a system may be identified with a *flow*, that is a continuous (differentiable) map $\Phi(x, t) = \Phi^t(x)$, defined on an open set of $X \times \mathbb{R}$ which contains $X \times \{0\}$, and satisfying the *semigroup laws* $\Phi(\cdot, 0) = Id_X$, $\Phi(x, s + t) = \Phi(\Phi(x, s), t)$. In particular, each Φ^t is a homeomorphism (diffeomorphism) of X. When X is a differentiable manifold (or orbifold), a flow is in turn identifiable with an unique vectorfield on X.

Starting from a flow, it is sometimes possible to define an associated discrete system: a *(cross-)section* (when it exists) is a submanifold Y of X of codimension 1 that is transverse to the flow, and intersects the orbit $\Phi^t(x)$ of each fixed $x \in X$ for infinitely many $t \in \mathbb{R}$. With such hypotheses, a discrete dynamical system on Y is given by the *first return map* of the flow Φ^t to Y. Conversely, a diffeomorphism f of a manifold Y can be associated with its *suspension*, which is a continuous system such that f appears as a return map: let $X = Y \times \mathbb{R}/\sim$ where $\sim$ is the equivalence relation generated by $(y, t) \sim (f(y), t + 1)$. The desired flow is simply $\Phi^t([x, u]) = [x, u + t]$.

§ 0.2.B Topological asymptotic properties Suppose that X is a (compact) metric space, and $f \in \mathrm{Homeo}(X)$; in this case, several definitions are commonly used to describe the behaviour of the orbits of f. Recall that, given $x \in X$, its *α-limit* is the set of the points y of X such that the negative orbit $(f^{-n}(x))_{n\in\mathbb{N}}$ gets arbitrarily close to y for infinitely many $n \in \mathbb{N}$; similarly its *ω-limit* is the set of the points y of X such that the positive orbit $(f^n(x))_{n\in\mathbb{N}}$ gets arbitrarily close to y for infinitely many $n \in \mathbb{N}$. A typical kind of considered property is whether an orbit makes infinitely many returns near its initial point:

Definition 0.2.1. A point $x \in X$ is *positively recurrent* if it is contained in its ω-limit; *negatively recurrent* if it is contained in its α-limit; *recurrent* if it is contained in both.

Another kind of question is whether orbits visit all regions of the space:

Definition 0.2.2. A discrete dynamical system is called *topologically transitive* if one of its orbits is a dense subset of X; and it is called *minimal* if *each* orbit is dense.

These definitions make sense, with minor modifications, for continuous systems as well.

§ 0.2.C Measurable systems Questions in the same spirit as above are most frequently made in a *statistical* setting: that is, X is not only a metric space, but we also consider its Borel σ-algebra. In the case of a discrete system specified by some *measurable* $f : X \to X$, a frequent request on the considered Borel measures (or, often, probabilities) is the following:

Definition 0.2.3. A Borel measure μ is *f-invariant* if, for any measurable $E \subseteq X$, we have $\mu\left(f^{-1}(E)\right) = \mu(E)$. We also say that f *preserves* μ.

Note that this definition makes sense even if f is *not* invertible. In the case of a continuous system specified by some flow Φ, we say that μ is *Φ-invariant* if it is Φ^t-invariant for all fixed t.

A well-known fact about f-invariant probabilities is the following:

Theorem 0.2.4 (Weak Poincaré recurrence). *Let μ be a f-invariant probability measure. Then, for every measurable $E \subseteq X$ with $\mu(E) > 0$, the (positive) orbit of almost any point of E returns infinitely many times to E.*

We denote $\mathcal{M}(X)$ the set of the Borel probabilities on X, and $\mathcal{M}_f(X)$ the subset of the f-invariant ones. According to a theorem of Krylov and Bogolubov, if f is a continuous endomorphism of a compact metric space X then $\mathcal{M}_f(X)$ is always nonempty; and it is not difficult to see that it is a convex subset. A particular attention is given to its *vertices*, which are measures «not decomposable with respect to the dynamics»:

Definition 0.2.5. A f-invariant Borel probability measure on X is called *ergodic* if, for every measurable $E \subset X$ such that $f^{-1}(E) = E$, one has either $\mu(E) = 0$ or $\mu(E) = 1$.

Given a function $\phi : X \to \mathbb{R}$, we define for any $n \in \mathbb{N}$ its n-th *Birkhoff sum* as the function

$$S_n\phi(x) := \sum_{j=0}^{n-1} \phi\left(f^j(x)\right).$$

Ergodicity of probability measures is related with the asymptotic behaviour of Birkhoff sums:

Theorem 0.2.6 (Birkhoff ergodic theorem). *Let μ be an ergodic Borel probability measure on X, and let $\phi : X \to \mathbb{R}$ be a measurable function. Then*

$$\lim_{n\to+\infty} \frac{1}{n} S_n\phi(x) = \int_X \phi \mathrm{d}\mu$$

for μ-almost every $x \in X$.

When $E \subseteq X$ is measurable and we take $\phi = \chi_E$ is its characteristic function, $\frac{1}{n} S_n \chi_E(x)$ is the *frequency of visit* of x to E: Birkhoff theorem implies that, for μ-almost every x, this expression goes to $\mu(E)$ for $n \to +\infty$. In particular two distinct ergodic measures are always mutually singular. Note that the Poincaré recurrence theorem states that the orbit of a generic point of E returns to E infinitely many times; whereas ergodicity implies that the orbit of a generic point *in the whole* X enters E infinitely many times.

A particular case of ergodicity is the following:

Definition 0.2.7. A measurable $f : X \to X$ is *uniquely ergodic* if $\mathcal{M}_f(X)$ consists of only one probability measure μ.

In this case μ is necessarily ergodic. Birkhoff theorem now has a stronger statement:

Theorem 0.2.8. *Let X be a compact metric space, and let $f : X \to X$ be an homeomorphism. The map f is uniquely ergodic if and only if, for every continuous function $\phi : X \to \mathbb{R}$, there is a constant c_ϕ such that*

$$\lim_{n\to+\infty} \frac{1}{n} S_n\phi(x) = c_\phi \text{ uniformly for all } x \in X.$$

In this case the unique f-invariant measure μ is the one such that $c_\phi = \int_X \phi \, \mathrm{d}\mu$.

All the discussions above only regarded discrete dynamical systems, but are easily adapted to continuous ones. In particular Birkhoff sums are replaced by integrals on finite segments of orbit, $\int_0^T \phi\left(\Phi^t(x)\right) \mathrm{d}t$.

0.3 Flat tori, rotations, and the modular surface

This work is concerned with the study of objects like *translation surfaces*, the *Teichmüller flow*, *interval exchange maps*. Since their definitions, and the relations between them, may appear rather technical, we try to clarify them with a quick description of the simplest case, which involves *flat tori*, the *geodesic flow on the modular surface*, and the *rotations of* $\mathbb{T}$.

§ 0.3.A Flat tori Take $\zeta_\beta, \zeta_\alpha \in \mathbb{R}^2 \cong \mathbb{C}$ a positive basis of $\mathbb{R}^2$; and consider the quotient

$$\mathbb{C} / \mathbb{Z}\zeta_\beta \oplus \mathbb{Z}\zeta_\alpha.$$

Topologically this is a closed 2-manifold of genus 1, but several other structures are specified on it:

- a preferred couple of generators of its fundamental group;
- a flat Riemannian metric;
- a structure of Riemann (i.e. complex) surface;
- a canonical complex 1-form, obtained by projection of the form $\mathrm{d}z$ on $\mathbb{C}$;
- for each direction ϑ, a partition (called *foliation*) of the torus in (oriented) parallel lines in direction ϑ—inherited from straight lines of $\mathbb{R}^2$.

We call such a structure a *flat torus*; flat tori are parametrised by the couple $(\zeta_\beta, \zeta_\alpha)$, so their set can be identified with $GL^+(2, \mathbb{R})$.

When two couples are related by $(\zeta_\beta, \zeta_\alpha) = v(\zeta'_\beta, \zeta'_\alpha)$ for some $v \in \mathbb{C}^*$, the underlying complex structures are *isotopic*. This implies that the set of the complex structures one can put on a topological torus, considered up to isotopies, has a one-to-one correspondence with

$$GL^+(2, \mathbb{R}) / \text{omotheties and rotations} \cong SL(2, \mathbb{R}) / SO(2, \mathbb{R}),$$

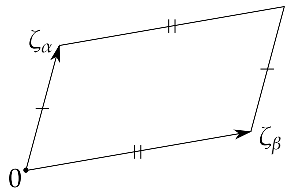

Figure 0.1: A fundamental domain for a flat torus

but, on the other hand, it has a bijection with the possible ratios $w = \zeta_\alpha/\zeta_\beta$ (seen as complex numbers) up to conjugation: therefore

$$SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \{w \in \mathbb{C} | \Im(w) > 0\} = \mathbb{H}^2.$$

If $\zeta_\beta = (a_\beta, b_\beta)$ and $\zeta_\alpha = (a_\alpha, b_\alpha)$, the corresponding element of $GL^+(2,\mathbb{R})$ is $\begin{pmatrix} a_\beta & a_\alpha \\ b_\beta & b_\alpha \end{pmatrix} \in GL^+(2,\mathbb{R})$. For any $t \in \mathbb{R}$, let us consider multiplication on the left by $\mathrm{diag}(e^t, e^{-t}) \in SL(2,\mathbb{R})$; this gives new vectors $\zeta_\alpha(t), \zeta_\beta(t)$, and their ratio as complex numbers is

$$w(t) = \frac{e^{2t}a_\alpha + ib_\alpha}{e^{2t}a_\beta + ib_\beta},$$

which parametrises a *geodesic* of $\mathbb{H}^2$; if we vary $\zeta_\alpha, \zeta_\beta$ so that their ratio $w \in \mathbb{H}^2$ stays constant — i.e. we consider all the flat tori inducing the complex structure which corresponds to w — all the non-costant geodesics passing through the point $w \in \mathbb{H}^2$ are obtained. Therefore we can identify the space of flat tori, namely $GL^+(2,\mathbb{R})$, with the *(complex) tangent bundle* (deprived of the zero section) to the space of Riemann tori, identified with $\mathbb{H}^2$; and the *geodesic flow* on the tangent bundle can be seen as a deformation of a flat torus: its horizontal foliation is progressively 'stretched', while its vertical one is 'shrunk' by the same factor.

§ 0.3.B The modular surface Suppose two couples $(\zeta_\beta, \zeta_\alpha)$, $(\zeta'_\beta, \zeta'_\alpha) \in GL^+(2,\mathbb{R})$ generate the same lattice: $\mathbb{Z}\zeta_\beta \oplus \mathbb{Z}\zeta_\alpha = \mathbb{Z}\zeta'_\beta \oplus \mathbb{Z}\zeta'_\alpha$; this is equivalent to say that a matrix $A \in SL(2,\mathbb{Z})$ exists such that

$$\left(\zeta_\beta \middle| \zeta_\alpha\right) A = \left(\zeta'_\beta \middle| \zeta'_\alpha\right).$$

The structures listed in paragraph 0.3.A that are induced on each of the two generated flat tori, except for the basis for the fundamental group, correspond under a *diffeomorphism* between them. Therefore the space of flat tori up to diffeomorphisms is

$$GL^+(2,\mathbb{R})/SL(2,\mathbb{Z})$$

and, similarly, the space of Riemann tori up to diffeomorphisms is

$$\mathbb{H}^2/SL(2,\mathbb{Z}) = \mathbb{H}^2/\mathbb{P}SL(2,\mathbb{Z}),$$

where the action of $SL(2,\mathbb{Z})$ is performed through Möbius tranformations (hence the equality between the two quotients). The latter quotient is called the *modular surface* and it is an *orbifold*, that is, it is a manifold «with finite-type singularities», which arise at the points obtained by projecting $i, e^{i\pi/3} \in \mathbb{H}^2$. Indeed the action of $SL(2,\mathbb{Z})$ at these two points is non-free, the point i being fixed by an element of order 2, and the point $e^{i\pi/3}$ being fixed by an element of order 3 (this means that the corresponding Riemann structures have non-trivial automorphisms). Moreover the modular surface has a *cusp*.

The modular surface inherits a geodesic flow from $\mathbb{H}^2$. However, any geodesic in $\mathbb{H}^2$

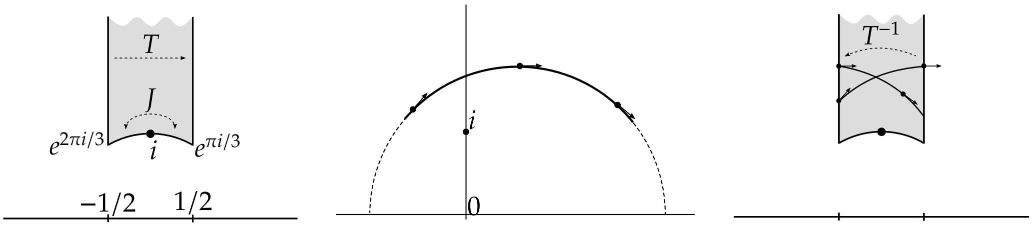

Figure 0.2: The first picture shows a fundamental domain for the action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$, and how are its edges identified under $T(z) = z + 1$ and $J(z) = -1/z$ (which generate $SL(2,\mathbb{Z})$) to obtain the modular surface. The second and the third picture compare the behaviours of a geodesic of $\mathbb{H}^2$ and of its projection in the modular surface.

simply goes to infinity, whereas all geodesics in the modular surface which are not obtained by projection of a vertical geodesic of $\mathbb{H}^2$ make infinitely many 'excursions' to the cusp and then get back. Figure 0.2 shows a fundamental domain in $\mathbb{H}^2$ for the definition of the modular surface, and an example of geodesic.

§ 0.3.C **Vertical flows and rotations of $\mathbb{T}$** Let us consider, for instance, a flat torus obtained by taking $\zeta_\alpha = 1$: the curve obtained from projection of the segment $[0,1] \subset \mathbb{C}$ can be considered as an embedding of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ in the torus. Starting from a point $x \in \mathbb{T}$, one can follow the vertical foliation of the flat torus upwards, until one gets back to $\mathbb{T}$, and precisely in a point $x + \vartheta$. One sees easily that ϑ does not depend on x, therefore the first return map of the vertical flow on $\mathbb{T}$ is a rotation r_ϑ.

The dynamical properties of r_ϑ are very different according to ϑ: when $\vartheta \in \mathbb{Q}/\mathbb{Z}$, every point is *q-periodic*, where q is the least possible denominator of ϑ. Otherwise — therefore for *almost every* ϑ in the sense of Lebesgue measure — it is a well-known fact that each orbit of r_ϑ is dense, that is the map r_ϑ is minimal. This implies that the vertical flow on the flat torus has the same property.

Furthermore, again for irrational ϑ, each orbit is *equidistributed*, that is for each continuous function $\phi : \mathbb{T} \to \mathbb{R}$ and any $x \in \mathbb{T}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(x + j\vartheta) = \int_{\mathbb{T}} \phi(x)\mathrm{d}x.$$

According to the stronger version of the Birkhoff Theorem 0.2.8, this means that r_ϑ is uniquely ergodic, and the Haar (i.e. Lebesgue) measure on $\mathbb{T}$ is the only f-invariant one.

The rotation r_ϑ can also be regarded as a self-map T of the fundamental domain $I = [0,1) \subset \mathbb{R}$. Let us consider I divided into two sub-intervals $I^t_\alpha = [0, 1-\vartheta)$ and $I^t_\beta = [1-\vartheta, 1)$; and also in $I^b_\beta = [0,\vartheta)$ and $I^b_\alpha = [\vartheta, 1)$: the map T translates the sub-interval I^t_α onto I^b_α, and the sub-interval I^t_β onto I^b_β; that is, it exchanges the two sub-intervals I^t_α and I^t_β.

§ 0.3.D **Exchanges of two intervals** Motivated by the viewpoint described above, let us consider a map T on a generic interval $I = [0, \lambda_\alpha + \lambda_\beta)$ which translates the sub-interval $I^t_\alpha = [0, \lambda_\alpha)$ onto $I^b_\alpha = [\lambda_\beta, \lambda_\alpha + \lambda_\beta)$; and the sub-interval $I^t_\beta = [\lambda_\alpha, \lambda_\alpha + \lambda_\beta)$ onto $I^b_\beta = [0, \lambda_\beta)$.

A similar map is derived from T with the following proceeding of *truncation*, valid for $\lambda_\alpha \neq \lambda_b$ (see Figure 0.3): remove from I the shorter between its sub-intervals on the right, I^t_β

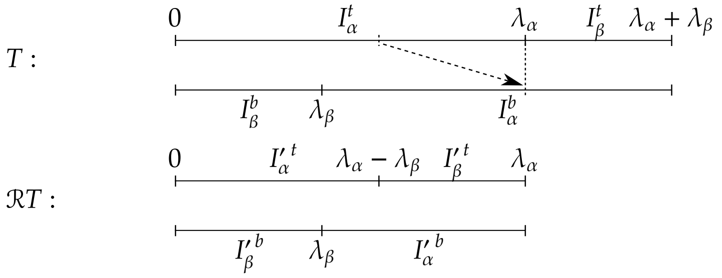

Figure 0.3: Graphical representation of the process of truncation, in the case $\lambda_\alpha > \lambda_\beta$.

and I^b_α, and call I' the resulting interval. Let $\mathcal{R}T : I' \to I'$ be the *first retun map* of T in I', that is for $x \in I'$ we define $\mathcal{R}T(x)$ as the first point of its positive orbit under T that lies again in I'. According to the Poincaré recurrence Theorem 0.2.4, this is a good definition for almost every $x \in I'$; in this case, actually, the map is defined for *every* point.

If $\lambda_\beta > \lambda_\alpha$, the map $\mathcal{R}T$ is again an exchange between the sub-intervals $[0, \lambda_\alpha)$ and $[\lambda_\alpha, \lambda_\beta)$; if $\lambda_\beta < \lambda_\alpha$, $\mathcal{R}T$ exchanges the sub-intervals $[0, \lambda_\beta)$ and $[\lambda_\beta, \lambda_\alpha)$. When $\lambda_\beta/\lambda_\alpha$ is irrational, the truncation algorithm can be iterated infinitely many times.

We can also consider a *faster version*, that is: if $\lambda_\alpha > \lambda_\beta$ we call r the largest positive integer such that $\lambda'_* := \lambda_\alpha - (r-1)\lambda_\beta > 0$. We set $I' = [0, \lambda'_*)$ and we denote $\mathcal{R}^*T : I' \to I'$ the first return map of T; that is, we set $\mathcal{R}^*T = \mathcal{R}^r T$. In the case $\lambda_\beta > \lambda_\alpha$ we proceed in a similar way.

§ 0.3.E A suspension construction Given T an exchange of two intervals whose lengths are λ_α and λ_β respectively, one can construct a flat torus with a marked horizontal segment such that T appears as a return map.

Take two real numbers $\tau_\alpha > 0 > \tau_\beta$, and set $\zeta_\alpha := \lambda_\alpha + i\tau_\alpha$, $\zeta_\beta := \lambda_\beta + i\tau_\beta$. Then $\mathbb{C}/(\mathbb{Z}\zeta_\alpha \oplus \mathbb{Z}\zeta_\beta)$ is a flat torus with the following property: let H be the horizontal segment on the torus obtained from projection of $[0, \lambda_\alpha + \lambda_\beta) \subset \mathbb{R} \subset \mathbb{C}$. Then the map T on H is the first return map of the vertical upgoing flow on the torus, i.e. its foliation in direction $\pi/2$. This is easily seen if one considers the torus as the quotient of the parallelogram whose sides are specified by ζ_α and ζ_β (see Figure 0.4 on the left).

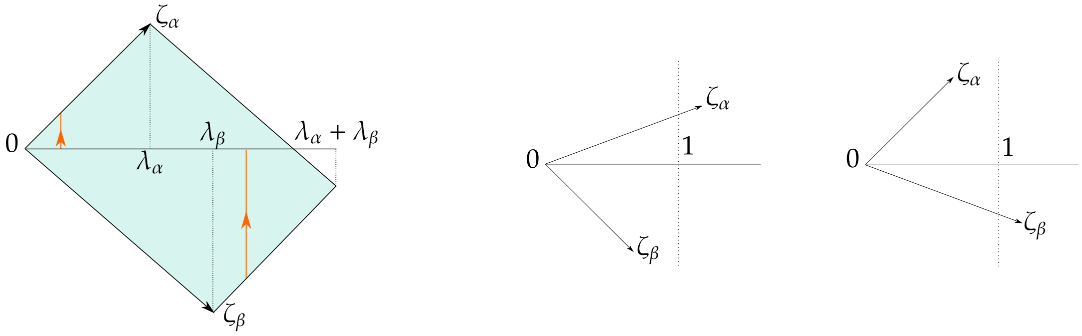

Figure 0.4: The picture on the left shows an example of suspension construction; the pictures on the right illustrate the two possibilities for a preferred basis as defined in Lemma 0.3.1.

This construction also establishes a relationship between geodesic flow on the *unit* tangent bundle of the modular surface (namely $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$, or the space of the flat tori up to diffeomorphisms) and the truncation algorithm above described. An element of the former can be identified with a lattice in $\mathbb{C}$ (without any preselected basis) with covolume 1; we restrict our attention to *irrational* lattices, i.e. lattices intersecting $\mathbb{R}$ and $i\mathbb{R}$ only in 0 (which are the typical ones).

For each lattice as above we select a preferred basis, according to the following

Lemma 0.3.1. *For each Λ irrational lattice in $\mathbb{C}$ with covolume 1 there exists an unique basis $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$, $\zeta_\beta = \lambda_\beta + i\tau_\beta$ such that one of the two following conditions holds:*
- $\lambda_\alpha \geq 1 > \lambda_\beta > 0$ *and* $0 < \tau_\alpha < -\tau_\beta$;
- $\lambda_\beta \geq 1 > \lambda_\alpha > 0$ *and* $0 < -\tau_\beta < \tau_\alpha$.

The two possibilities are shown in Figure 0.4 on the right. In particular a preferred basis is one that makes a flat torus appear as obtained from suspension of an exchange of two intervals. How do those preferred bases behave under the geodesic flow? Recall that, at the time t (which we suppose > 0), it stretches horizontal lengths (and in particular the λ's) by a factor e^t, while it shrinks vertical ones (and in particular the τ's) by the same factor. Denote g^t the geodesic flow, and $\zeta(t)$ the preferred basis for the flat torus $g^t(\mathbb{C}/\Lambda)$, where Λ is a lattice as above.

For sufficiently small $t > 0$, the preferred basis $\zeta(t)$ is simply $g^t_*\zeta(0)$. In particular this is true for all $t \in [0, t_0)$, where $t_0 := -\log\left(\min\{\lambda_\alpha, \lambda_\beta\}\right) > 0$. At the time t_0 this is not true anymore, as the real part of both vectors of the basis is at least 1.

Suppose for instance that $\lambda_\alpha > \lambda_\beta$, so that the preferred basis for Λ satisfies the first of the two conditions in the Lemma; we set

$$\zeta'_\alpha := g^{t_0}_*(\zeta_\alpha - r\zeta_\beta)$$

where r is the maximum positive integer such that ζ'_α still has a positive real part; and we set $\zeta'_\beta := g^{t_0}_*(\zeta_\beta)$. This basis for the lattice $g^{t_0}_*(\Lambda)$ is the preferred one, as it satisfies the second condition in the Lemma: and it has been obtained following the same instructions that define the fast version of truncation. The same holds in the case $\lambda_\beta > \lambda_\alpha$. In any case, since we started from an irrational lattice, infinite iteration of truncation is allowed.

To summarise: after we have selected a preferred basis for each irrational element of $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$, we have a way to define a map from «almost all» the unit tangent bundle of the modular surface to the set of the exchanges of two intervals. If we follow an orbit of the geodesic flow, the associated exchange map is piecewise constant; where a discontinuity takes place, it changes according to the fast truncation $\mathcal{R}^*$, up to a scale factor.

1 What is Teichmüller dynamics?

As we already announced, our first aim is to generalise the arguments developed in section 0.3. More precisely we want to consider:

- orientable surfaces whose genus is more than 1;
- maps of the interval that behave as an exchange of more that two sub-intervals.

This chapter focuses on the former; the following one will focus on the latter, and on their relationship with the geometrical objects we are going to build.

The path towards generalisation requires us to go beyond several obstacles. Of course one can define the space of the complex structures on a topological surface up to isotopies, the so-called *Teichmüller space*, but in the case of the torus it has been largely helpful that this space is naturally identified with the hyperbolic plane $\mathbb{H}^2$: a structure of Riemannian manifold, and therefore a notion of geodesic, was already given.

We will see that there is a natural topology on the Teichmüller space, according to which it is a manifold (actually, homeomorphic to some $\mathbb{R}^n$). Teichmüller space is *not* endowed with a Riemannian structure, but we will get close to it (at least for what concerns us): it is a *geodesic metric space*, with special properties.

While working to define a distance on the Teichmüller space, we will encounter a sort of *tangent space* at every point, which specifies the possible 'directions' of geodesics starting at that point. In the case of complex tori, we already know that this role is played by the set of the flat tori inducing the considered complex structure. For surfaces of higher genus this is still true: a notion of *flat surface* can be defined, even if we are forced (for instance, because of the Gauss-Bonnet theorem) to admit *singularities* on those structures. The *geodesic flow*, made up by the curves such that the length of their short portions equals the distance between their endpoints, results to be described by the action of $\mathrm{diag}(e^t, e^{-t})$ on flat structures, as it is the case for tori.

Another similarity with $\mathbb{H}^2$ emerges: this Teichmüller geodesic flow does not present any recurrence property. In the case of the torus we preferred to project that flow on (the tangent bundle to) the modular surface, therefore to interpret it as a flow among flat tori up to diffeomorphism. In the same way, we project the Teichmüller flow on the *moduli spaces*, namely the spaces of complex structures up to diffeomorphisms.

Similarly as the modular surface, moduli spaces are orbifolds, and are noncompact as they own a cusp. There is plenty of invariant subsets of the tangent bundle which are invariant under the geodesic flow (such subsets are called *strata*); our attention will be directed in particular to the ones made up of flat structures with a property of *orientability*: this is necessary for the formalism we are going to develop.

This chapter is not a complete exposition of the standard Teichmüller theory: our purpose is to present it rapidly in order to motivate our further discussions. We will mostly adapt the approach used in [FM11] and [Mar12].

1.1 Teichmüller and moduli spaces: classic definitions

§ 1.1.A Riemann structures up to homotopy Let S be a smooth closed oriented surface. A *Riemann structure* on S is a pair (X, ϕ) where X is a Riemann surface (i.e. a complex 1-manifold) and $\phi : S \longrightarrow X$ (called a *marking* for X) is an orientation-preserving diffeomorphism. We say that two Riemann structures (X_1, ϕ_1) and (X_2, ϕ_2) on S are *homotopic* if there exists a biholomorphism $f : X_1 \longrightarrow X_2$ such that the following diagram commutes up to homotopy of maps:

$$\begin{array}{ccc} & S_g & \\ {}^{\phi_1}\swarrow & & \searrow^{\phi_2} \\ X_1 & \xrightarrow{\quad f \quad} & X_2 \end{array} \tag{1.1}$$

This notion of homotopy is an equivalence relation on the set $\mathrm{RS}(S)$ of the Riemann structures on S, so we can define:

Definition 1.1.1. The *Teichmüller space* of a surface S is the quotient set

$$\mathrm{Teich}(S) := {}^{\mathrm{RS}(S)}\big/_{\text{homotopy}}.$$

Equivalently, let $\mathrm{Diff}^+(S)$ be the Lie group of the orientation-preserving diffeomorphisms of S; and let $\mathrm{Diff}_0(S) < \mathrm{Diff}(S)$ be the connected component of Id_S, i.e. the subgroup of the diffeomorphisms which are isotopic to identity. $\mathrm{Diff}(S)$ naturally acts on $\mathrm{RS}(S)$ on the right by $(X, \phi) \cdot f = (X, \phi \circ f)$, and we have

$$\mathrm{Teich}(S) = {}^{\mathrm{RS}(S)}\big/_{\mathrm{Diff}_0(S)}.$$

Remark 1.1.2. When $S = S_1$ is the torus, this formal definition of Teichmüller spaces actually corresponds to the construction given in paragraph 0.3.A. Indeed, let us fix a basis $\gamma_\alpha, \gamma_\beta$ for $\pi_1(S_1) \cong \mathbb{Z}^2$. A positive basis $\zeta_\alpha, \zeta_\beta$ for $\mathbb{R}^2$ can be interpreted as the marking $\phi(S_1) \to \mathbb{C}/(\mathbb{Z}\zeta_\alpha \oplus \mathbb{Z}\zeta_\beta)$ which transforms γ_α in the closed curve obtained by quotient of the segment from 0 to ζ_α; and the same for γ_β (the homotopy class of such a marking is uniquely determined). The markings associated to $(\zeta_\alpha, \zeta_\beta)$ and $(\zeta'_\alpha, \zeta'_\beta)$ are homotopic if and only if $(\zeta'_\alpha, \zeta'_\beta) = v(\zeta_\alpha, \zeta_\beta)$ for some $v \in \mathbb{C}^*$; and each marked Riemann torus is, up to homotopy, obtained this way. Thus, recalling the arguments we developed, we have a natural identification $\mathrm{Teich}(S_1) \cong \mathbb{H}^2$. ◇

§ 1.1.B The algebraic topology We are talking about Teichmüller *spaces* and not simply about *sets*, as they can be endowed with several additional structures. First of all there is an intrinsic way to define a so-called *algebraic topology* on $\mathrm{Teich}(S)$. Rather than giving its formal definition, we will describe it in a more concrete way.

The algebraic topology on $\mathrm{Teich}(S_1)$ coincides with the topology of $\mathbb{H}^2$. For higher genus surfaces, let us recall that their Euler characteristic $\chi(S)$ is negative; and that such a surface can be always endowed with (several) *hyperbolic structures*, namely Riemannian structures with constant Gaussian curvature −1. By 'structure' we mean, as before, the specification of a *marking* from S to a hyperbolic surface diffeomorphic to S.

Every hyperbolic structure induces a *conformal* structure, therefore a complex one. This yields a *one-to-one* correspondence between the set of such structures and $\mathrm{RS}(S)$. Therefore we can identify

$$\mathrm{Teich}(S) = {}^{\{\text{hyperbolic structures on } S \text{ with finite area}\}}\big/_{\mathrm{Diff}_0(S)}.$$

Let g be the genus of S respectively, and let $\gamma_1, \dots, \gamma_{3g-3}$ be simple, closed, essential curves

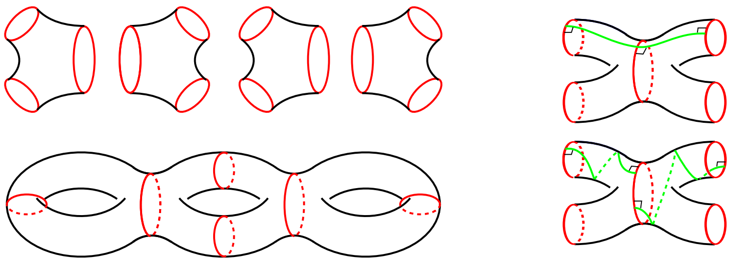

Figure 1.1: On the left: a pant decomposition for the closed surface of genus 3. On the right: two gluings of the same pants, but with different twist angle. The lines in the drawings are each one the only geodesic segment which is perpendicular to the boundary components it connects: it is evident that the hyperbolic metrics differ according to the twist angle.

on S which cut it in a collection of *pants* (i.e. topological spheres deprived of three open disks; see Figure 1.1 on the left).

It can be proved that a unique hyperbolic structure (up to isotopies) is determined on a pant if we require its boundary components to be geodesics, each of them with an assigned length. So, different hyperbolic structures are obtained on S according to the length we assign to each boundary component of the glued pants; but also according to how much we twist a collar neighbourhood of two boundary components before gluing (see Figure 1.1 on the right).

The *Fenchel-Nielsen coordinates* on $\mathrm{Teich}(S)$ are defined as follows. For each $\mathcal{X} = [(X,\phi)] \in \mathrm{Teich}(S)$ and $1 \leq j \leq 3g-3$, define $\ell_j(\mathcal{X}) \in \mathbb{R}_+$ as the length of the only hyperbolic closed geodesic on X which is homotopic to $\phi \circ \gamma_j$. A neighbourhood of this geodesic results from gluing collars of two boundary components of pants: we call $\vartheta_j(\mathcal{X})$ the angle we have to twist one of them in order to obtain the desired structure $\mathcal{X}$.

Such coordinates provide an extremely natural way to see the algebraic topology:

Theorem 1.1.3 (Fricke). *The map given by the Fenchel-Nielsen coordinates,*

$$\begin{array}{rccl} FN: & \mathrm{Teich}(S) & \longrightarrow & \mathbb{R}^{6g-6} \\ & \mathcal{X} & \longmapsto & (\log \ell_1(\mathcal{X}), \ldots, \log \ell_{3g-3}, \vartheta_1(\mathcal{X}), \ldots, \vartheta_{3g-3}(\mathcal{X})) \end{array}$$

is one-to-one. If $\mathrm{Teich}(S)$ *is endowed with the algebraic topology, it is also a homeomorphism.*

The Fenchel-Nielsen coordinates can be defined also for non-closed surfaces: if we consider a surface of genus g with b boundary components and n punctures, again with negative Euler characteristic, we will have $3g-3+2b+2n$ length coordinates, and $3g-3+b$ twist coordinates. Fricke's theorem will still hold.

§ 1.1.C Moduli spaces We now define the generalisation of the modular surface. Recall that the *mapping class group* of S is the quotient group $\mathrm{Mod} := \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$ whose elements are isotopy classes of diffeomorphisms of S.

Definition 1.1.4. The *moduli space* of S is the set

$$\mathrm{M}(S) := {}^{\mathrm{RS}(S)}\!/_{\mathrm{Diff}^+(S)} = {}^{\mathrm{Teich}(S)}\!/_{\mathrm{Mod}(S)}.$$

Put into words, $\mathrm{M}(S)$ is the set of the Riemann structures on S up to orientation-preserving diffeomorphisms.

It is a theorem of Fricke that the action of Mod(S) on Teich(S) is properly discontinuous, though it is not free in general. So, M(S) has a structure of *orbifold*, that is to say for each $\mathcal{X} \in \mathrm{M}(S)$ there is a local chart $\mathbb{R}^d/\Gamma_{\mathcal{X}} \to U_{\mathcal{X}}$ (with $d = 6g-6$), where $\Gamma_{\mathcal{X}} < O(n)$ is a finite group, and it is nontrivial only for a closed, nowhere dense subset of M(S). In this case of moduli spaces, those singular points come from the $[(X, \phi)] \in$ Teich(S) that admit some nontrivial automorphism.

1.2 Flat and translation structures

§ 1.2.A Flat surfaces and singular foliations We have seen in paragraph 0.3.A that a flat torus, obtained as the quotient of $\mathbb{C}$ by a lattice Λ, is a topological torus endowed with a flat Riemannian metric; moreover, for each $\vartheta \in S^1$, the family of parallel lines in $\mathbb{C}$ in direction ϑ induces a partition of the torus in parallel geodesics.

According to the Gauss-Bonnet and Poincaré-Hopf theorems, no closed surfaces of higher genus can be endowed with such structures. This leads to the following definitions, which introduce the possibility of having *singularities*.

Definition 1.2.1. Let S be a smooth closed surface. A *flat structure* on S is specified by:
1. a finite subset $\Sigma = \{p_1, \dots, p_s\} \subset S$, whose elements are called *singular points*;
2. an atlas $\zeta = \{(U_\alpha, \phi_\alpha)\}_\alpha$ for $S \setminus \Sigma$, made up of charts $\phi_\alpha : U_\alpha \to \mathbb{C}$ such that the coordinate changes are of the form $\phi_\beta \circ \phi_\alpha^{-1} : z \mapsto \pm z + c_{\beta\alpha}$, where $c_{\beta\alpha}$ are constants; we also require this atlas to be maximal among the ones with this property;
3. for each $j = 1, \dots, s$ there is an integer $k_j \geq 2$ such that the flat structure defined by means of the above atlas has a conical singularity of angle $k_j\pi$; moreover $k_j = 1$ is allowed only if p_j is a puncture for S.

A flat surface will be denoted (S, Σ, k, ζ), where $k = (k_1, \dots, k_s)$.

It is worth spending some words about what a *conical singularity* of angle $k_j\pi$ at a point p_j is exactly. Take k_j copies $\Pi_1, \dots, \Pi_k$ of a straight angle with *horizontal* sides, no matter which half-plane of $\mathbb{R}^2$ it spans; let a_i and b_i be their respective edges. Then glue each b_i with a_{i+1} for $i = 1, \dots, k-1$, and b_k with a_1. Call Π the resulting space, and 0_Π its point coming from the vertices of the straight angles Π_i; Π will have a natural flat metric out of 0_Π. The conical singularity is then described by a map $\phi_j : U_j \to \Pi$ such that: U_j is a neighbourhood of p_j in S not containing other points of Σ; $\phi_j(p) = 0_\Pi$; $\phi_j(U_j)$ is an open subset of Π and ϕ_j is a homeomorphism with its image; for each $V \subseteq U_j \setminus \{p_j\}$ open set such that $\phi_j(V)$ is contained in the union of two consecutive half-planes, the restriction $\phi_j|_V$ is a chart in the flat atlas ζ.

To be precise, since we are talking of a 'structure' on the topological surface S, we would expect that a marking is defined: indeed we can take the tautological one given by the identity map

$$S \text{ as a topological surface} \to (S, \Sigma, k, \zeta).$$

With this definition, a flat structure induces a partition in curves in every direction, but each of these ones has still singularities in Σ:

Definition 1.2.2. Let (S, Σ, k, ζ) be a flat structure, and let $\vartheta \in \mathbb{T}$. The *singular foliation* for that structure in direction ϑ is a partition of $S \setminus \Sigma$ in curves (called *leaves*) such that, for every chart $(U_\alpha, \phi_\alpha) \in \zeta$, they appear as the partition of $\phi_\alpha(U_\alpha)$ in straight lines in direction ϑ.
The foliation in direction $\vartheta = 0$ is called *horizontal*, the one in direction $\vartheta = \pi/2$ is called *vertical*. A leaf (in any direction ϑ) having an endpoint in a singularity is called a *separatrix*; a leaf connecting two singular points is called a *saddle connection*.

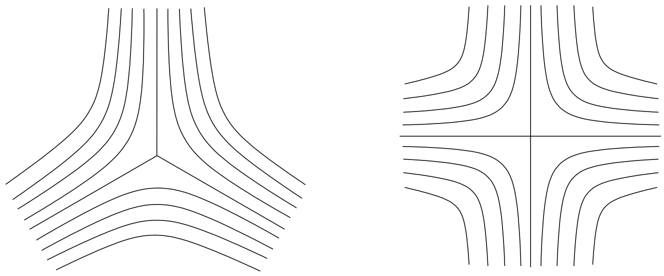

Figure 1.2: Vertical foliations in the neighbourhood of a singularity with $k_j = 3$ and $k_j = 4$, respectively.

In other words, in the neighbourhood of any regular point for the flat structure, a singular foliation simply appears as parallel lines. In the neighbourhood of a singular point, instead, each of the above foliations appears as the level curves of a saddle with k_j separatrices (see Figure 1.2). Note that flat structures give natural ways to calculate *lengths of curves* and *areas*.

Remark 1.2.3. A flat structure on a surface S specifies a Riemann atlas on $S \setminus \Sigma$; but its singularities in the points of Σ are all removable, so a (canonical) Riemann atlas on the whole S is induced. We attach to it again the tautological marking given by the identity map

$$S \text{ as a topological surface} \to S \text{ with this Riemann atlas;}$$

this way every flat structure induces an element of $\mathrm{RS}(S)$. ◇

A generalisation of the Poincaré-Hopf theorem gives the following constraint for combinatorial data of flat structures:

Proposition 1.2.4 (Euler-Poincaré formula). *Let (S, Σ, k, ζ) be a flat surface. The following relation holds:*

$$\sum_{j=1}^{s} (2 - k_j) = 2\chi(S) = 4(1 - g).$$

§ 1.2.B Quadratic differentials

Flat structures also have an alternative description:

Definition 1.2.5. Let (X, ϕ) be a Riemann structure on a closed surface S. A *holomorphic quadratic differential* q is a holomorphic section of $T^*X \otimes T^*X$ (where we see X as a complex 1-manifold), such that $q(p)$ is a symmetric 2-form for each $p \in X$.

For what we will need, if $\{w_\alpha\}$ is an atlas of local charts for X, q is simply a collection $\{q_\alpha(w_\alpha)\mathrm{d}w_\alpha^2\}_\alpha$ where q_α are holomorphic functions, defined on the images of the charts w_α, such that when applying a coordinate change we have

$$q_\beta(w_\beta)\left(\frac{\mathrm{d}w_\beta}{\mathrm{d}w_\alpha}\right)^2 = q_\alpha(w_\alpha) \tag{1.2}$$

From now on we will omit the word 'holomorphic'. Suppose $q \not\equiv 0$; then, for each $p \in X$, there is a chart z centred at p such that $q(z) = z^{k_p - 2}\mathrm{d}z^2$, for a $k_p \in \mathbb{Z}$, $k_p \geq 2$; actually $k_p > 2$ only if p is a zero for q. This is called a *natural chart*. The coordinate change between two natural charts has always the form $z \mapsto \pm z + c$.

If we set $\Sigma := \phi^{-1}(\{\text{zeros for } q\}) \subset S$, a flat structure on S with singularities at points of Σ is naturally induced: it suffices to take as an atlas the set of all the natural charts $z \circ \phi$ centred at points of $S \setminus \Sigma$; each point $\phi^{-1}(p) \in \Sigma$ is the vertex of a conical singularity of angle $k_p\pi$.

Conversely, start from a flat structure on S with atlas ζ: according to Remark 1.2.3, it induces a Riemann structure (X, Id_S) on S. A nonzero quadratic differential, holomorphic with respect to the Riemann structure X, is obtained on $X \setminus \Sigma$ by setting it locally equal to $\mathrm{d}z_\alpha^2$ for each chart $z_\alpha \in \zeta$; formula 1.2 gives an unique way of extending it to the complex charts centred at points of Σ. Therefore, the set of quadratic differentials on X is in one-to-one correspondence with the set of flat structures on S which induce the Riemann structure X.

We will denote QD(X) the vector space of all quadratic differentials on the Riemann surface X. Using complex-analytic arguments, it can be shown that

Theorem 1.2.6. *Let X be a closed Riemann surface of genus g. Then*

$$\dim_{\mathbb{C}} \mathrm{QD}(X) = 3g - 3.$$

§ 1.2.C Construction via polygons An easy way to construct flat structures is the following: every topological closed surface can be obtained from a polygon $P \subseteq \mathbb{R}^2$ (even a disconnected one) whose sides are parallel in pairs, by identifying each side with its parallel.

If we call S the quotient topological surface, and Σ the projection of the vertices of P, a translation structure is induced on S with Σ its singular set: it is enough to take the interior of P as one of the translation charts; and to add charts to cover the interiors of the sides of P.

Conical angles at points of Σ are necessarily multiples of π, the only attention required is that we would like them to be at least 2π wide. Anyway, it can be seen easily that every flat structure on every surface can be obtained with this construction.

§ 1.2.D The orientable version Let us introduce a slight modification in the definition of flat structure with stronger requests:

Definition 1.2.7. Let S be a closed surface. A *translation structure* on S is almost the same as a flat structure (Definition 1.2.1), except that:

- in condition 2, we require the coordinate changes to be of the form $\phi_\beta \circ \phi_\alpha^{-1} : z \mapsto z + c_{\beta\alpha}$ (i.e. to be translations in $\mathbb{C}$);
- angles at conical singularities have to be $2h_j\pi$, for some integers $h_j \geq 1$.

A translation surface will be denoted (S, Σ, h, ζ), where $h = (h_1, \ldots, h_s)$. We will sometimes call h the *vector of indices*.

The notion of conical singularity is now better described if we take as a reference model the gluing Π of h_j copies of a full angle whose sides coincide with the positive real half-line in $\mathbb{C}$; a ramified covering $\rho : \Pi \to \mathbb{C}$ is naturally defined. The structure of conical angle at p_j will be specified by a map $\phi_j : U_j \to \Pi$ which satisfies the same requests as before, except for the last one: we will require that for each open subset $V \subseteq U_j \setminus \{p_j\}$ such that $(\rho \circ \phi_j)|_V$ is injective we have $(\rho \circ \phi_j)|_V \in \zeta$.

These new hypotheses imply that foliations are *oriented*: that is, each leaf in the ϑ-foliation can be oriented so that, in every chart of ζ, it appears directed in direction ϑ (rather than $-\vartheta$). In other words, leaves in direction ϑ become *geodesics* of the flat surface, in direction ϑ and with unit speed. In a small neighbourhood of each singularity, they appear as trajectories around a saddle point with an *even* number of separatrices; these ones are alternatively incoming the singularity and outgoing from it (see Figure 1.3). From now on, by *vertical flow* we will always mean the one in direction $\vartheta = \pi/2$ i.e. the one going upwards; by *horizontal flow* we will mean the one in direction $\vartheta = 0$ i.e. the one going rightwards.

Translation structures inducing a Riemann structure X on S naturally correspond to *Abelian differentials* on X, namely holomorphic 1-forms: a translation structure induces the 1-form ω

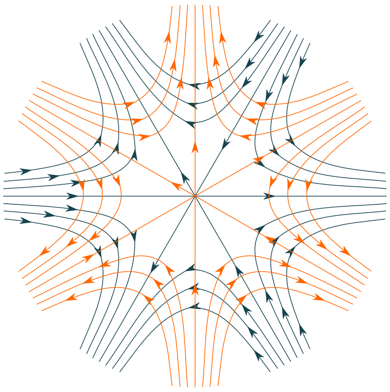

Figure 1.3: The horizontal ($\vartheta = 0$) and vertical ($\vartheta = \pi/2$) flows on a translation surface, in the neighbourhood of a singularity with $h_j = 3$.

that can be written as $\mathrm{d}z_\alpha$ for each chart $z_\alpha \in \zeta$, and extended to the whole S; conversely, each 1-form ω admits natural charts in which it can be written as $z^{h_p - 1}\mathrm{d}z$, which induce a translation structure. If $f_\alpha(w_\alpha)\mathrm{d}w_\alpha$ is a local expression for an Abelian differential ω in some chart for X, its *square* $f_\alpha(w_\alpha)^2\mathrm{d}w_\alpha^2$ is a local expression for the quadratic differential corresponding to the same flat structure.

On a translation surface not only lengths of curves are well-defined, but each of them is associated with a vector in $\mathbb{C}$:

Definition 1.2.8. Let $\gamma : I \to (S, \Sigma, h, \zeta)$ be a simple, smooth curve; let ω the Abelian differential associated to the translation structure. The *holonomy* of γ is the number

$$\mathrm{Hol}(\gamma) := \int_\gamma \omega = \int_I \gamma^* \omega.$$

1.3 The Teichmüller flow

§ 1.3.A Teichmüller maps It is time to recover, on the Teichmüller space of a closed surface of arbitrary genus, structures which resemble the ones of $\mathbb{H}^2$ we used in an essential way in section 0.3.

Definition 1.3.1. Let Flat(S) be the set of all flat structures that can be put on S; and let $\Sigma = \{p_1, \ldots, p_s\} \subset S$; $k = (k_1, \ldots, k_s)$; $\varepsilon = \pm 1$. The *stratum of flat structures* related to the triple (Σ, k, ε) is the set $\mathrm{Flat}(S, \Sigma, k, \varepsilon) \subset \mathrm{Flat}(S)$ whose elements are flat structures with Σ as set of singular points, πk_j the conical angle at p_j for each j, which are also, or are not, translation structures, according to ε (of course, if k has an odd entry and $\varepsilon = 1$ the stratum is empty).

Linear transformations of $\mathbb{R}^2$ act on each of these sets on the left: for $A \in GL(2, \mathbb{R})$ and $\zeta \in \mathrm{Flat}(S)$, we define $A \cdot \zeta$ as the translation atlas obtained from ζ by replacing each of its charts ϕ_α with $A \circ \phi_\alpha$. This action preserves each stratum; moreover, if we restrict it to $A \in SL(2, \mathbb{R})$, also areas of flat structures are preserved.

In particular, for $K \in \mathbb{R}_+$, the matrix

$$A_K := \begin{pmatrix} \sqrt{K} & 0 \\ 0 & 1/\sqrt{K} \end{pmatrix} \in SL(2, \mathbb{R})$$

has (similarly as for flat tori) the effect of stretching the horizontal foliation of ζ by a factor $\sqrt{K}$ (we are supposing $K > 1$), and of shrinking the vertical one by the same factor. The identity map $(S, \Sigma, h, \zeta) \to (S, \Sigma, h, A_K \cdot \zeta)$, i.e. the natural map from the old flat surface to the newly created one, is called a *Teichmüller map*; K is its *dilatation factor*.

Such a map also changes the underlying Riemann structure X induced on S by the flat structure ζ; we denote $A_K \cdot X$ the new Riemann structure obtained by completing the complex atlas $A_K \cdot \zeta$ to cover the whole S.

Teichmüller maps are canonical representatives for homotopy classes of homeomorphisms between Riemann surfaces. We state this result directly in the language of Teichmüller spaces:

Theorem 1.3.2 (Teichmüller's existence and uniqueness theorems). *X and Y be closed Riemann surfaces of genus $g > 1$, and let $f : X \to Y$ be a homeomorphism between them. Suppose a marking $\phi : S \to X$ is fixed. Then there exist, and are unique:*
- *a Riemann structure (Y', ψ') on S with $[(Y', \psi')] = [(Y, f \circ \phi)] \in \mathrm{Teich}(S)$;*
- *a couple of quadratic differentials $q_X \in \mathrm{QD}(X)$ and $q_{Y'} \in \mathrm{QD}(Y')$*

such that the corresponding flat structures on S are obtained one from another with a Teichmüller map (which is itself unique).

The statement still holds for $g = 1$, except that we have uniqueness up to translations of complex tori.

§ 1.3.B The metric We can now define a metric on Teich(S). Fix two elements $\mathcal{X} = [(X, \phi)], \mathcal{Y} = [(Y, \psi)] \in \mathrm{Teich}(S)$; then, according to the Teichmüller's theorem above, there is a unique way of taking another representative (Y', ψ') of $\mathcal{Y}$, and two flat structures on S which induce the Riemann structures (X, ϕ) and (Y', ψ') respectively, such that there exists a Teichmüller map from the first to the second. Let K be its dilatation factor: the *Teichmüller distance* between $\mathcal{X}$ and $\mathcal{Y}$ is then

$$d_{\mathrm{Teich}}(\mathcal{X}, \mathcal{Y}) := \frac{1}{2} |\log(K)|.$$

The Teichmüller theorem implies that it is a good definition; and it is easily seen that it actually gives a distance (triangular inequality holds essentially because dilation factors are sub-multiplicative under composition of maps). Moreover, if S is the torus, it coincides with the metric on $\mathbb{H}^2$.

Remark 1.3.3. We completely hid a notable fact in the discussions above. The distance d_{Teich} is a quantitative answer to the question: given two (isotopy classes of) Riemann structures $\mathcal{X}$ and $\mathcal{Y}$ on the surface S, how much is a homeomorphism between them far from being a biholomorphism?

Recall that holomorphic maps coincide with *conformal* maps, i.e. maps «which preserve angles». One says that a map is *quasi-conformal* if it «distorts angles by a bounded factor».

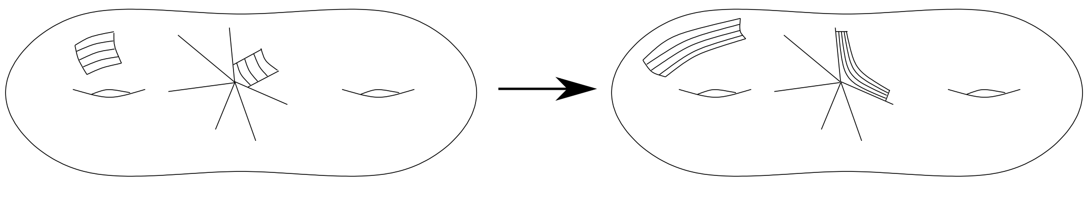

Figure 1.4: The action of a Teichmüller map on two portions of the horizontal foliation.

The dilatation factor K actually derives from this bounded factor; in the statement of Theorem 1.3.2, we didn't mention that Teichmüller maps are the ones which minimise K within their homotopy class: this is the real reason why they are preferred representatives. ◇

It can be proved that

Proposition 1.3.4. *The distance d_{Teich} is complete and induces the algebraic topology on* Teich(S).

§ **1.3.C Geodesics** Even if Teichmüller spaces are only topological manifolds with an additional structure of metric space, a geodesic flow, called the *Teichmüller flow*, can be specified on them, using flat structures and Teichmüller maps.

Definition 1.3.5. Let $\mathcal{X} = [(X, \phi)] \in \text{Teich}(S)$, and let $q \in \text{QD}(X) \setminus \{0\}$. We consider on X the flat structure given by q. The *Teichmüller line* starting at $\mathcal{X}$ in direction q is the map

$$\begin{array}{rccc} \mathcal{L}_{\mathcal{X},q}: & \mathbb{R} & \longrightarrow & \text{Teich}(S) \\ & t & \longmapsto & [(A_{e^{2t}} \cdot X, \phi)] \end{array}$$

We note that, for each $t_1, t_2 \in \mathbb{R}$ we have $d_{\text{Teich}}\left(\mathcal{L}_{\mathcal{X},q}(t_1), \mathcal{L}_{\mathcal{X},q}(t_2)\right) = |t_2 - t_1|$: therefore Teichmüller lines are geodesic, in the sense of metric spaces. Now we define what we would like to be an *exponential map* at a point $\mathcal{X}$ (we don't care whether it parametrizes each Teichmüller line the right way):

$$\begin{array}{rccc} \mathcal{E}_{\mathcal{X}}: & \text{QD}(X) & \longrightarrow & \text{Teich}(S) \\ & q & \longmapsto & \mathcal{L}_{\mathcal{X},q}(1) \end{array}$$

where we are setting conventionally $\mathcal{L}_{\mathcal{X},0}(1) = \mathcal{X}$. A lemma required for the proof of the Teichmüller existence theorem states that

Proposition 1.3.6. *The map $\mathcal{E}_{\mathcal{X}}$ is a homeomorphism.*

Moreover, we have the following result:

Proposition 1.3.7. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in$ Teich(S) be distinct. The following assertions are equivalent:*

- *$d(\mathcal{X}, \mathcal{Y}) + d(\mathcal{Y}, \mathcal{Z}) = d(\mathcal{X}, \mathcal{Z})$;*
- *there exists a Teichmüller line containing the three points $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$; and $\mathcal{Y}$ lies between the other ones.*

In this case, such a line is unique.

To sum up: the mentioned results imply that geodesic segments in Teich(S) for the distance d_{Teich} are exactly the segments of Teichmüller lines. In particular, all geodesic lines are defined on the whole $\mathbb{R}$, according to the completeness of Teich(S), in an Hopf-Rinow Theorem's fashion. Moreover two elements in Teich(S) are always connected by a geodesic because of Teichmüller's Theorem 1.3.2; and this geodesic is unique.

The direction of a Teichmüller line is specified by a quadratic differential: so, for each point $\mathcal{X} = [(X, \phi)]$, the vector space QD($X$) can be identified with the *tangent space* of Teich(S) at $\mathcal{X}$; actually, it is generally seen as a *cotangent space*.

Proposition 1.3.6 implies that a geodesic on Teich(S) cannot have any recurrence properties. Since each geodesic is the image under the map $\mathcal{E}_{\mathcal{X}}$ of a line in $\text{QD}(X) \cong \mathbb{R}^{6g-6}$, it simply «goes to infinity» in both directions. The Teichmüller flow has much more interesting properties when projected to the moduli space.

§ 1.3.D Switching to moduli spaces It can be proven that Mod(S) acts through isometries on Teich(S): this means that M(S) inherits the Teichmüller metric, as well as a notion of geodesic. The asymptotic properties of geodesics will be one of our main concerns in the remainder of this work (even if with a different formalism, more suitable for our investigations). In this paragraph we talk about some similarities of M(S) with the modular surface. First of all:

The moduli space of a closed surface of any genus ≥ 1 *has an infinite diameter.*

At the beginning of this chapter we anticipated that «infinity is in a single direction», since the moduli space has sort of a cusp, like the modular surface.

It is useful to recall that, if S is a closed surface of genus $g \geq 2$, its moduli space M(S) coincides with the set of the hyperbolic structures on S up to diffeomorphisms. Therefore, for each $\mathcal{X} \in$ M(S) represented by some hyperbolic structure X, we can define $\ell(\mathcal{X})$ as the minimum length of a closed geodesic in X. We have

Theorem 1.3.8 (Mumford). *For any* $\varepsilon > 0$ *the subspace*

$$\mathrm{M}_\varepsilon(S) := \{\mathcal{X} \in \mathrm{M}(S) | \, \ell(\mathcal{X}) \geq \varepsilon\}$$

is compact. Therefore the family $\{\mathrm{M}_{1/n}(S) | \, n \in \mathbb{N}\}$ *is an exhaustion in compact sets for* M(S).

So, a sequence in M(S) «goes to infinity», i.e. escapes from all compact subsets, if and only if evaluation of ℓ on this sequence goes to zero.

The following remarkable result is that:

Fix any $\varepsilon > 0$ *and suppose* $\mathcal{X}, \mathcal{Y} \in$ Teich(S) *are such that their projections in* M(S) *are not contained in* $\mathrm{M}_\varepsilon(S)$. *Then* $\mathcal{X}$ *and* $\mathcal{Y}$ *are connected by a path in* Teich(S) *such that its projection in* M(S) *lies entirely outside* $\mathrm{M}_\varepsilon(S)$.

This fact implies easily that M(S) has a 'cusp', in the sense of the definition below.

Definition 1.3.9. A locally compact topological space X is said to have *one end* if, for any compact subset $K \subseteq X$, the complement $X \setminus K$ has only one connected component such that its closure in X is noncompact.

The cusp of the modular surface has the same interpretation: it suffices to consider the unit area flat structures on the torus rather than the hyperbolic ones, and repeat the same arguments as above.

§ 1.3.E A setting for the Teichmüller flow Our discussions so far suggest that the quotient spaces

$$\mathrm{Flat}(S)/\mathrm{Diff}_0(\mathrm{S}) \quad \text{and} \quad \mathrm{Flat}(S)/\mathrm{Diff}^+(\mathrm{S}).$$

provide natural notions for *tangent bundles*, respectively, to Teich(S) and M(S), deprived of their zero sections. We have seen, indeed, that geodesics coincide with Teichmüller lines, which can be lifted to these two spaces in a natural way, resulting in a *Teichmüller flow*. However, in paragraph 1.3.A we mentioned that there is plenty of subsets of Flat(S), the *strata*, which are invariant under action of $GL(2, \mathbb{R})$; so, it is more natural to take such sets as the theatre for the Teichmüller flow. Definitions may differ from an author to another one, but identification of quadratic differentials with tangent vectors to Teichmüller spaces is usually forgotten. Following a well-established custom, the Teichmüller flow at the time $t \in \mathbb{R}$ will be denoted g^t on any of the spaces on which it is defined.

We describe the spaces used in our main reference texts, [Yoc07] and [Mar10]. They only regard translation structures and Abelian differentials: indeed in the next chapter we will introduce a comfortable formalism to study Teichmüller dynamics, but it works *only in this setting*.

We fix a (topological, or smooth) surface S, a singular locus $\Sigma = \{p_1, \ldots, p_s\}$ and a vector h of indices of singularities; we will quotient the related stratum of translation structures in a sligthly different way than prescribed in the classic definitions of Teichmüller and moduli spaces. The notation $\mathrm{Diff}(S, \Sigma)$ will stand for the Lie group of diffeomorphisms of S which are the identity on Σ; $\mathrm{Diff}^+(S, \Sigma)$, $\mathrm{Diff}_0(S, \Sigma)$, and $\mathrm{Mod}(S, \Sigma)$ will have analogous definitions.

Definition 1.3.10. The set

$$\mathcal{T}(S, \Sigma, h) := {\mathrm{Flat}(S, \Sigma, 2h, 1)}\big/{\mathrm{Diff}_0(S, \Sigma)}$$

is called a *stratum in the Teichmüller space of translation structures*; and

$$\mathcal{H}(S, \Sigma, h) := {\mathrm{Flat}(S, \Sigma, 2h, 1)}\big/{\mathrm{Diff}^+(S, \Sigma)} = {\mathcal{T}(S, \Sigma, h)}\big/{\mathrm{Mod}(S, \Sigma)}$$

is called a *stratum in the moduli space of translation structures*.

Recall also that, if we restrict the action to $SL(2, \mathbb{R})$, areas of flat structures are also preserved. In particular areas are preserved under the Teichmüller flow, so it makes sense to define an «unit area version»of strata. Let $\mathrm{Flat}^{(1)}(S, \Sigma, k, \varepsilon) \subset \mathrm{Flat}(S, \Sigma, k, \varepsilon)$ be the subset of the unit area flat structures belonging to a fixed stratum; we define the *normalised strata* of Teichmüller and moduli spaces as

$$\mathcal{T}^{(1)}(S, \Sigma, h) := {\mathrm{Flat}^{(1)}(S, \Sigma, 2h, 1)}\big/{\mathrm{Diff}_0(S, \Sigma)};$$

$$\mathcal{H}^{(1)}(S, \Sigma, h) := {\mathrm{Flat}^{(1)}(S, \Sigma, 2h, 1)}\big/{\mathrm{Diff}^+(S, \Sigma)} = {\mathcal{T}^{(1)}(S, \Sigma, h)}\big/{\mathrm{Mod}(S, \Sigma)}.$$

Remark 1.3.11. When dealing with strata of flat non-translation surfaces, most authors also admit conical singularities of angle π, which correspond to *simple poles* for the associated quadratic differential. Such quadratic differentials are to be included in the cotangent bundle to $\mathrm{Teich}(S)$ in order to generalise the theory developed in this chapter to *punctured* surfaces.◇

2 The formalism of interval exchange maps

In the previous chapter we only accomplished the first half of the work needed to generalise section 0.3. We are going to introduce *interval exchange maps*, namely self-maps of an interval that can be described as a rearrangement of sub-intervals (in general, more than 2), and therefore are completely determined once a set of parameters, belonging to some parameter space, has been chosen: the following exposition will be particularly detailed about the notations used for parameters. A property of ‘irrationality’ can be defined for such maps as well as for rotations (even if in this case it is only a *sufficient* condition for minimality); and all our arguments will need us to restrict to the i.e.m.s which satisfy this property.

The truncation proceeding we introduced in paragraph 0.3.D for exchanges of two intervals can be generalised without difficulties to the *Rauzy-Veech algorithm* for i.e.m.s. Its iteration originates a dynamical system in the parameter space: in other words, we have two levels of dynamical systems. The situation resembles what happens for translation structures: there are flows *on each* structure, and there is also a flow *among* structures. In the following chapters we will obtain results that mostly regard *generic* i.e.m.s, in the sense of Lebesgue measure in the parameter space; and it is remarkable that the kind of arguments we will use will be more about the dynamics in the parameter space, than about the dynamics of i.e.m.s themselves.

Let us consider a translation surface, and a horizontal segment on it: the vertical flow on the surface induces a return map on the selected segment that is an i.e.m.; and, if we consider the return map on an appropriate initial sub-segment of the previous one, the new i.e.m. is obtained from the previous one with the Rauzy-Veech algorithm. Conversely, starting from an irrational i.e.m., one can apply a generalised version of the suspension construction in paragraph 0.3.E, and find a translation surface such that the considered i.e.m. appears as a return map. The parameters associated to this i.e.m. play the role of «horizontal lengths» in this construction, therefore one can apply it only after some additional parameter, specifying «vertical lengths», has been chosen.

This construction gives a way for passing from translation surfaces to i.e.m.s and conversely; so, an *augmented* parameter space for i.e.m.s can play the role of a local chart for a stratum, and also carry the Lebesgue measure on it. The way charts are glued altogether can also be deduced. In particular one may restrict to a subset, called *box*, of the augmented parameter spaces, with two boundary components. There is a way of gluing boxes altogether along their boundaries; in so doing, we obtain a model for almost all of a moduli stratum.

When considering this model, the relationship between translation surfaces and i.e.m.s appears again between the dynamical systems at the upper level: that is, the Rauzy-Veech algorithm coincides with the return map for the Teichmüller flow on the boxes’ boundaries.

This chapter follows mainly [Yoc07]. All the proofs omitted in this chapter can be found there, unless otherwise specified.

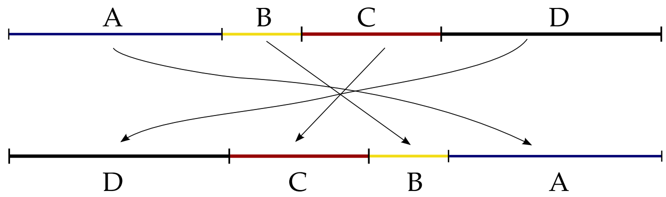

Figure 2.1: An interval exchange map of 4 sub-intervals. Its restriction to each sub-interval is a translation.

2.1 Interval exchange maps. Rauzy-Veech iteration

We give immediately the formal definition of the real protagonist of this work:

Definition 2.1.1. Let $I = (0, \lambda^*) \subset \mathbb{R}$ be a bounded open interval. An *interval exchange map (i.e.m.)* on I is a one-to-one map $T : I \setminus A_t \longrightarrow I \setminus A_b$ such that:
- $A_t, A_b \subset I$ are two finite subsets with the same cardinality;
- when T is restricted to each connected component of $I \setminus A_t$, it is a translation onto some connected component of $I \setminus A_b$.

The points of A_t are called *singularities* of T (so, the points of A_b are the singularities of T^{-1}).

REMARK 2.1.2. The way of treating singularities may differ slightly according to the author: someone, for instance, prefers to define an i.e.m. as a bijective self-map of $I = [0, \lambda^*)$: this is subdivided in left-open, right-closed intervals such that restriction of the i.e.m. to each of them is a translation. In other words, each singularity is regarded as attached to the interval on its right.

From now on, for a simpler exposition, we will not exclude singularities explicitly: we may talk about i.e.m.s and their iterations as self-maps of intervals, even if they shall be always meant to be defined out of a finite set of singularities. ◇

§ 2.1.A Parameters for an i.e.m.

An i.e.m. T is completely described by some combinatorial data, namely the lengths of its sub-intervals and the way they are arranged.

Definition 2.1.3. A *marked permutation* on d letters is a triple $(\mathcal{A}, \pi_t, \pi_b)$ where:
- $\mathcal{A}$, called *alphabet*, is a set of d elements, called *letters*;
- $\pi_t, \pi_b : \mathcal{A} \to \{1, \dots, d\}$ are bijections.

A marked permutation may be represented concisely as

$$\pi = \begin{pmatrix} (\pi_t)^{-1}(1) & \cdots & (\pi_t)^{-1}(d) \\ (\pi_b)^{-1}(1) & \cdots & (\pi_b)^{-1}(d) \end{pmatrix}.$$

Each i.e.m. T can be associated with a marked permutation $(\mathcal{A}, \pi_t, \pi_d)$, where $\mathcal{A}$ is an alphabet of $d = \#A_t + 1$ letters and, for each $\alpha \in \mathcal{A}$, T sends the $\pi_t(\alpha)$-th connected component of $I \setminus A_t$ (counting from the left) onto the $\pi_b(\alpha)$-th connected component of $I \setminus A_b$. Two marked permutations $(\mathcal{A}^1, \pi_t^1, \pi_b^1)$ and $(\mathcal{A}^2, \pi_t^2, \pi_b^2)$ represent the same permutation of sub-intervals if and only if there exists a bijection $i : \mathcal{A}^1 \to \mathcal{A}^2$ such that $\pi_t^1 = \pi_t^2 \circ i$ and $\pi_b^1 = \pi_b^2 \circ i$. Anyway, from now on, we will not consider this indefiniteness: every time we will consider an i.e.m., it will always come with a selected marked permutation.

REMARK 2.1.4. The rearrangement of sub-intervals performed by T could be described also using a 'standard' permutation i.e. a self-bijection of $\{1, \dots, d\}$. The advantage of a marked

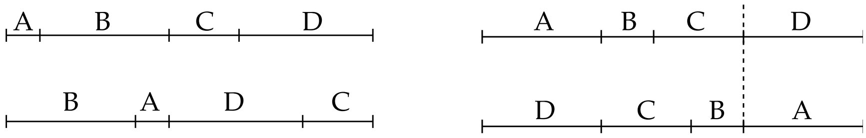

Figure 2.2: The i.e.m. on the left is not admissible. The i.e.m. on the right is obtained by varying lengths of the reference i.e.m. in Figure 2.1, and in particular it is admissible; but it has a connection $(u^t_D, u^b_A, 0)$.

permutation is that it keeps a symmetry between T and T^{-1}, indeed if $(\mathcal{A}, \pi_t, \pi_b)$ is a marked permutation associated to T, then $(\mathcal{A}, \pi_b, \pi_t)$ is associated to T^{-1}. The subscripts t and b stand for «top (row)» and «bottom (row)» respectively. ◇

For each $\alpha \in \mathcal{A}$, we denote I^t_α the $\pi_t(\alpha)$-th connected component of $I \setminus A_t$, and I^b_α the $\pi_b(\alpha)$-th connected component of $I \setminus A_b$. We denote $\lambda_\alpha = |I^t_\alpha| = |I^b_\alpha|$ their length, and define the *length vector* associated to T as $\lambda := (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}_+$. The i.e.m. T is thus completely determined by, and can be identified with, the pair (π, λ). We denotw $\Delta_\pi := \mathbb{R}^{\mathcal{A}}_+$ the space of the (parameters for) i.e.m.s which are associated with the marked permutation π: we will write indifferently $T \in \Delta_\pi$ or $\lambda \in \Delta_\pi$.

We also establish a standard notation for singularities of $T^{\pm 1}$: for each $\alpha \in \mathcal{A}$, we will denote $u^t_\alpha \in A_t \cup \{0\}$ the left endpoint of I^t_α; and $u^b_\alpha \in A_b \cup \{0\}$ the left endpoint of I^b_α. Obviously,

$$u^t_\alpha = \sum_{x:\pi_t(x)<\pi_t(\alpha)} \lambda_x \quad \text{and} \quad u^b_\alpha = \sum_{x:\pi_b(x)<\pi_b(\alpha)} \lambda_x.$$

The number $\delta_\alpha := u^b_\alpha - u^t_\alpha$ is the amount of the translation of I^t_α under T; the vector $\delta = (\delta_\alpha)_{\alpha\in\mathcal{A}}$ is called the *translation vector* associated to T.

It is also useful to denote $u^t_1 < \ldots < u^t_{d-1}$ the ordered elements of A_t; and $u^b_1 < \ldots < u^b_{d-1}$ the ordered elements of A_b: in other words, $u^t_j = u^t_{(\pi_t)^{-1}(j+1)}$; and $u^b_j = u^b_{(\pi_b)^{-1}(j+1)}$, for $j = 1, \ldots, d-1$ (we may also denote $u^t_0 = u^b_0 := 0$ and $u^t_d = u^b_d := \lambda^*$).

Definition 2.1.5. A marked permutation $(\mathcal{A}, \pi_t, \pi_b)$ with $\#\mathcal{A} = d$ is *admissible* (or *irreducible*) if, for each $1 \leq d' < d$, $(\pi_t)^{-1}(\{1, \ldots, d'\})$ and $(\pi_b)^{-1}(\{1, \ldots, d'\})$ are distinct subsets of $\mathcal{A}$. An i.e.m. is admissible if the associated marked permutation is.

Admissibility means that the marked permutation cannot be considered as the juxtaposition of marked permutations on smaller alphabets (see Figure 2.2 on the left). From now on, unless otherwise specified, we will only consider admissible marked permutations and i.e.m.s.

Definition 2.1.6. A *connection* for an i.e.m. T is a triple $(u^b, u^t; n)$, with $n \in \mathbb{N}$, $u^b \in A_b$, $u^t \in A_t$ such that

$$T^n u^b = u^t.$$

If T has no connections, it is said to satisfy the *Keane's property*.

Note that Keane's property implies admissibility.

§ 2.1.B Rauzy-Veech algorithm We now give a formal definition of the truncation proceeding already introduced in paragraph 0.3.D. The *Rauzy-Veech algorithm* applied on an admissible i.e.m. T gives a new i.e.m. $T' = \mathcal{R}T$ defined on the interval

$$I' := \left(0, \max\{u^t_{d-1}, u^b_{d-1}\}\right)$$

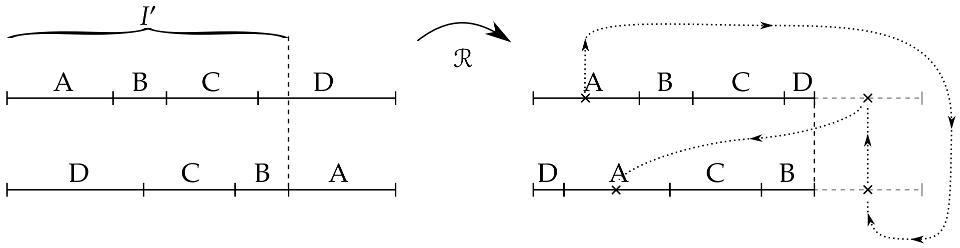

Figure 2.3: The Rauzy-Veech applied on an i.e.m. T with the same combinatorial data as in Figure 2.1. In this case a reduction of top type takes place, with winner D and loser A. Therefore two iterations of the map T are required to let the points of I^t_A return in the new interval I'.

as the first return map of the original i.e.m. T on I'. This is not a good definition for *every* i.e.m., but only for generic ones (in the sense of the Lebesgue measure on Δ_π), as the following explicit construction proves.

Let α_t and α_b the rightmost letters for π_t and π_b respectively. If $\lambda_{\alpha_t} > \lambda_{\alpha_b}$, we have a reduction of *top type*, with α_t its *winner* and α_b its *loser*. Conversely, if $\lambda_{\alpha_t} < \lambda_{\alpha_b}$, we have a reduction of *bottom type*, with α_b its winner and α_t its loser. We do not define the algorithm for the case $\lambda_{\alpha_t} = \lambda_{\alpha_b}$.

In the first case, I' is obtained by eliminating from $I \setminus A_b$ its rightmost connected component. The first return map of T to I' is therefore given by

$$T'(x) := \begin{cases} T^2(x) & \text{if } x \in I^t_{\alpha_b}; \\ T(x) & \text{if } x \in I' \setminus I^t_{\alpha_b}. \end{cases} \tag{2.1}$$

A correction on the associated marked permutation is induced: it is natural to take $\mathcal{A}' = \mathcal{A}$ and $\pi'_t = \pi_t$; but the old sub-interval $I^b_{\alpha_t}$ is now split in two: the left part is still associated with the letter α_t, the right one is the new bottom interval associated with α_b. That is,

$$\pi'_b(\alpha) = \begin{cases} \pi_b(\alpha) & \text{if } \pi_b(\alpha) \le \pi_b(\alpha_t); \\ \pi_b(\alpha_t) + 1 & \text{if } \alpha = \alpha_b; \\ \pi_b(\alpha) + 1 & \text{if } \pi_b(\alpha_t) < \pi_b(\alpha) < d. \end{cases} \tag{2.2}$$

and the new length data are obviously

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_t} - \lambda_{\alpha_b} & \text{if } \alpha = \alpha_t; \\ \lambda_\alpha & \text{otherwise.} \end{cases} \tag{2.3}$$

In the second case instead, I' is obtained by eliminating from $I \setminus A_t$ its rightmost connected component. The first return map of T to I' is best described by giving its inverse (here the symmetries of marked permutations become useful):

$$(T')^{-1}(x) := \begin{cases} T^{-2}(x) & \text{if } x \in I^t_{\alpha_b}; \\ T^{-1}(x) & \text{if } x \in I' \setminus I^t_{\alpha_b}. \end{cases} \tag{2.4}$$

Now we take $\pi'_b = \pi_b$, and the old $I^t_{\alpha_b}$ is now split in two, so that

$$\pi'_t(\alpha) = \begin{cases} \pi_t(\alpha) & \text{if } \pi_t(\alpha) \le \pi_t(\alpha_b); \\ \pi_t(\alpha_b) + 1 & \text{if } \alpha = \alpha_t; \\ \pi_t(\alpha) + 1 & \text{if } \pi_t(\alpha_b) < \pi_t(\alpha) < d. \end{cases} \tag{2.5}$$

and the new length data are

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_b} - \lambda_{\alpha_t} & \text{if } \alpha = \alpha_b; \\ \lambda_\alpha & \text{otherwise.} \end{cases} \tag{2.6}$$

§ 2.1.C Iteration of the algorithm The Rauzy-Veech algorithm is usually applied *iteratively*, thus a family of i.e.m.s — an infinite one, under the conditions stated below — is generated. The i.e.m. generated after r iterations is usually denoted $T^{(r)}$; and similarly all the objects related to it are denoted adding a superscript $^{(r)}$ to the notations introduced in paragraph 2.1.A (for instance, $I^{(r)}$ is the interval where $T^{(r)}$ acts, $\lambda_\alpha^{(r)}$ are its length data, $u_\alpha^{(r),t}$ its singularities and so on).

Remark 2.1.7. For each r, $T^{(r)}$ is the return map of T on $I^{(r)}$. In particular, say that, at the r-th step, we have a reduction of top type (with α_t and α_b the rightmost letters before the reduction). Then all the top singularities $u_\alpha^{(r-1),t}$ coincide with the respective $u_\alpha^{(r),t}$, whereas $u_{\alpha_b}^{(r-1),b}$ has become the right endpoint of $I^{(r)}$. As a consequence of formula 2.1, we have $u_{\alpha_b}^{(r),b} = T^{(r-1)} u_{\alpha_b}^{(r-1),b}$ (while the other bottom singularities stay unchanged from the $(r-1)$-th step to the r-th).

It is easily checked by induction on the number of steps that

$$u_\alpha^{(r),b} = T^l u_\alpha^b$$

where $l \geq 0$ is the *first entry time* of u_α^b into $I^{(r)}$ under iteration of T (i.e. the least integer j such that $T^j u_\alpha^b \in I^{(r)}$). Symmetrically,

$$u_\alpha^{(r),t} = T^{-h} u_\alpha^t$$

where $h \geq 0$ is the first entry time of u_α^t into $I^{(r)}$ under iteration of T^{-1}.

Actually, the same holds for the whole sub-intervals of $I^{(r)}$ starting at those singularities:

$$I_\alpha^{(r),b} = T^l(u_\alpha^b, u_\alpha^b + \lambda_\alpha^{(r)}) \quad \text{and} \quad I_\alpha^{(r),t} = T^{-h}(u_\alpha^t, u_\alpha^t + \lambda_\alpha^{(r)})$$

while $T^k(u_\alpha^b, u_\alpha^b + \lambda_\alpha^{(r)}) \cap I^{(r)} = \emptyset$ for $0 \leq k < l$ (provided that $l > 0$), and similarly $T^{-k}(u_\alpha^t, u_\alpha^t + \lambda_\alpha^{(r)}) \cap I^{(r)} = \emptyset$ for $0 \leq k < h$ (provided that $h > 0$). ◇

§ 2.1.D Rauzy diagrams and classes Starting from an alphabet $\mathcal{A}$, a graph can be defined taking as its vertices all the possible admissible marked permutations with alphabet $\mathcal{A}$: each of those vertices $\pi = (\pi_t, \pi_b)$ will be the starting point of two arrows:

- a *top type* arrow arriving at $\pi' = \mathcal{R}_t \pi$, defined by setting $\pi'_t = \pi_t$ and π'_b as in formula 2.2;
- a *bottom type* arrow arriving at $\pi' = \mathcal{R}_b \pi$, defined by setting $\pi'_b = \pi_b$ and π'_t as in formula 2.5.

That is, the arrows represent abstractly the two possible outcomes of the Rauzy-Veech algorithm starting from different length data. Each arrow has a *winner* and a *loser*, defined similarly as in paragraph 2.1.B. It can be shown that $\mathcal{R}_t$ and $\mathcal{R}_b$ are bijective maps, so each π is also *endpoint* for exactly a top type arrow and a bottom type one.

The connected component of this diagram to whom π belongs is called the *Rauzy diagram* for π; its vertices make up the *Rauzy class* of π. A concatenation of arrows in a Rauzy diagram is called a *path*.

The set of all possible finite paths in a Rauzy diagram $\mathcal{D}$ will be denoted $\Pi(\mathcal{D})$; it will be endowed with a partial ordering: $\gamma_0 \leq \gamma_1 \Leftrightarrow \gamma_1$ begins with γ_0. Similarly, $\Pi_\pi(\mathcal{D})$ will denote the subset of paths starting from a fixed $\pi \in \mathcal{C}$ the Rauzy class underlying the Rauzy diagram $\mathcal{D}$.

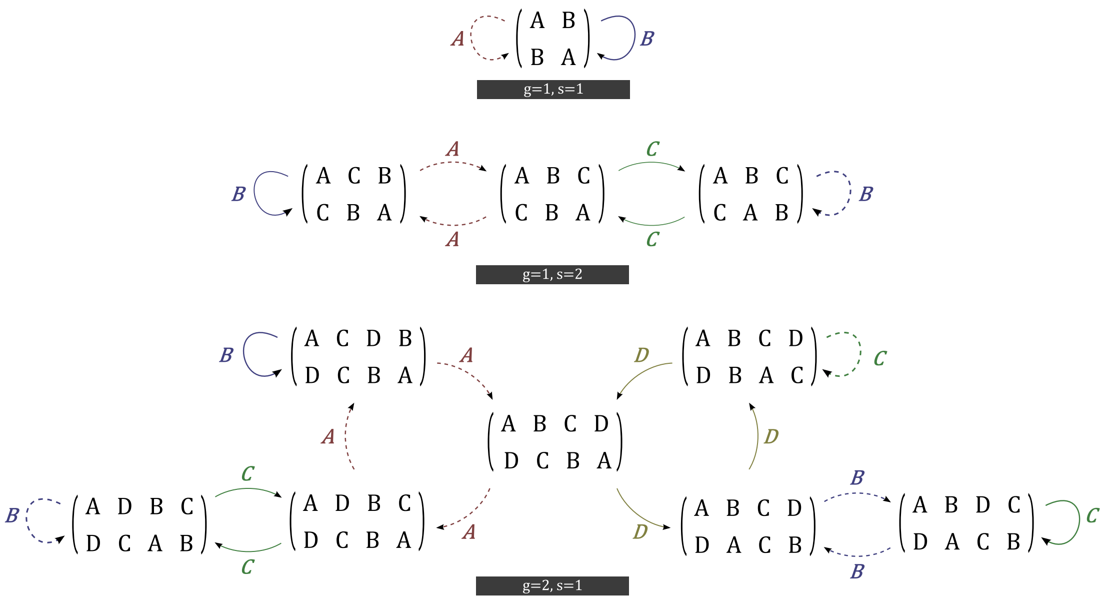

Figure 2.4: Three examples of Rauzy diagram (the third one includes our reference marked permutation); top arrows are represented with a continuous stroke, bottom arrows with a dotted one; the letter which accompanies an arrow is its winner. The numbers in black boxes indicate the genus g and the number of singularities s of the translation surface obtained by means of the Veech construction (see section 2.2) from i.e.m.s associated with marked permutations which belong to the considered diagram.

A family of paths $\Gamma \subseteq \Pi(\mathcal{D})$ will be called *disjoint* if, for any distinct $\gamma_0, \gamma_1 \in \Gamma$, neither $\gamma_0 \prec \gamma_1$ nor $\gamma_1 \prec \gamma_0$ holds.

Remark 2.1.8. Each time we fix a Rauzy class $\mathcal{C}$, all $\pi \in \mathcal{C}$ have the same leftmost letters on both the top and the bottom row. We will denote them by $t_{\mathcal{C}}$ and $b_{\mathcal{C}}$, respectively. ◇

Lemma 2.1.9. *If π and π' belong to the same Rauzy class, then there exist a path from π to π' and another one in the opposite direction.*

A natural question at this point could be: can any path in a Rauzy diagram be obtained by reiterating the Rauzy-Veech algorithm on some suitable i.e.m.? The answer is the following:

Definition 2.1.10. A path γ in a Rauzy diagram $\mathcal{D}$ on some alphabet $\mathcal{A}$ is called *complete* if every letter in $\mathcal{A}$ is the winner of at least an arrow in γ.
An infinite path γ is *∞-complete* if every letter in $\mathcal{A}$ is the winner of infinitely many arrows in γ; or, equivalently, if γ is obtained by concatenation of infinitely many complete paths.

Proposition 2.1.11. *Let γ be an infinite path in some Rauzy diagram. The path γ is ∞-complete if and only if there exists an i.e.m. T such that Rauzy-Veech algorithm can be iterated infinitely many times on T, and the n-th marked permutation touched by γ is the marked permutation associated with $\mathcal{R}^n T = T^{(n)}$.*

So, each *finite* path can be obtained by applying the iteration to some appropriate i.e.m. T (because it can be continued in such a way to become ∞-complete). Two sub-products of this result are:

Corollary 2.1.12. *If the Rauzy-Veech algorithm can be iterated infinitely many times on an admissible i.e.m. T, the length of the interval on which $\mathcal{R}^n T$ acts goes to 0 as $n \to \infty$.*

Corollary 2.1.13. *The Rauzy-Veech algorithm can be iterated infinitely many times on an admissible i.e.m. T if and only if T satisfies the Keane's property.*

We will denote $\gamma_T(r)$ the finite path originated by iterating the Rauzy-Veech algorithm r times, starting from the i.e.m. T (when this is possible); and if T satisfies the Keane's property, we will denote γ_T the path obtained by iterating the algorithm to infinity.

2.2 Veech construction

We now introduce the *zippered rectangle construction*, due to Veech, that is the generalisation to the suspension construction described in paragraph 0.3.E for exchange maps of two intervals. Given an admissible i.e.m. T, this construction produces translation surfaces such that T is the return map of the vertical flow on an horizontal segment on the surface.

§ 2.2.A Details of the construction Similarly as in the case of two intervals, in order to construct a surface we have to choose further parameters than the ones carried by T:

Definition 2.2.1. Let T be an i.e.m. inducing an admissible marked permutation $\pi = (\mathcal{A}, \pi_t, \pi_b)$; we will use the notations specified in paragraph 2.1.A. A *suspension vector* for T is a vector $\tau \in \mathbb{R}^{\mathcal{A}}$ such that

$$\sigma^t_\alpha := \sum_{x:\pi_t(x)<\pi_t(\alpha)} \tau_x > 0 \quad \text{and} \quad \sigma^b_\alpha := \sum_{x:\pi_b(x)<\pi_b(\alpha)} \tau_x < 0 \quad \text{for all } \alpha \in \mathcal{A}.$$

Such a vector always exists: the canonical one is given by $\tau^{can}_\alpha = \pi_b(\alpha) - \pi_t(\alpha)$. We denote $\sigma^* := \sum_{x\in\mathcal{A}} \tau_x$. The set of all possible suspension data for the marked permutation π will be denoted Θ_π.

Once $T = (\pi, \lambda)$ and $\tau \in \Theta_\pi$ have been chosen, the construction fixes some rectangles in $\mathbb{C} \cong \mathbb{R}^2$; then identifies couples of them and of portions of their boundaries. For each $\alpha \in \mathcal{A}$, set:

- $\zeta_\alpha := \lambda_\alpha + i\tau_\alpha$;
- $\eta_\alpha := \sigma^t_\alpha - \sigma^b_\alpha > 0$;
- $\xi^t_\alpha := \sum_{x:\pi_t(x)<\pi_t(\alpha)} \zeta_x = u^t_\alpha + i\sigma^t_\alpha$; and $\xi^b_\alpha := \sum_{x:\pi_b(x)<\pi_b(\alpha)} \zeta_x = u^b_\alpha + i\sigma^b_\alpha$.

We define d rectangles in the upper half plane of $\mathbb{C} \cong \mathbb{R}^2$ having the intervals I^t_α as edges, and d in the lower plane having the intervals I^b_α as edges:

$$R^t_\alpha := [u^t_\alpha, u^t_\alpha + \lambda_\alpha] \times [0, \eta_\alpha]; \quad R^b_\alpha := [u^b_\alpha, u^b_\alpha + \lambda_\alpha] \times [-\eta_\alpha, 0].$$

Roughly speaking, we would like to obtain a translation surface using the rectangles in the upper plane, gluing their upper sides with portions of I according to how T exchanges sub-intervals; and in so doing, we want the points ξ^t_α, and $\xi^* := \lambda^* + i\sigma^*$, to be projected to singularities.

Starting from the *disjoint* union $R = \bigsqcup_{\alpha\in\mathcal{A};\varepsilon\in\{t,b\}} R^\varepsilon_\alpha$, we perform the following identifications on it (see Figure 2.5):

- every time a point of $\bar{I} \times \{0\}$ appears in two rectangles, we identify these points;
- the same for every point of $\{u^t_\alpha\} \times [0, \sigma^t_\alpha]$ and of $\{u^b_\alpha\} \times [\sigma^b_\alpha, 0]$, for every $\alpha \in \mathcal{A}$;
- for each $\alpha \in \mathcal{A}$, the whole rectangle R^t_α is identified with R^b_α through a translation of $\theta_\alpha := \xi^b_\alpha - \xi^t_\alpha$;

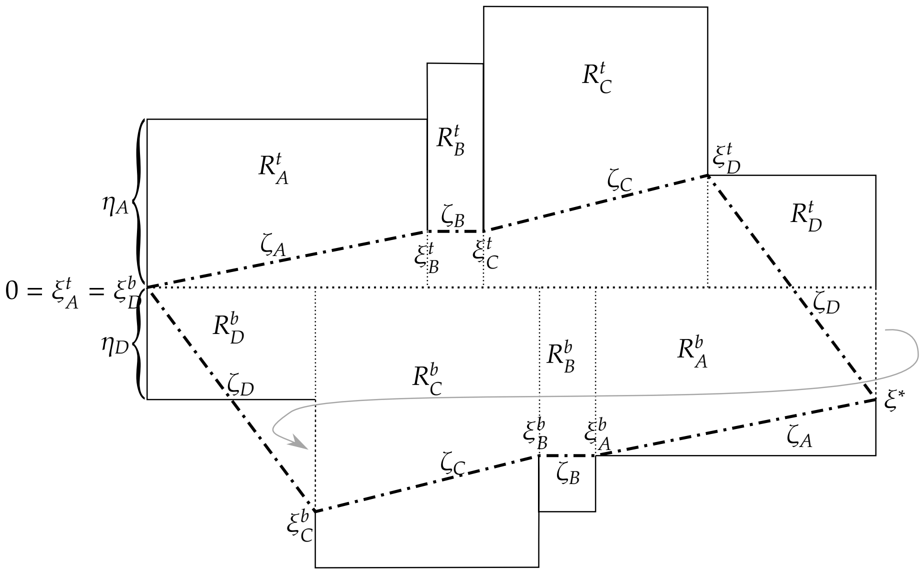

Figure 2.5: The Veech construction performed on an i.e.m. with the same marked permutation as in Figure 2.1 (but different length data), for which a suspension vector τ has been chosen. Dotted lines stand for portions of border of adjoining rectangles that are glued together; moreover each of the top rectangles is identified with the bottom one called by the same letter; and the grey arrow shows the last identification of the list.

- if $\sigma^* = 0$, we are done; otherwise: if $\sigma^* > 0$, we identify $\{\lambda^*\} \times [0, \sigma^*]$ with its $-\theta_{\pi_b^{-1}(d)}$-translated, as we do with the bottom rectangle $R^b_{\pi_b^{-1}(d)}$; and if $\sigma^* < 0$, we identify $\{\lambda^*\} \times [\sigma^*, 0]$ with its $\theta_{\pi_t^{-1}(d)}$-translated, as we do with the top rectangle $R^t_{\pi_t^{-1}(d)}$.

The quotient is a closed orientable topological surface, which we denote S: indeed, it is simple to check that the boundary of each rectangle disappears when performing some of the prescribed gluings. Let $\Sigma = \Sigma(\pi, \lambda, \tau) \subset S$ be the set of the point obtained by projection of the $\xi^{t,b}_\alpha$ and ξ^*.

Note that $\bigsqcup_\alpha R^t_\alpha$ is a fundamental domain for the identifications we perform. So, the interior of each top rectangle can be used as domain for natural local coordinates for S; and there is a natural way to complete this construction to a translation atlas for $S \setminus \Sigma$ (it is a different version of the instructions given in paragraph 1.2.C; in this case we are able to define charts «around the vertices of the rectangles», but we have to exclude the points $\xi^{t,b}_\alpha$ and ξ^*).

The translation structure we just defined — from now on we will denote it $X(\pi, \lambda, \tau)$ — has conical singularities at points of Σ: indeed, if we start turning around one of (the preimages of) such points, remembering to switch from a rectangle to another one according to the identifications specified above; and we remember the Euler-Poincaré formula (Proposition 1.2.4), we recognize that:

Lemma 2.2.2. *Let $X(\pi, \lambda, \tau)$ be the translation surface obtained with the Veech construction from an i.e.m. $T = (\pi, \lambda)$ and a suspension vector τ. The genus g of $X(\pi, \lambda, \tau)$, its number of conical singularities s and the associated vector of indices h only depend on π. Moreover, the following* *equality holds:*

$$d - 1 = \sum_{j=1}^{s} h_j, \text{ therefore } d = 2g + s - 1.$$

Now, let $\tilde{I} \subset S$ be the horizontal segment obtained by projection of $I \subset \mathbb{C}$. It is easy to check that the vertical flow on S according to the translation structure just defined induces a return map on $\tilde{I}$ that is exactly T.

Remark 2.2.3. One might think that the same surface is obtained if one does not use the rectangles $R_\alpha^{t,b}$, but simply joins the points $\xi_\alpha^{t,b}$ and ξ^* with segments, that is considers the polygon traced in Figure 2.5, and then identifies the couples of sides corresponding to the same ζ_α.

This is true if $\sum_\alpha \tau_\alpha = 0$: otherwise it may happen that this polygon presents auto-intersections, as in Figure 2.6. ◇

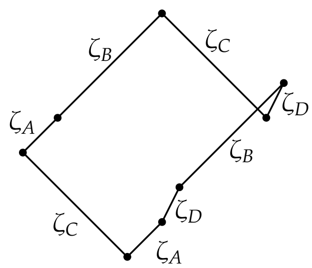

Figure 2.6: Autointersection in the polygon obtained from the marked permutation $\begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}$ and the choice of some 'critical' length and suspension vectors.

Remark 2.2.4. A technical remark: for each $\alpha \in \mathcal{A}$, the point ξ_α^t is always contained in ∂R_α^t; but, let α' be the letter just before α in the top row of π (we suppose α not to be the leftmost one): when does $\xi_\alpha^t \in \partial R_{\alpha'}^t$? This equivalent to say $\sigma_\alpha \leq \eta_{\alpha'}$, and performing some calculations, one sees that this does not happen only in *one* case, that is, when α' is the rightmost letter in the bottom row of π, and $\sum_x \tau_x > 0$.

Similarly, let α'' be the letter just before α in the bottom row of π (when it is not the leftmost). Then $\xi_\alpha^b \notin \partial R_{\alpha''}^b$ only if α'' is the rightmost letter of the top row, and $\sum_x \tau_x < 0$.

In conclusion, at most one among the points $\xi_\alpha^{t,b}$ does not belong to the boundary of the rectangle on its left. ◇

§ 2.2.B **Rauzy-Veech iteration for suspension data** The Rauzy-Veech algorithm can be defined not only on (almost all) admissible i.e.m.s, but also on an *augmented parameter space* made up by triples (π, λ, τ) where π is an admissible marked permutation, $\lambda \in \Delta_\pi$, $\tau \in \Theta_\pi$. It suffices to transform τ the same way as λ according to the type of the reduction; that is

- if $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ we set $\tau'_{\alpha_t} := \tau_{\alpha_t} - \tau_{\alpha_b}$; and $\tau'_\alpha = \tau_\alpha$ for all $\alpha \neq \alpha_t$;
- if $\lambda_{\alpha_t} < \lambda_{\alpha_b}$ we set $\tau'_{\alpha_b} := \tau_{\alpha_b} - \tau_{\alpha_t}$; and $\tau'_\alpha = \tau_\alpha$ for all $\alpha \neq \alpha_b$.

We denote $(\pi', \lambda', \tau') = \mathcal{R}(\pi, \lambda, \tau)$ as before. It is worth noting two facts:

Lemma 2.2.5. *If there is an arrow from π to π', then the Rauzy-Veech algorithm sends Θ_π onto:*

- $\Theta_{\pi'} \cap \{\sum_\alpha \tau'_\alpha < 0\}$ *if the arrow is of top type;*
- $\Theta_{\pi'} \cap \{\sum_\alpha \tau'_\alpha > 0\}$ *if the arrow is of bottom type.*

Lemma 2.2.6. *There is a canonical isomorphism between the surfaces $X(\pi, \lambda, \tau)$ and $X(\pi', \lambda', \tau')$ obtained by means of the Veech construction from the two triples.*

This isomorphism can be recognised directly from the construction (see Figure 2.7): application of the Rauzy-Veech algorithm to the data (π, λ, τ) means, for instance in the case of a reduction of top type, that $R^t_{\alpha_t}$ is vertically cut in two rectangles; the one on the right has width λ_{α_b}, and it is removed from there, to be placed above the rectangle $R^t_{\alpha_b}$. The rectangle $R^b_{\alpha_t}$ is accordingly cut into two rectangles, and below the one on the right we glue $R^b_{\alpha_b}$, which has been removed from its original, rightmost position.

The identifications we have to perform between couples of these new rectangles are obtained in the most natural way from the ones performed between the old ones. So, the canonical «piecewise isomorphism» between the two sets of rectangles, which also makes each marked point $\xi^{b,t}_\alpha$ correspond with the relative ${\xi'}^{b,t}_\alpha$, passes to an isomorphism between $X(\pi, \lambda, \tau)$ and $X(\pi', \lambda', \tau')$.

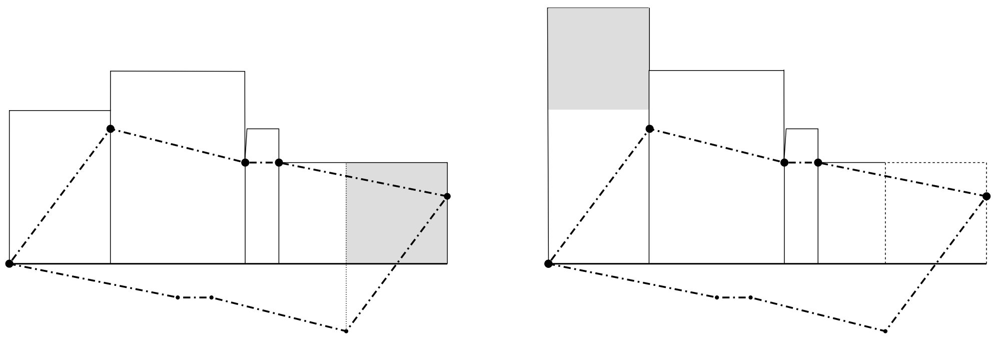

Figure 2.7: The Rauzy-Veech algorithm for suspension data: what happens to the top rectangles in the Veech construction when a step of the algorithm (in this case of top type) is applied.

§ 2.2.C **Keane's theorem** By means of the Veech construction, a result regarding *minimality* of i.e.m.s can be easily proven. In the setting of translation surfaces, the following proposition holds:

Proposition 2.2.7. *Let X be a translation surface of genus > 1. If there is no vertical saddle connection on X, then the vertical flow is minimal.*

The proof of this result is completely geometrical: if an half-orbit is not dense, there exists an horizontal segment disjoint from it. But one proves that the absence of vertical connections implies that the union of the orbits which are disjoint from this segment is a cylinder. If we enlarge it until we meet some obstacle (i.e. a singularity), its boundary will be made up of vertical connections, and this is absurd.

If T is an admissible i.e.m. and we consider a translation surface X built from it after some suspension vector has been chosen, there is a natural correspondence between connections for T and vertical saddle connections for X. And, since minimality for a flow implies minimality for return maps,

Theorem 2.2.8 (Keane). *If an i.e.m. has no connection, then it is minimal.*

This is the simplest example of how the Veech construction is exploited to make the setting of translation surfaces communicate with the one of i.e.m.s: and this is a very common way to investigate both of them.

REMARK 2.2.9. We have seen that an exchange map of two intervals is identifiable, up to a scale factor, with a rotation of $\mathbb{T}$; in particular, minimal i.e.m.s can be identified with rotations by an angle having an irrational ratio with π. So, Keane's property can be seen as a generalisation of the concept of irrationality for general i.e.m.s.

Furthermore, let us fix a marked permutation π: the set of the i.e.m.s $T \in \Delta_\pi$ such that $(u^t_\alpha, u^b_\beta, n)$ is a connection for T has codimension 1, and in particular its Lebesgue measure is zero. Thus, the set of the i.e.m.s which satisfy the Keane's property is complementary to a countable union of sets with measure zero: therefore *almost every* i.e.m. is minimal, as it happened in the case $d = 2$. ◇

In this context, another notable result is:

Proposition 2.2.10. *If the length data of an i.e.m. T are independent over $\mathbb{Q}$, then T satisfies the Keane's property.*

2.3 Additional structures on strata

The Veech construction provides the most natural way to parametrise locally the Teichmüller strata, and to define a measure on them. Indeed the construction is, to some extent, invertible:

Proposition 2.3.1. *Let (S, Σ, h, ζ) be a translation surface without any saddle connections for its vertical foliation; and let H_∞ be an outgoing separatrix for the horizontal flow. Suppose that there exist an open bounded initial segment $H \subseteq H_\infty$ such that its right endpoint lies on a vertical separatrix which does not meet H. If this is not true, then H_∞ is a horizontal separatrix, and we set $H := H_\infty$. In both cases, the return map of the vertical flow to H is an admissible i.e.m. $T = (\pi, \lambda)$ on H such that the given translation surface is isomorphic to $X(\pi, \lambda, \tau)$ for some appropriate $\tau \in \Theta_\pi$.*

We now define a local parametrisation of $\mathcal{T}(S, \Sigma, h)$ based on the parameters chosen for the Veech construction; it will be applicable everywhere except for a negligible subset, whose elements are the translation surfaces with vertical saddle connections.

§ 2.3.A Veech coordinates

Let us fix a topological surface S, a finite subset $\Sigma \subset S$ and an associated vector of indices h. Suppose that, starting from a translation structure on (S, Σ, h), the proposition above gives an i.e.m. with an associated (admissible) marked permutation π. For each $\alpha \in \mathcal{A}$ we set $\lambda^{can}_\alpha = 1$, and τ^{can}_α as immediately after Definition 2.2.1; according to Lemma 2.2.2, the topological pair $(X(\pi, \lambda^{can}, \tau^{can}), \Sigma(\pi, \lambda^{can}, \tau^{can}))$ is homeomorphic with (S, Σ); and we fix once and for all an identification between them.

Now, let us consider the translation surface $X(\pi, \lambda, \tau)$ obtained by choosing another pair $\lambda \in \Delta_\pi$, $\tau \in \Theta_\pi$. We would like to see it as another translation structure on the *same* topological couple (S, Σ) as before, in order to derive from it an element of $\mathcal{T}(S, \Sigma, h)$; in other words, we want to find a canonical homeomorphism $\phi_{\lambda,\tau}$ between (S, Σ) — or, to better say, $(X(\pi, \lambda^{can}, \tau^{can}), \Sigma(\pi, \lambda^{can}, \tau^{can}))$ — and $(X(\pi, \lambda, \tau), \Sigma(\pi, \lambda, \tau))$. Then we use $\phi_{\lambda,\tau}$ to pull the translation structure on $X(\pi, \lambda, \tau)$ back to $X(\pi, \lambda^{can}, \tau^{can})$

For each $\alpha \in \mathcal{A}$, we denote γ_α the curve in $X(\pi, \lambda, \tau)$ with endpoints in $\Sigma(\pi, \lambda, \tau)$ which is the projection, under the identifications prescribed by the Veech construction, of a curve $\tilde{\gamma}_\alpha$ in $\mathbb{C}$, determined as follows. If $\alpha \neq \alpha_t$, let α' be the letter on its right.

- Suppose α' is the letter appearing after α in the top row of π and that $\xi^t_{\alpha'} \notin \partial R^t_\alpha$ (this happens for at most one letter α, according to Remark 2.2.4). Then $\tilde{\gamma}_\alpha$ is the polygonal path obtained by joining the segment in R^t_α from ξ^t_α to the top right vertex of the rectangle, to the segment from this point to $\xi^t_{\alpha'}$.
- Otherwise, $\tilde{\gamma}_\alpha$ is simply the segment in R^t_α from ξ^t_α to $\xi^t_{\alpha'}$.

For $\alpha = \alpha_t$, we have again to distinguish between two cases:

- If $\sum_x \tau_x$, we let $\tilde{\gamma}_{\alpha_t}$ be the polygonal path obtained by joining the segment in $R^t_{\alpha_t}$ from $\xi^t_{\alpha_t}$ to the bottom right vertex λ^*, to the segment from this point to ξ^*.
- Otherwise, $\tilde{\gamma}_{\alpha_t}$ is the segment in $R^t_{\alpha_t}$ from $\xi^t_{\alpha_t}$ to ξ^*.

In all cases, the projection of each of the curves $\tilde{\gamma}_\alpha$ on $X(\pi, \lambda, \tau)$ is well-defined. We will need the following, easy to prove, result:

The first relative homology group $H_1(S, \Sigma; \mathbb{Z})$ is isomorphic with $\mathbb{Z}^{2g+s-1} = \mathbb{Z}^d$. A basis for it is given by the elements $[\gamma_\alpha]$ represented by the curves defined above.

For $\alpha \in \mathcal{A}$, let γ^{can}_α be the curve obtained, the same way as above, on the surface $X(\pi, \lambda^{can}, \tau^{can})$. We are now able to determine $\phi_{\lambda,\tau}$ up to isotopy saying that, for each $\alpha \in \mathcal{A}$, we want the curves $\phi_{\lambda,\tau}(\gamma^{can}_\alpha)$ and γ_α on $X(\pi, \lambda, \tau)$ to be isotopic with fixed endpoints.

The *Veech coordinates* related to the marked permutation π are defined as

$$\begin{array}{rccl} \mathcal{I}_\pi: & \Delta_\pi \times \Theta_\pi & \longrightarrow & \mathcal{T}(S, \Sigma, h) \\ & (\lambda, \tau) & \longmapsto & \left[\phi^*_{\lambda,\tau} X(\pi, \lambda, \tau)\right]. \end{array}$$

§ **2.3.B A measure on Teichmüller strata** The above discussion leads to a natural definition for a measure on $\mathcal{T}(S, \Sigma, h)$: we would like the translation structures with vertical connections to be a zero measure subset; and to identify the other ones with the triples (π, λ, τ) which produce them, in order to put the $2d$-dimensional Lebesgue measure on them.

Recall Definition 1.2.8, fixing an Abelian differential (therefore a translation structure) ω. We notice that the holonomy of γ is invariant under the action of $\mathrm{Diff}_0(S)$ on ω; and, moreover, being ω a closed 1-form, it is also invariant when γ varies among the possible representatives of $[\gamma] \in H_1(S, \Sigma; \mathbb{Z})$. So, a linear map is well-defined

$$\mathrm{Hol}_\omega : H_1(S, \Sigma; \mathbb{Z}) \longrightarrow \mathbb{C}$$

which can be identified with an element of $H^1(S, \Sigma; \mathbb{C}) \cong \mathbb{C}^d$. The so-called *period map* is

$$\begin{array}{rccl} \mathcal{P}: & \mathcal{T}(S, \Sigma, h) & \longrightarrow & H^1(S, \Sigma; \mathbb{C}) \\ & [\omega] & \longmapsto & \mathrm{Hol}_\omega. \end{array}$$

Proposition 2.3.2. *The period map is a local homeomorphism. In particular, $\mathcal{T}(S, \Sigma, h)$ is a complex manifold of dimension $d = 2g + s - 1$ (and $\mathcal{H}(S, \Sigma, h)$ is an orbifold of the same complex dimension).*

To be precise, so far we haven't defined yet any explicit topology on the strata $\mathcal{T}(S, \Sigma, h)$: we may consider it determined by this proposition. Let dLeb be the standard volume form on $H^1(S, \Sigma; \mathbb{C})$, normalised so that the lattice $\mathrm{Hom}\,(H_1(S, \Sigma; \mathbb{Z}); \mathbb{Z} \oplus i\mathbb{Z})$ has covolume 1. Then $\mathrm{d}m := \mathcal{P}^*\mathrm{dLeb}$ is a volume form on $\mathcal{T}(S, \Sigma, h)$, which carries a measure m, called the *Masur-Veech measure*.

If a translation structure, identified with an Abelian differential ω, has a vertical connection, then Hol_ω takes a purely imaginary value on some elements of $H_1(S, \Sigma; \mathbb{Z})$. For each $x \in$

$H_1(S,\Sigma;\mathbb{Z})$, the request $\mathrm{Hol}_\omega(x) \in i\mathbb{R}$ defines a subset of $H^1(S,\Sigma;\mathbb{C})$ with measure zero; and $H_1(S,\Sigma;\mathbb{Z})$ is countable. Therefore the set of the translation structures with a vertical connection has zero measure in $\mathcal{T}(S,\Sigma,h)$.

The period map is strictly related to the local Veech coordinates defined in the previous paragraph. Fix any marked permutation π such that the Veech construction gives a translation surface $X(\pi,\lambda^{can},\tau^{can})$ which is homeomorphic with (S,Σ) (thus we fix an identification between the two pairs); call $v_\pi : H^1(S,\Sigma;\mathbb{C}) \to \mathbb{C}^{\mathcal{A}}$ the isomorphism given by evaluation of a linear form on the basis for $H_1(S,\Sigma;\mathbb{Z})$ given by the $[\gamma_\alpha]$. Then the composition

$$v_\pi \circ \mathcal{P} \circ \mathcal{I}_\pi : \Delta_\pi \times \Theta_\pi \to \mathcal{T}(S,\Sigma,h) \to H^1(S,\Sigma;\mathbb{C}) \to \mathbb{C}^{\mathcal{A}}$$

is simply the identity map of its domain: indeed it takes each pair (λ,τ) to $\zeta = \lambda + i\tau$.

§ 2.3.C Derived measures One sees that the action of $\mathrm{Mod}(S,\Sigma)$ on $\mathcal{T}(S,\Sigma,h)$ preserves the volume form $\mathrm{d}m$; therefore $\mathcal{H}(S,\Sigma,h)$ inherits a volume form $\mathrm{d}\mu$. Moreover, let $A : \mathcal{T}(S,\Sigma,h) \to \mathbb{R}$ be the area function. It can be proven to be a smooth function (after we have verified that $\mathcal{T}(S,\Sigma,h)$ is a differential manifold; but this is not the point of this work), so we can decompose

$$\mathrm{d}m = \mathrm{d}m^{(1)} \wedge \frac{\mathrm{d}A}{A}$$

and in so doing we obtain a volume form $\mathrm{d}m^{(1)}$ on $\mathcal{T}^{(1)}(S,\Sigma,h)$, which in turn can be pushed forward to a measure on $\mathcal{H}^{(1)}(S,\Sigma,h)$.

All the measures above defined, which are still called Masur-Veech measures, not only are the most natural suggested by Veech construction, but *are all preserved by the Teichmüller flow on the respective spaces*. So, we are allowed to study the latter in the language of ergodic theory.

§ 2.3.D A finite covering of strata Since the Veech construction always provides translation surfaces with a selected outgoing separatrix for their first singularity, a better representation of this construction is obtained by also considering this separatrix. Imitating what we did in paragraph 1.3.E, let $\widetilde{\mathrm{Flat}}(S,\Sigma,2h,1)$ be the set of the translation structures on S with singularities in Σ, conical angles $2\pi h_j$ wide, and a marked outgoing horizontal separatrix H_∞ which starts at the conical singularity $p_1 \in \Sigma$.

We define the *marked strata* in Teichmüller and moduli spaces as

$$\tilde{\mathcal{T}}(S,\Sigma,h) := {\widetilde{\mathrm{Flat}}(S,\Sigma,2h,1)}\big/{\mathrm{Diff}_0(S,\Sigma)}, \quad \text{and}$$

$$\tilde{\mathcal{H}}(S,\Sigma,h) := {\widetilde{\mathrm{Flat}}(S,\Sigma,2h,1)}\big/{\mathrm{Diff}^+(S,\Sigma)} = {\tilde{\mathcal{T}}(S,\Sigma,h)}\big/{\mathrm{Mod}(S,\Sigma)}.$$

In these formulae, we mean that two marked translation structures project to the same point if there is a map that makes the two translation atlases correspond, and also takes the marked separatrix of the first structure to the marked separatrix of the second one.

It is easily seen that the obvious map $\tilde{\mathcal{T}}(S,\Sigma,h) \to \mathcal{T}(S,\Sigma,h)$ is a covering of degree h_1; and $\tilde{\mathcal{H}}(S,\Sigma,h) \to \mathcal{H}(S,\Sigma,h)$ is a covering between orbifolds of the same degree; moreover, $\tilde{\mathcal{H}}(S,\Sigma,h)$ is a manifold, because no element of $\tilde{\mathcal{T}}(S,\Sigma,h)$ may possess nontrivial automorphisms. Of course, it is also possible to define the normalised marked Teichmüller and moduli strata $\tilde{\mathcal{T}}^{(1)}(S,\Sigma,h)$ and $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$.

The covering maps can be used to pull the Masur-Veech measures, as defined above, back to these new spaces. Moreover, since the Veech's construction naturally induces an horizontal separatrix for the first singularity of the obtained surface, the Veech local coordinates can be lifted to $\tilde{\mathcal{T}}(S,\Sigma,h)$. If we restrict appropriately the domain of some Veech coordinates, they can also be used as local coordinates for $\tilde{\mathcal{H}}(S,\Sigma,h)$, because the projection map is a local homeomorphism.

2.4 Rauzy dynamics as a return map

This sections shows how can the Veech local charts coming from a selected Rauzy class be glued altogether, in order to describe a stratum entirely in terms of augmented parameter spaces for i.e.m.s. The construction will also provide us a section for the Teichmüller flow such that the return map can be described in terms of the Rauzy-Veech algorithm.

§ 2.4.A (Normalised) Rauzy dynamics Fix a Rauzy class $\mathcal{C}$, with $\mathcal{D}$ the associated Rauzy diagram. We have seen (cfr. paragraphs 2.1.B, 2.1.D, and 2.2.B) that the Rauzy-Veech algorithm defines *Lebesgue-almost everywhere* a self-map $\mathcal{R} : \tilde{\Omega}(\mathcal{D}) \to \tilde{\Omega}(\mathcal{D})$, the *Rauzy dynamics*, of the augmented parameter space, which is defined as

$$\tilde{\Omega}(\mathcal{D}) := \bigsqcup_{\pi\in\mathcal{C}} \{\pi\} \times \Omega_\pi, \text{ where } \Omega_\pi := \Delta_\pi \times \Theta_\pi.$$

Alternatively, the algorithm defines a self-map $\mathcal{R} : \tilde{\Delta}(\mathcal{D}) \to \tilde{\Delta}(\mathcal{D})$ of the standard parameter space $\tilde{\Delta}(\mathcal{D}) := \bigsqcup_{\pi\in\mathcal{C}}\{\pi\} \times \Delta_\pi$. Anyway, since iteration of the Rauzy-Veech algorithm causes lengths to decrease to zero (Corollary 2.1.12), neither of these maps can have any remarkable recurrence property.

It is then useful to proceed to a *renormalisation*. For each $\pi \in \mathcal{C}$ we define

$$\Delta^{(1)}_\pi := \left\{\lambda \in \Delta_\pi \,\middle|\, \sum \lambda_\alpha = 1\right\} \quad \text{and} \quad \Omega^{(1)}_\pi := \left\{(\lambda, \tau) \in \Delta^{(1)}_\pi \times \Theta_\pi \,\middle|\, \mathrm{Area}\,(X(\pi, \lambda, \tau)) = 1\right\};$$

In both cases, the restricted parameter spaces are zero loci of smooth functions on the original ones (cfr. paragraph 3.1.B for the area's formula): therefore they inherit the Lebesgue measure. Now we set

$$\Omega(\mathcal{D}) := \bigsqcup_{\pi\in\mathcal{C}} \{\pi\} \times \Omega^{(1)}_\pi \quad \text{and} \quad \Delta(\mathcal{D}) := \bigsqcup_{\pi\in\mathcal{C}} \{\pi\} \times \Delta^{(1)}_\pi.$$

The map $\mathcal{R}_\Omega : \Omega(\mathcal{D}) \to \Omega(\mathcal{D})$ will be defined Lebesgue-almost everywhere by

$$\mathcal{R}_\Omega(\pi, \lambda, \tau) := \left(\pi', \left(\sum \lambda'_\alpha\right)^{-1} \lambda', \left(\sum \lambda'_\alpha\right) \tau'\right) \quad \text{where } (\pi', \lambda', \tau') = \mathcal{R}(\pi, \lambda, \tau).$$

A similar formula without suspension data defines $\mathcal{R}_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$. Renormalisation could also be defined using the projectivisation of Δ_π and Θ_π, rather than introducing corrective constants as above.

Up to zero measure subsets, $\mathcal{R}_\Delta$ is two-to-one, because each of its components Δ_π is reached both by a top type arrow and by a bottom type one. On the other hand, $\mathcal{R}_\Omega$ is (almost everywhere) a bijective map, because according to Lemma 2.2.5, almost each triple (π', λ', τ') is reached only by a top arrow or a bottom one, depending on the sign of $\sum_x \tau'_x$. The maps $\mathcal{R}_\Omega$ and $\mathcal{R}_\Delta$ define the *renormalised Rauzy dynamics*.

§ 2.4.B Globalisation of Veech coordinates Rauzy dynamics induces instructions to 'glue' altogether the Veech local charts coming from the vertices of a fixed Rauzy diagram $\mathcal{D}$; this time we will use them as coordinates for a *marked* Teichmüller stratum $\tilde{\mathcal{T}}(S, \Sigma, h)$. Let us fix an admissible marked permutation π such that $X(\pi, \lambda^{can}, \tau^{can})$ belongs to this stratum, and fix once and for all an identification of the topological pair (S, Σ) with $(X(\pi, \lambda^{can}, \tau^{can}), \Sigma(\pi, \lambda^{can}, \tau^{can}))$. Let $\mathcal{C}$ be the Rauzy class of π, with $\mathcal{D}$ the related Rauzy diagram.

If $\gamma : \pi' \to \pi''$ is an arrow in $\mathcal{D}$, let $\lambda' \in \Delta_{\pi'}$, $\tau'' \in \Theta_{\pi''}$ be parameters such that $\mathcal{R}(\pi', \lambda', \tau^{can}_{\pi'}) = (\pi'', \lambda^{can}_{\pi''}, \tau'')$. According to Lemma 2.2.6, a canonical isomorphism exists between the translation surfaces obtained with the Veech construction from these parameters;

of course, we can also regard it simply as a homeomorphism.

The latter induces a homeomorphism between the Teichmüller strata related to the two (topological) surfaces:

$$j_\gamma : \tilde{\mathcal{T}}(X(\pi', \lambda', \tau^{can}_{\pi'}), \Sigma(\pi', \lambda', \tau^{can}_{\pi'}), h) \longrightarrow \tilde{\mathcal{T}}(X(\pi'', \lambda^{can}_{\pi''}, \tau''), \Sigma(\pi'', \lambda^{can}_{\pi''}, \tau''), h).$$

We may consider as well $\gamma^{-1} : \pi'' \to \pi'$ the reverse of the arrow γ, even if it is not an arrow of $\mathcal{D}$; and define $j_{\gamma^{-1}} := j_\gamma^{-1}$. When γ is a 'generalised path' made up of arrows and their reverses, we also define j_γ by composition of the homeomorphisms related to each arrow of γ.

We set

$$U(\pi) := \bigcup_{\substack{\gamma:\pi\to\pi' \\ \text{generalised path}}} j_\gamma^{-1}\left(\mathcal{I}_{\pi'}(\Omega_{\pi'})\right) \subseteq \tilde{\mathcal{T}}(S, \Sigma, h)$$

where the Veech maps $\mathcal{I}_{\pi'}$ are not exactly the same as defined above, since they must be directed towards the 'right' Teichmüller space, namely the image of j_γ. In words, $U(\pi)$ is the subset of the considered Teichmüller stratum that we are able to cover using all the Veech coordinates related to a Rauzy diagram $\mathcal{D}$.

Proposition 2.4.1. *Let C be a connected component of a marked stratum $\tilde{\mathcal{T}}(S, \Sigma, h)$; and let $U \subseteq C$ be the set of the translation structures that are obtained by means of the Veech construction (applied on the i.e.m. specified by Proposition 2.3.1). Then U is an open subset of codimension ≥ 2; and there exists some π on $d = 2g + s - 1$ letters such that it is possible to identify (topologically) $S = X(\pi, \lambda^{can}, \tau^{can})$; and $U = U(\pi)$.*

In other words, for each connected component of a Teichmüller stratum there exists an open subset with full measure that is covered by the Veech charts related with a single Rauzy diagram.

§ 2.4.C Veech boxes Connected components of marked moduli spaces can be covered in a similar, but better way than before, as they can be obtained by gluing portions of parameter spaces along their *boundaries*. Moreover in the previous construction we had to consider a chart for each generalised path; in the one we are going to introduce, instead, one for each element of $\mathcal{C}$ is sufficient.

We fix (S, Σ, h) as usual, as well as a connected component of $\tilde{\mathcal{H}}(S, \Sigma, h)$. Its preimage in $\tilde{\mathcal{T}}(S, \Sigma, h)$ will consist of one or more components; we take an admissible marked permutation π as in Proposition 2.4.1 above for one of these connected components.

Let $\mathcal{C}$ be the Rauzy class π belongs to, and let $\mathcal{D}$ be its relative diagram. For each $\pi' \in \mathcal{C}$ we only consider a part of the possible length data:

$$\hat{\Delta}_{\pi'} := \{\lambda \in \Delta_{\pi'} | 1 \leq \textstyle\sum_x \lambda_x \leq 1 + \min\{\lambda_{\alpha_t}, \lambda_{\alpha_b}\}\}$$

(where, as usual, α_t and α_b are the rightmost letters in the top and bottom row of π', respectively). The spaces $\hat{\Omega}_{\pi'} := \hat{\Delta}_{\pi'} \times \Theta_{\pi'} \subseteq \Omega_{\pi'}$ are called *Veech boxes*. Each of them has an *upper boundary* $U_{\pi'} := \{\sum \lambda_x = 1 + \min\{\lambda_{\alpha_t}, \lambda_{\alpha_b}\}\}$, and a *lower boundary* $L_{\pi'} := \{\sum \lambda_x = 1\}$.

Let us identify each Veech box $\hat{\Omega}_{\pi'}$ with its image in the Teichmüller stratum under $\mathcal{I}_{\pi'}$. The Teichmüller flow will act by exponentially stretching the λ component and shrinking the τ component; therefore it rises from $L_{\pi'}$ to $U_{\pi'}$. If we apply $\mathcal{R}$ to the point of $U_{\pi'}$ we have reached, we get a point in the lower boundary of some other Veech box $\hat{\Omega}_{\pi''}$; but, according to Lemma 2.2.6, these two points represent the same translation structure. The flow continues towards the upper boundary of this box, and so on.

We now make this construction formal. For each π', the upper boundary of $\hat{\Omega}'_\pi$ is divided into a *top half* $Ut_{\pi'} := \{(\lambda, \tau) \in U_{\pi'} | \lambda_{\alpha_t} > \lambda_{\alpha_b}\}$; and a *bottom half* $Ub_{\pi'} := \{(\lambda, \tau) \in U_{\pi'} | \lambda_{\alpha_b} > \lambda_{\alpha_t}\}$

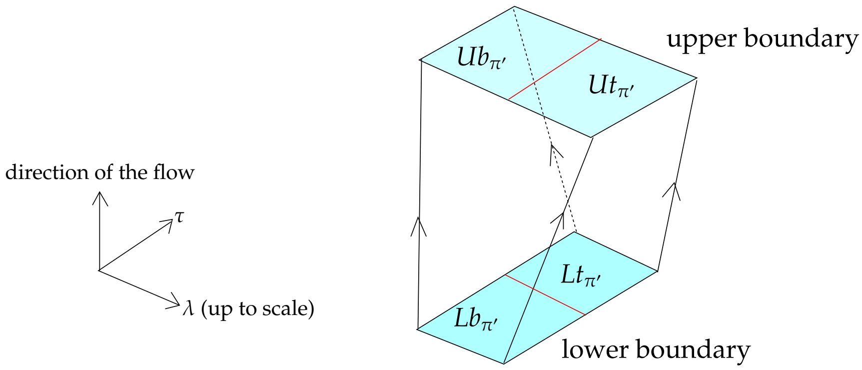

Figure 2.8: Graphical representation of a Veech box.

(the remainder is a subset of measure zero). The lower boundary is also divided in a *top half* $Lt_{\pi'} := \left\{(\lambda,\tau) \in L_{\pi'} \,\middle|\, \sum_x \tau_x < 0\right\}$, and a *bottom half* $Lb_{\pi'} := \left\{(\lambda,\tau) \in L_{\pi'} \,\middle|\, \sum_x \tau_x > 0\right\}$ (the remainder has again measure zero; the reason for such a subdivision is Lemma 2.2.5).

Let us consider $E = \bigsqcup_{\pi' \text{ vertex of } \mathcal{D}} \hat{\Delta}_{\pi'} \times \hat{\Theta}_{\pi'}$. For each π' vertex of $\mathcal{D}$,

- if $(\lambda,\tau) \in Ut_{\pi'}$, then $\mathcal{R}(\pi',\lambda,\tau) = (\mathcal{R}_t\pi',\lambda'',\tau'')$ for some parameters $(\lambda'',\tau'') \in Lt_{\mathcal{R}_t\pi'}$; we glue $Ut_{\pi'}$ and $Lt_{\mathcal{R}_t\pi'}$ identifying each pair (λ,τ) with its respective (λ'',τ'');
- if $(\lambda,\tau) \in Ub_{\pi'}$, then $\mathcal{R}(\pi',\lambda,\tau) = (\mathcal{R}_b\pi',\lambda'',\tau'')$ for some parameters $(\lambda'',\tau'') \in Lb_{\mathcal{R}_b\pi'}$; we glue $Ub_{\pi'}$ and $Lb_{\mathcal{R}_b\pi'}$ identifying each pair (λ,τ) with its respective (λ'',τ'').

Under such identifications we obtain a quotient set $\mathcal{H}(\mathcal{D})$.

Proposition 2.4.2. *Using the notations established in paragraph 2.4.B, set*

$$V(\pi) := \bigcup_{\substack{\gamma:\pi\to\pi' \\ \text{generalised path}}} j_\gamma^{-1}\left(\mathcal{I}_{\pi'}(\hat{\Omega}_{\pi'})\right).$$

The set $U(\pi) \setminus V(\pi)$ has zero measure, and there exists a unique continuous map

$$p : V(\pi) \longrightarrow \mathcal{H}(\mathcal{D})$$

such that, locally where defined, each composition

$$p \circ j_\gamma^{-1} \circ \mathcal{I}_{\pi'} : \hat{\Omega}_{\pi'} \to \mathcal{H}(\mathcal{D})$$

is the inclusion. Moreover, p is a covering map, which identifies $\mathcal{H}(\mathcal{D})$ with the quotient of $V(\pi)$ under action of the subgroup $\mathrm{Mod}_0(S,\Sigma) \leq \mathrm{Mod}(S,\Sigma)$, whose elements are the mapping classes which stabilise the connected component C of $\tilde{\mathcal{T}}(S,\Sigma,h)$ which contains $U(\pi)$.

In other words, $\mathcal{H}(\mathcal{D})$ can be identified with an open subset of $\tilde{\mathcal{H}}(S,\Sigma,h)$ with full measure. Now, let us restrict to unit area structures: the definition of Veech box can be adapted to this case, adding restrictions on the parameters τ according to λ; and, if we repeat the construction, we will obtain a space $\mathcal{H}^{(1)}(\mathcal{D})$ which can be identified with a subset of full measure of a connected component of $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$. The following result is now straightforward:

Corollary 2.4.3. *The identification, for each $\pi' \in \mathcal{C}$, of $\Omega_{\pi'}$ with the lower boundary $L_{\pi'} \subset \hat{\Omega}^{(1)}_{\pi'}$ induces a canonical inclusion $\Omega(\mathcal{D}) \hookrightarrow \mathcal{H}^{(1)}(\mathcal{D}) \subseteq \tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$.*
The return map of the Teichmüller flow on the section $\Omega(\mathcal{D})$ coincides with the normalised map $\mathcal{R}_\Omega$ given in paragraph 2.4.A.

3 Classical and preliminary results

Now that we have introduced properly all the fundamentals of the theory of i.e.m.s and translation surfaces, it is time to look at some results obtained in this field. We do not move immediately on our main concern, that is generalisations to the Khinchin theorem: this chapter is dedicated to mention the most famous results in the subject, as well as to introduce technical tools we will need later.

The first section explains how it is possible to deal with parameters related to i.e.m.s, and with the Rauzy dynamics, in terms of linear applications. This formalism is able to describe concisely a wide range of quantities related to i.e.m.s.; it is ubiquitous in the literature about i.e.m.s, and we cannot disregard it.

The second one lists the results which are unanimously considered the classics of the subject: the ones which prove ergodicity. First of all, it has been proved that almost every i.e.m., and almost every flow on a translation surface, is uniquely ergodic. But there are also results about the ergodicity of the «dynamics at the higher level», namely Rauzy and Teichmüller dynamics. However, the former needs to be considered in a different, 'accelerated' version to be properly considered an ergodic dynamics.

The third section is a quick overview of the Kontsevich-Zorich cocycle, namely a linear dynamical system defined *over* the Rauzy/Teichmüller dynamics: it describes how are lengths/elements of homology transformed under the dynamics. We will only mention few facts about its Lyapounov exponents, but we could add much more.

Finally, with the fourth section we prepare our work to generalise the Khinchin theorem: we define the objects of our interest in this generalisation, and see how they are related with the Rauzy-Veech algorithm.

Our main references for the first three sections are [Yoc07] and [MY12]. The fourth is taken from Marchese's paper [Mar11].

3.1 A linear-algebraic formalism

Almost everything in the definitions of i.e.m.s and the Rauzy-Veech iteration is, in some adequate sense, linear; it is therefore natural to set up a language based on vectors in $\mathbb{R}^{\mathcal{A}}$ and matrices in $\mathbb{R}^{\mathcal{A}\times\mathcal{A}}$. We consider a fixed Rauzy class $\mathcal{C}$ with diagram $\mathcal{D}$. If $v \in \mathbb{R}^{\mathcal{A}}$, $\|v\|$ will always denote the 1-norm $\sum_{\alpha\in\mathcal{A}} |v_\alpha|$; if M is a matrix, $\|M\| := \max\left\{\|Mv\|/\|v\| \,\middle|\, v \in \mathbb{R}^{\mathcal{A}},\ v \neq 0\right\}$ will denote the induced norm.

§ 3.1.A The matrices B_γ If $\gamma : \pi \to \pi'$ is an arrow of $\mathcal{D}$, such that the letter α wins and β loses, we set $B_\gamma := Id + E_{\beta\alpha} \in GL(\mathbb{R}^{\mathcal{A}})$, where the only nonzero entry of $E_{\beta\alpha}$ is an 1 at the β-th row and the α-th column.

Suppose γ is the arrow crossed when one applies the Rauzy-Veech algorithm on an i.e.m. $T = (\pi, \lambda)$, resulting in $\mathcal{R}T = (\pi', \lambda')$: then $\lambda = {}^{T}B_\gamma \lambda'$; and the behaviour is identical for suspension data.

Take $\gamma = \gamma_1 \cdots \gamma_r \in \Pi(\mathcal{D})$ (where the γ_j's are arrows) a path starting at π and ending at $\pi^{(r)}$: we set $B_\gamma := B_{\gamma_r} \cdots B_{\gamma_1}$. Of course all entries of B_γ are in $\mathbb{N}$ and, by induction on r, we have as before

$$\lambda = {}^T B_\gamma \lambda^{(r)} \quad \text{and} \quad \tau = {}^T B_\gamma \tau^{(r)}.$$

So, one can easily check that

The set $\Delta_\gamma := {}^T B_\gamma \Delta_{\pi'}$ is the cone of the i.e.m.s $T \in \Delta_\pi$ such that the Rauzy-Veech iteration on T is defined for at least r steps, and $\gamma_T(r) = \gamma$. The set $\Delta_\gamma^{(1)} := \Delta_\gamma \cap \Delta_\pi^{(1)}$ is a simplex.

In particular, a family of paths $\Gamma \subseteq \Pi_\pi(\mathcal{D})$ is disjoint (in the sense of paragraph 2.1.D) if and only if, for any distinct $\gamma_0, \gamma_1 \in \Gamma$, we have $\Delta_{\gamma_0} \cap \Delta_{\gamma_1} = \emptyset$.

§ 3.1.B The matrices Q_π For each $\pi \in \mathcal{C}$ we define the skew-symmetric matrix $Q_\pi \in \mathbb{R}^{\mathcal{A}\times\mathcal{A}}$ whose entries are

$$(Q_\pi)_{\alpha\beta} := \begin{cases} +1 & \text{if } \pi_b(\beta) < \pi_b(\alpha) \text{ and } \pi_t(\beta) > \pi_t(\alpha); \\ -1 & \text{if } \pi_b(\beta) > \pi_b(\alpha) \text{ and } \pi_t(\beta) < \pi_t(\alpha); \\ 0 & \text{otherwise.} \end{cases}$$

This matrix represents the translations performed by an i.e.m. $T = (\pi, \lambda)$, in the sense that the translation vector δ defined in paragraph 2.1.A is obtained as $\delta = Q_\pi \lambda$. Recalling the notations specified in paragraph 2.2.A for the Veech construction, in the same way we have

$$\eta = -Q_\pi \tau \qquad \text{and} \qquad \theta = Q_\pi \zeta.$$

Furthermore the area of the resulting surface is ${}^T\eta\lambda = -{}^T\tau {}^T Q_\pi \lambda = {}^T\tau Q_\pi \lambda$.

One may check that, for each arrow γ of $\mathcal{D}$ from a vertex π to a vertex π', it holds that $Q_{\pi'} = B_\gamma Q_\pi {}^T B_\gamma$; and arguing by induction, the same holds for any path. Let us consider the trivial vector bundle $\Delta(\mathcal{D}) \times \mathbb{R}^d$ over $\Delta(\mathcal{D})$: each of its fibres $\{(\pi, \lambda)\} \times \mathbb{R}^d$ is endowed with the singular 2-form given by the corresponding Q_π; and the transformation ${}^T B_\gamma^{-1}$ takes $\operatorname{Ker} Q_\pi$ onto $\operatorname{Ker} Q_{\pi'}$: thus a map $\{(\pi, \lambda)\} \times \mathbb{R}^d/\operatorname{Ker} Q_\pi \to \{(\pi', \lambda')\} \times \mathbb{R}^d/\operatorname{Ker} Q_{\pi'}$ is induced, and it is symplectic with respect to the structures induced by Q_π, $Q_{\pi'}$ respectively. If we identify $\mathbb{R}^d/\operatorname{Ker} Q_\pi \cong \operatorname{Im} Q_\pi$ and $\mathbb{R}^d/\operatorname{Ker} Q_{\pi'} \cong \operatorname{Im} Q_{\pi'}$, this transformation is represented by B_γ. It defines the Kontsevich-Zorich cocycle, that we will discuss in section 3.3.

§ 3.1.C The vectors c^γ For $\gamma \in \Pi(\mathcal{D})$ going from a vertex π to some π', we define

$$c^\gamma := B_\gamma \vec{1}, \quad \text{where } \vec{1} = (1, \ldots, 1) \in \mathbb{R}^{\mathcal{A}}.$$

When it is possible to deduce the path γ from the context, and $\gamma_0 < \gamma$ has length $r_0 \leq r$, we will also denote $c^{(r_0)} = c^{\gamma_0}$. Since the entries of the matrices B_{γ_j} are non-negative, for each $\xi \in \mathcal{A}$ and $r_0 < r_1$ we have $c_\xi^{(r_0)} \leq c_\xi^{(r_1)}$.

Some computations show that the entries of c^γ are the right 'renormalization factors' for the Rauzy-Veech algorithm:

Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be the standard basis of $\mathbb{R}^{\mathcal{A}}$, thus the set of the vertices of $\Delta_{\pi'}^{(1)}$. Then the vertices of $\Delta_\gamma^{(1)}$ are $v_\alpha = (c_\alpha^\gamma)^{-1} {}^T B_\gamma e_\alpha$, for $\alpha \in \mathcal{A}$.

It is a matter of calculation also showing that they are *return times*:

Proposition 3.1.1. *Take any i.e.m. $T \in \Delta_\gamma$, where the length of γ is r. Then the entry of B_γ in position (α, β) is the time that any point of $I_\alpha^{(r),t}$ spends in I_β^t under iteration of T, before coming back to $I^{(r)}$. In particular the first return time to $I^{(r)}$ for such points equals c_α^γ.*

Remark 3.1.2. It is worth noting that, for $0 \leq k < c^{\gamma}_{\alpha}$, $T^k I^{(r),t}_{\alpha}$ cannot contain singularities for T, as they would have the effect of 'splitting' $I^{(r),t}_{\alpha}$ before it returns to $I^{(r)}$; and this contradicts the properties of the Rauzy-Veech algorithm. Similarly, for $0 < k \leq c^{\gamma}_{\alpha}$, $T^k I^{(r),t}_{\alpha}$ cannot contain singularities for T^{-1}.

Recalling Remark 2.1.7, observe that

$$T^{-l} I^{(r),b}_{\alpha} = (u^b_{\alpha}, u^b_{\alpha} + \lambda^{(r)}_{\alpha}) = T(u^t_{\alpha}, u^t_{\alpha} + \lambda^{(r)}_{\alpha}) = T^{h+1} I^{(r),t}_{\alpha}.$$

So, being $c^{(r)}_{\alpha}$ the first return time of points of $I^{(r),t}_{\alpha}$ into $I^{(r)}$, it must equal $l + h + 1$. ◇

3.2 Ergodic properties

§ 3.2.A Unique ergodicity of i.e.m.s and foliations In paragraph 0.3.D we recalled that all irrational rotations of $\mathbb{T}$ are both minimal and ergodic. For generic i.e.m.s things are not so simple, since it is not true that *all* minimal i.e.m.s are also ergodic. However:

Theorem 3.2.1. *For each admissible marked permutation π and for almost any choice of the length data $\lambda \in \Delta_\pi$, the resulting i.e.m. T is uniquely ergodic (with respect to the adequate renormalization of Lebesgue measure on the interval where T acts).*

This has been proved independently by Masur ([Mas82]) and Veech ([Vee82]); we give quickly the idea for a possible proof (see [Yoc07] and [MY12]). Let γ be the infinite Rauzy path originated by Rauzy-Veech iteration on T, and let $\Delta_\gamma = \bigcap_{\gamma' < \gamma \text{ finite path}} \Delta_{\gamma'}$ be the subset of Δ_π whose elements are the i.e.m. which still originate the path γ. Each T-invariant measure on $I = (0, \lambda^*)$ can be seen as the Lebesgue measure on another interval $\tilde{I}$ where a new i.e.m. $\tilde{T} \in \Delta_\gamma$ acts: it suffices to distort the lengths of the sub-intervals. So, T is uniquely ergodic if and only if Δ_γ has a unique element with $\lambda^* = 1$, i.e. Δ_γ is only a ray in Δ_π.

One can show that, for almost any T, there exists a finite path γ_0 which appears infinitely many times in γ (i.e. $\gamma = \gamma_1 \gamma_0 \gamma_2 \gamma_0 \gamma_3 \cdots$) and such that B_{γ_0} has only positive entries: therefore for each path ν starting where γ_0 ends, $\Delta_{\gamma_0 \nu}$ is strictly 'narrower' than Δ_ν. In particular $\Delta_\gamma = \bigcap_n B_{\gamma_0} B_{\gamma_n} \cdots B_{\gamma_1} B_{\gamma_0}$ must be a single ray because it is the intersection of countably many cones, each one obtained by shrinking the previous one by a factor bounded from below.

As one could expect, there exist also results of almost sure unique ergodicity for translation structures; we cite the following by Kerckhoff, Masur, and Smillie ([KMS86]):

Theorem 3.2.2. *Let S be a Riemann surface, and let q be a quadratic differential on S. Then, for almost every $\vartheta \in \mathbb{T}$, the foliation induced by q on S in direction ϑ is uniquely ergodic.*

This theorem also covers *non-orientable* foliations (in this case, ergodicity and invariance of the Lebesgue measure are meant with respect to the natural *local* flow which parametrises the foliation). Its proof *does not make use of i.e.m.s*, in fact i.e.m.s relate only with translation surfaces.

§ 3.2.B Zorich acceleration for a finite volume We would also like to study the ergodic properties of the dynamical systems we defined «on the higher level», namely the Teichmüller flow on moduli spaces and the renormalised Rauzy dynamics $\mathcal{R}_\Delta$ and $\mathcal{R}_\Omega$. In paragraph 2.4.C we saw that the Teichmüller flow on each connected component of $\tilde{\mathcal{H}}^{(1)}(S, \Sigma, h)$ admits as a section $\Omega(\mathcal{D})$, where $\mathcal{D}$ is an appropriate Rauzy diagram; and $\mathcal{R}_\Omega$ is the related return map. Thus the dynamical properties of the Teichmüller flow and of Rauzy dynamics are strictly related.

In section 2.3 we defined the Masur-Veech measure on strata (of any kind), which is merely a globalisation of the Lebesgue measure on the parameter spaces Ω_π for the Veech construction.

It is easily checked that it is invariant under the Teichmüller flow, and one can prove that

Proposition 3.2.3. $\mathcal{H}^{(1)}(\mathcal{D})$ *(as well as the corresponding connected component of* $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$*) has a finite volume.*

Starting from this measure, a $\mathcal{R}_\Omega$-invariant measure on $\Omega(\mathcal{D})$, equivalent to the Lebesgue measure, can be defined: but, unfortunately, this one has an infinite volume: therefore ergodic-theoretical observations have less significance with respect to it. For this reason, a *smaller* section for the Teichmüller flow is defined, and it corresponds to an *acceleration* of the Rauzy-Veech algorithm.

From Proposition 2.1.11 we know that iteration of Rauzy-Veech algorithm on an initial T with the Keane's property generates an ∞-complete Rauzy path γ: in particular it must happen infinitely many times that γ arrives to a marked permutation π' with a top arrow, and leaves it with a bottom one; or vice versa. Let r be the least positive integer such that the type of the $(r+1)$-th arrow of γ is the opposite of type of the r-th one: we set $\mathcal{R}^*T = \mathcal{R}^r T$. The map $\mathcal{R}^*$ is called *Zorich acceleration* of the Rauzy-Veech algorithm. The renormalised $\mathcal{R}^*_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ is naturally defined.

For what concerns acceleration of the algorithm with suspension data, $\Omega(\mathcal{D})$ has a subset such that Zorich acceleration appears as a first return map. Suppose that iteration of $\mathcal{R}_\Omega$, at a certain stage, is passing through an element $(\pi', \lambda', \tau') \in \Omega(\mathcal{D})$. Then the couple (π, λ) determines the type of the next arrow; but, moreover, recalling Lemma 2.2.5, we know from the sign of $\sum_\alpha \tau'_\alpha$ what type has been the last arrow. Therefore, we consider the subset $\Omega^*(\mathcal{D}) \subset \Omega^*(\mathcal{D})$ made up of the points such that the arriving arrow and the leaving one are of different types: the return map of $\mathcal{R}_\Omega$ to $\Omega^*(\mathcal{D})$ will be a renormalised version of Zorich acceleration.

The explicit construction of $\Omega^*(\mathcal{D})$ is as follows: for each fixed $\pi \in \mathcal{C}$, let α_t and α_b be the rightmost letters in its top and bottom row respectively. Set

$$\Omega^{(1),tb}_\pi = \left\{(\lambda,\tau) \in \Omega^{(1)}_\pi \,\middle|\, \textstyle\sum_x \tau_x < 0, \lambda_{\alpha_t} < \lambda_{\alpha_b}\right\} \text{ and } \Omega^{(1),bt}_\pi = \left\{(\lambda,\tau) \in \Omega^{(1)}_\pi \,\middle|\, \textstyle\sum_x \tau_x > 0, \lambda_{\alpha_t} > \lambda_{\alpha_b}\right\}.$$

The above mentioned subset is then $\Omega^*(\mathcal{D}) := \bigsqcup_{\pi\in\mathcal{C}} \{\pi\} \times (\Omega^{(1),tb}_\pi \sqcup \Omega^{(1),bt}_\pi)$.

§ 3.2.C Uniqueness of the a.c.i.p. A $\mathcal{R}^*_\Omega$-invariant measure can be defined on $\Omega^*(\mathcal{D})$, again regarding it as a section for the Teichmüller flow in some $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$. This measure is again equivalent to the Lebesgue one, and now it holds that

$\Omega^*(\mathcal{D})$ *has a finite volume.*

The study of the ergodic properties of the invariant, finite measures the spaces of our interest are provided with has been performed by Veech ([Vee86]) and Zorich ([Zor96]). We cite the main result(s):

Theorem 3.2.4. *The space* $\Omega^*(\mathcal{D})$ *admits an unique absolutely continuous* $\mathcal{R}^*_\Omega$*-invariant probability; this one is, actually, equivalent to the Lebesgue measure.*

This probability measure is, of course, (a scaling of) the one mentioned above. The uniqueness of the a.c.i.p. immediately implies its ergodicity (if it were not ergodic, there would be two complementary invariant subsets with nonzero measure: by restricting and scaling the a.c.i.p. previously found, two other ones would be obtained). It follows easily:

Corollary 3.2.5. *The map* $\mathcal{R}^*_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ *admits an unique a.c.i.p. (actually equivalent to the Lebesgue measure).*

and recalling that $\Omega^*(\mathcal{D})$ is a section for the Teichmüller flow:

Theorem 3.2.6. *Each connected component of a marked moduli stratum of unit area translation surfaces, $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$, has an unique a.c.i.p.. It is a scaling of the Masur-Veech measure.*

Remark 3.2.7. To be precise, the map $\mathcal{R}_\Omega : \Omega(\mathcal{D}) \to \Omega(\mathcal{D})$, and the map $\mathcal{R}_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$, also have an unique continuous invariant measure (as a consequence of the uniqueness of the a.c.i.p. for the Teichmüller flow), even if it has not finite mass. So, these two maps are also 'ergodic', in a weaker sense. ◇

3.3 The Kontsevich-Zorich cocycle

§ 3.3.A The continuous version The *continuous Kontsevich-Zorich cocycle* is a way to keep track of the automorphisms induced by Teichmüller flow on the homology of the considered topological surface. Its definition goes as follows: the Teichmüller flow g^t on a marked Teichmüller stratum $\tilde{\mathfrak{T}}^{(1)}(S,\Sigma,h)$ can be extended trivially to a flow KZ^t on the product

$$\tilde{\mathfrak{T}}^{(1)}(S,\Sigma,h) \times H_1(S,\Sigma,h)$$

simply setting its first component equal to g^t, and the second one to be the identity. Let us consider the (relative) mapping class group $\mathrm{Mod}(S,\Sigma)$: we already know how its acts on $\tilde{\mathfrak{T}}^{(1)}(S,\Sigma,h)$; and a natural, nontrivial action on $H_1(S,\Sigma;\mathbb{Z})$ is defined as well. Therefore the space

$$\tilde{\mathcal{H}}^{+}(S,\Sigma,h) := \tilde{\mathfrak{T}}^{(1)}(S,\Sigma,h) \times H_1(S,\Sigma;\mathbb{Z}) \big/ \mathrm{Mod}(S,\Sigma)$$

is a vector bundle over the marked moduli stratum $\tilde{\mathcal{H}}^{(1)}(S,\Sigma,h)$, with fibres isomorphic to $\mathbb{R}^d$ (where $d = 2g + s - 1$). The Kontsevich-Zorich cocycle is then the flow obtained as projection of KZ^t on this bundle.

Instead of considering relative homology, one could also take the absolute one $H_1(S;\mathbb{Z}) \cong \mathbb{R}^{2g}$ (which can be seen as a subgroup of the former): this defines the *restricted* version of the cocycle.

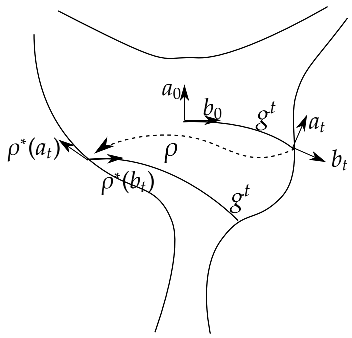

Figure 3.1: A pictorial representation of the continuous Kontsevich-Zorich cocycle: consider a fundamental domain for the action of the mapping class group on a Teichmüller stratum. Starting from two (independent) elements $a_0 b_0 \in H_1(S,\Sigma;\mathbb{Z})$, the Teichmüller flow carries them in a trivial way until it reaches the boundary of the fundamental domain. Afterwards, if we want to see the flow in the same fundamental domain, we need to apply some $\rho \in \mathrm{Mod}(S,\Sigma)$, which transforms a_t, b_t into two new (independent) elements of the homology group.

§ 3.3.B The discrete version There is also a *discrete* version of the Kontsevich-Zorich cocycle, in defining which we will be more explicit. In general, given a measurable map $f : X \to X$, where X is endowed with an f-invariant measure, a *linear cocycle* over f is defined formally as a map

$$\begin{array}{ccc} X \times \mathbb{R}^d & \longrightarrow & X \times \mathbb{R}^d \\ (x, v) & \longmapsto & (f(x), A(x)v) \end{array}$$

where d is a positive integer and $A : X \to GL(d, \mathbb{R})$ is a measurable map. We denote $A^n(x) := A(f^{n-1}(x)) \cdots A(x)$.

When the dynamical system considered is some version of the renormalised Rauzy dynamics, we have a natural choice for the map A: the *extended discrete Kontsevich-Zorich cocycle* over some $\Delta(\mathcal{D})$ is the map we already introduced in paragraph 3.1.B

$$\begin{array}{cccc} KZ_\Delta : & \Delta(\mathcal{D}) \times \mathbb{R}^d & \longrightarrow & \Delta(\mathcal{D}) \times \mathbb{R}^d \\ & ((\pi, \lambda), v) & \longmapsto & \left(\mathcal{R}_\Delta(\pi, \lambda), B_\gamma v\right) \end{array}$$

where $d = \#\mathcal{A}$, and γ is the arrow of $\mathcal{D}$ that is crossed when applying the Rauzy-Veech algorithm on (π, λ). Of course we can define in the same way a cocycle KZ_Ω over the augmented parameter space $\Omega(\mathcal{D})$. Moreover, according to what we saw in paragraph 3.1.B, we can define a *restricted* version of the cocycle (which is the interesting one) by restricting each fibre to the corresponding $\operatorname{Im} Q_\pi$.

The continuous and discrete versions of the cocycle are related:

Let us consider the inclusion $\Omega(\mathcal{D}) \hookrightarrow \tilde{\mathcal{H}}^{(1)}(S, \Sigma, h)$ of Corollary 2.4.3; and let $\Omega^+(\mathcal{D})$ be the pre-image of $\Omega(\mathcal{D})$ in the bundle $\tilde{\mathcal{H}}^+(S, \Sigma, h)$. Then we can identify $\Omega^+(\mathcal{D})$ with $\Omega(\mathcal{D}) \times \mathbb{R}^d$ in such a way that the return map of the continuous Kontsevich-Zorich cocycle KZ^t on $\Omega^+(\mathcal{D})$ coincided with the discrete version KZ_Ω. This identification also makes the restricted versions correspond.

We can also define the Zorich-accelerated versions of the cocycle: for instance,

$$\begin{array}{cccc} KZ^*_\Delta : & \Delta(\mathcal{D}) \times \mathbb{R}^d & \longrightarrow & \Delta(\mathcal{D}) \times \mathbb{R}^d \\ & ((\pi, \lambda), v) & \longmapsto & \left(\mathcal{R}^*_\Delta(\pi, \lambda), B_\gamma v\right) \end{array}$$

where γ if again the path spanned in $\mathcal{D}$ when applying the accelerated algorithm, made up of arrows which are all of the same type, and have the same winner.

§ 3.3.C Lyapounov exponents The most natural question about a cocycle is, what happens to a vector $v \in \mathbb{R}^d$ under reiteration of the map? We recall the statement of the classical *multiplicative ergodic theorem*:

Theorem 3.3.1 (Oseledets). *Let X be a probability space; suppose that $f : X \to X$ is an ergodic transformation, and that a linear cocycle over f is determined by a map $A : X \to GL(d, \mathbb{R})$ such that $\log \|A\|$ and $\log \|A^{-1}\|$ are both integrable; then the following property holds.*
For almost every $x \in X$, there exists an unique filtration $\mathbb{R}^d = E_0(x) \supsetneq E_1(x) \supsetneq \ldots \supsetneq E_r(x) = \{0\}$ such that:

- *r is independent of x;*
- *the E_j are measurably dependent from x, and such that $A(x)E_j(x) \subseteq E_j(f(x))$;*
- *r real numbers $\lambda_1 > \cdots > \lambda_r$ exist such that, for almost every $x \in X$ and any $v \in E_{j-1} \setminus E_j$, we have*

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_j.$$

The λ_j's are called *Lyapounov exponents* of the cocycle. If A were a constant map, they would correspond to the logarithms of the absolute values of the eigenvalues.

The (extended discrete) Kontsevich-Zorich cocycle, when defined on the accelerated maps $\mathcal{R}^*_\Delta$ or $\mathcal{R}^*_\Omega$ (beacuse a finite total mass is needed), can be seen to satisfy the property of integrability required to apply the Oseledets' theorem. Moreover the following can be proven:

For each π vertex of $\mathcal{D}$, a basis of $\mathbb{R}^d/\mathrm{Im}\,(Q_\pi)$ can be chosen such that, under such choice of bases, for each $\gamma : \pi \to \pi'$ arrow in $\mathcal{D}$, the application $B_\gamma : \mathbb{R}^d/\mathrm{Im}\,(Q_\pi) \to \mathbb{R}^d/\mathrm{Im}\,(Q_{\pi'})$ is expressed with the identity matrix.

Thus, the cocycle obtained with this quotient has 0 as unique Lyapounov exponent; whereas (see [AV07])

Theorem 3.3.2 (Avila-Viana). *The Lyapounov spectrum associated to the restricted Kontsevich-Zorich cocycle is simple.*

3.4 Reduced triples, detection and production

We are done with presenting the most classical facts about i.e.m.s and Teichmüller dynamics; we finally prepare the ground to give a generalisation of the Khinchin theorem. Before doing this, we introduce the main tool we will use, namely *reduced triples* for i.e.m.s.

§ 3.4.A Terminology We fix a Rauzy class $\mathcal{C}$ with diagram $\mathcal{D}$, and consider i.e.m.s whose associated marked permutation belongs to $\mathcal{C}$. When not explicitly stated, we will use all the notations for elements of an i.e.m. established in the previous chapter.

We will be particularly interested in sub-intervals with a good behaviour under the Rauzy-Veech iteration:

Definition 3.4.1. Let T be an admissible i.e.m., acting on an interval I. We will call an element $(\beta, \alpha; n) \in \mathcal{A}^2 \times \mathbb{N}$ a *triple* if $\beta \neq b_\mathcal{C}$ the leftmost letter of the bottom row, $\alpha \neq t_\mathcal{C}$ the leftmost letter of the top row (see Remark 2.1.8), and $(u^b_\beta, u^t_\alpha; n')$ is not a connection for any $n' \leq n$. We denote $I(\beta, \alpha; n)$ the open sub-interval of I whose endpoints are u^t_α and $T^n u^b_\beta$.
Moreover, we say that a triple $(\beta, \alpha; n)$ is *reduced* if, for any $j \in \{0, \dots, n\}$, $T^{-j}\,(I(\beta, \alpha; n))$ does neither contain any singularity of T, nor of T^{-1}.

Roughly speaking, a triple is reduced if the first n iterations of T^{-1} take $I(\beta, \alpha; n)$ onto intervals, whose endpoints are $T^{n-j}u^b_\beta$ and $T^{-j}u^t_\alpha$.

Reduced triples are 'seen' by the Rauzy-Veech algorithm: let us introduce two definitions.

Definition 3.4.2. Let $(\beta, \alpha; n)$ be a triple as above. We say that the triple is *detected* by Rauzy-Veech iteration at the r-th step if, using the standard notations for the obtained i.e.m.s, we have

$$u^{(r),b}_\beta - u^{(r),t}_\alpha = T^n u^b_\beta - u^t_\alpha.$$

We say that the triple is *produced* by the Rauzy-Veech iteration at the r-th step if there exists a letter $\xi \in \mathcal{A}$ such that

$$|I(\beta, \alpha; n)| = \lambda^{(r)}_\xi.$$

§ 3.4.B Reduced triples are detected *All* reduced triples are seen by the algorithm in the first sense:

Proposition 3.4.3. *Let T be an i.e.m. satisfying the Keane's property, and let $(\beta, \alpha; n)$ be a reduced triple for T. Then $(\beta, \alpha; n)$ is detected by the Rauzy-Veech iteration on T, at some step r.*

Proof. Let $m \in \{0, \dots, n\}$ such that $T^{n-m}u^b_\beta$ is leftmost among the $T^j u^b_\beta$ varying $j \in \{0, \dots, n\}$. Since the considered triple is reduced, $T^{-m}u^t_\alpha$ is also leftmost among the $T^{-j}u^t_\alpha$ varying $j \in \{0, \dots, n\}$. We will prove that the triple is produced at the step

$$r := \max\{k | I^{(k)} \text{ contains both } T^{n-m}u^b_\beta \text{ and } T^{-m}u^t_\alpha\}.$$

For simplicity, let us suppose $T^{n-m}u^b_\beta > T^{-m}u^t_\alpha$, the other case being totally similar. Therefore $I^{(r+1)} \not\ni T^{n-m}u^b_\beta$, and this interval is obtained from $I^{(r)}$ truncating it at the rightmost singularity of $(T^{(r)})^{\pm 1}$; in particular, each such singularity has to be $\leq T^{n-m}u^b_\beta$.

Recalling Remark 2.1.7, since $T^{n-m}u^b_\beta$ (resp. $T^{-m}u^t_\alpha$) is a point of $I^{(r)}$ obtained by iteration of T (resp. T^{-1}) on u^b_β (resp. u^t_α), $L, H \in \mathbb{N}$ can be found such that

$$T^{n-m}u^b_\beta = (T^{(r)})^L u^{(r),b}_\beta \quad \text{and} \quad T^{-m}u^t_\alpha = (T^{(r)})^{-H} u^{(r),t}_\alpha.$$

If we show that $L = H = 0$, we are done. By contradiction, if $L > 0$, then

$$u^{(r),b}_\beta = T^{n-m-l'}u^b_\beta > T^{n-m}u^b_\beta$$

where, in the first equality, the existence of such a $0 < l' < n - m$ is again a consequence of Remark 2.1.7; and the inequality follows from the definition of m. But the expression above contradicts that any singularity for $(T^{(r)})^{\pm 1}$ must not be on the right of $T^{n-m}u^b_\beta$. And if $H > 0$, then

$$T^{n-m}u^b_\beta > u^{(r),t}_\alpha = T^{-m+h'}u^t_\alpha > T^{-m}u^t_\alpha$$

where the first inequality is again because of how singularities of $(T^{(r)})^{\pm 1}$ are placed, and the other ones are as above (with $0 < h' < m$). But this contradicts the reducedness of $(\beta, \alpha; n)$. □

§ 3.4.C How to produce reduced triples When dealing with $(\beta, \alpha) \in \mathcal{A}^2$ with $\alpha \neq t_{\mathcal{C}}$, $\beta \neq b_{\mathcal{C}}$, the arguments we will use in our proofs differ according to the existence of a marked permutation in $\mathcal{C}$ that shows them in particular positions.

Definition 3.4.4. Let (β, α) be as above. We say that this couple is of:

- *type A* if a $\pi \in \mathcal{C}$ exists with its top row ending with α, and its bottom row ending with β;
- *type B* if it is not of type A and a $\pi \in \mathcal{C}$ exists such that the bottom row ends with α, the top one ends with a letter $v \neq \alpha, \beta$, and

$$\{x \in \mathcal{A} | \pi_b(x) \leq \pi_b(\beta)\} = \{x \in \mathcal{A} | \pi_t(x) \leq \pi_t(\alpha)\} \cup \{v\}.$$

In both cases, we will call π a *preferred* marked permutation for (β, α).

In the second case, the letter just before β in the bottom row of π will be called q: we will have $q \neq v$, otherwise π is not admissibile. To sum up, in this case π appears like:

$$\pi = \begin{pmatrix} \cdots & q & & \cdots & \alpha & \cdots & & v \\ \cdots & & v & \cdots & q & \beta & \cdots & \alpha \end{pmatrix}$$

The technical fact below will be essential for several arguments in what follows (we omit its proof):

Proposition 3.4.5. *Each couple $(\beta, \alpha) \in \mathcal{A}^2$ with $\alpha \neq t_{\mathcal{C}}, \beta \neq b_{\mathcal{C}}$ is either of type A or of type B.*

Our definitions of type A and B exclude each other only for a simpler exposition.

Now, we fix (β, α) and explain how to find an n and a set of i.e.m.s that have $(\beta, \alpha; n)$ as a reduced triple produced by the Rauzy-Veech algorithm.

Let us start from type A: we fix a finite path $\eta = \eta(\beta, \alpha) \in \Pi(\mathcal{D})$ such that a preferred marked permutation π appears in η, reached with an arrow of bottom type; then the letter α wins; and then it loses letting another letter $w \in \mathcal{A}$ win; at this stage, η ends.

Lemma 3.4.6. *Let (β,α) be a couple of type A and let $\eta(\beta,\alpha)$ be defined as above. Let $\gamma \in \Pi(\mathcal{D})$ be any path ending with η, made up of r arrows. Then a positive integer $n \leq \|c^\gamma\|$ exists such that, for any $T \in \Delta_\gamma$, the triple $(\beta,\alpha;n)$ is reduced for T, and $|I(\beta,\alpha;n)| = \lambda_\alpha^{(r)}$.*

Proof. Let us fix $T \in \Delta_\gamma$. Since in the last arrow of γ the letter α loses against a letter w, and in the previous one it has won against the letter β, we must have $\pi_b(w) = d-1$; and $\lambda_\alpha^{(r)} = \lambda_\alpha^{(r-1)} = \lambda_\alpha^{(r-2)} - \lambda_\beta^{(r-2)} = u_\beta^{(r-2),b} - u_\alpha^{(r-2),t}$. According to Remark 2.1.7, positive integers l and h exist such that $u_\alpha^{(r-2),t} = T^{-h}u_\alpha^t$; and $u_\beta^{(r-2),b} = T^l u_\beta^b$. So, we have that

$$T^l u_\beta^b - T^{-h}u_\alpha^t = \lambda_\alpha^{(r)}$$

Set $n := l+h$. We have $\|c^\gamma\| \geq \|c^{(r-2)}\| > l+h$ (because of Remark 3.1.2). Now we only have to prove that the open interval J bounded by $T^l u_\beta^b$ and $T^{-h}u_\alpha^t$ is such that T^kJ does not contain any singularity of $T^{\pm 1}$ for any $-l \leq k \leq h$: in this case we have $I(\beta,\alpha;n) = T^hJ$ and assertions about its length and the reducedness of the triple are obvious.

We already know what are the winner and the loser of two steps of the Rauzy-Veech algorithm starting from $T^{(r-2)}$. If $w \neq \beta$, this knowledge implies $u_w^{(r-2),b} < u_\alpha^{(r-2),t} < u_\beta^{(r-2),b}$, with $u_\alpha^{(r-2),t}$ the rightmost singularity for T and $u_w^{(r-2),b}, u_\beta^{(r-2),b}$ the two rightmost for T^{-1}; so $J \subset I_\alpha^{(r-2),t} \cap I_w^{(r-2),b}$. Otherwise $u_\alpha^{(r-2),b} < u_\alpha^{(r-2),t} < u_\beta^{(r-2),b}$, with $u_\alpha^{(r-2),t}$ the rightmost singularity for $T^{(r-2)}$ and $u_\alpha^{(r-2),b}, u_\beta^{(r-2),b}$ the two rightmost for $(T^{(r-2)})^{-1}$; so $J \subset I_\alpha^{(r-2),t} \cap I_\alpha^{(r-2),b}$.

In the first case: from Remark 3.1.2, we know that $T^kJ \subset T^kI_\alpha^{(r-2),t}$ contains no singularities of T for $0 \leq k < c_\alpha^{(r-2)}$, hence it does not contain singularities of T^{-1} for $0 < k \leq c_\alpha^{(r-2)}$; using the same argument on T^{-1}, we get that $T^kJ \subset T^kI_\beta^{(r-2),b}$ contains no singularities of T^{-1} for $0 \geq k > -c_w^{(r-2)}$, nor singularities of T for $0 > k \geq -c_w^{(r-2)}$.

If $h < c_\alpha^{(r-2)}$ and $l < c_w^{(r-2)}$ both hold, we are done. The first inequality is again a consequence of Remark 3.1.2; whereas, for the second one, we proceed by contradiction. If $l \geq c_w^{(r-2)}$, set $l' = l - c_w^{(r-2)} \geq 0$. As l is the first entry time of u_β^b in $I^{(r-2)}$ under iteration of T, all the iterates T^k for $0 \leq k \leq l$ must be continuous in a neighbourhood of u_β^b. But, $u_\beta^{(r-2),b}$ being the right endpoint of $I_w^{(r-2),b}$, $T^{l'}u_\beta^b$ should be (by local continuity) the right endpoint of $I_w^{(r-2),t}$, so it should be a point of $I^{(r-2)}$: and this is absurd.

In the second case: the arguments above stay almost unchanged, except that in this case $l' \geq 0$ implies that $T^{l'}u_\beta^b$ is the right endpoint of $I^{(r-2)}$, and this implies that π has been reached with an arrow of top type, contrarily to our assumptions on $\eta(\beta,\alpha)$. □

An analogous result holds for couples of type B: in this case, let again π be a preferred marked permutation for (β,α); we will consider a path $\eta = \eta(\beta,\alpha) \in \Pi(\mathcal{D})$ such that its last arrow is the arrow starting from π with winner α. The same result as above holds, provided we add an hypothesis to the considered i.e.m.s:

Lemma 3.4.7. *Let (β,α) be a couple of type B, and let $\eta(\beta,\alpha)$ be defined as above. Let $\gamma \in \Pi(\mathcal{D})$ be any path ending with η, made up of r arrows. Then a positive integer $n \leq \|c^\gamma\|$ exists such that, for any $T \in \Delta_\gamma$ yielding $\lambda_v^{(r)} < \lambda_q^{(r)}$, the couple $(\beta,\alpha;n)$ is reduced for T and $|I(\beta,\alpha;n)| = \lambda_v^{(r)}$.*

Proof. Let T be as in the statement. From the Rauzy-Veech dynamics we have $\lambda_v^{(r)} = \lambda_v^{(r-1)}$; the definitions of, π, v and γ imply $u_\alpha^{(r-1),t} + \lambda_v^{(r-1)} = u_\beta^{(r-1),b}$. Again because of Remark 2.1.7 two non-negative integers l and h exist such that $u_\alpha^{(r-1),t} = T^{-h}u_\alpha^t$; and $u_\beta^{(r-1),b} = T^l u_\beta^b$; therefore

$$T^l u_\beta^b - T^{-h} u_\alpha^t = \lambda_v^{(r)}.$$

The same way we did for type A, we set $n := l + h$, so $\|c^\gamma\| \geq \|c^{(r-1)}\| > l + h$ as desired; and we have to prove that the open interval J bounded by $T^l u_\beta^b$ and $T^{-h}u_\alpha^t$ is such that $T^k J$ does not contain singularities of T, for all $-l \leq k < h$.

Now, the fact that α wins on v allows us to say that $\lambda_v^{(r-1)} < \lambda_\alpha^{(r-1)}$. On the one hand, this yields $u_\alpha^{(r-1),t} < u_\beta^{(r-1),b} < u_\alpha^{(r-1),t} + \lambda_\alpha^{(r-1)}$. On the other hand, the bottom singularity preceeding $u_\beta^{(r-1),b}$ is $u_q^{(r-1),b} = u_\beta^{(r-1),b} - \lambda_q^{(r-1)} < u_\beta^{(r-1),b} - \lambda_v^{(r-1)} = u_\alpha^{(r-1),t}$. So, this time the interval J is contained in $I_\alpha^{(r-1),t}$ and in $I_q^{(r-1),b}$. The proof goes on similarly as in the first case for the type A. □

§ 3.4.D Sequences of produced reduced triples

The result we can obtain about the production of reduced triples is someway weaker than about their detection:

Proposition 3.4.8. *Let $(\beta, \alpha) \in \mathcal{A}^2$ be such that $\beta \neq b_\mathfrak{C}$ and $\alpha \neq t_\mathfrak{C}$; and fix $\pi_0 \in \mathfrak{C}$. Then, for Lebesgue-almost any $T \in \Delta_{\pi_0}$, two sequences of positive integers, $(r_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$, can be found such that: the first one is increasing; and for all $j \in \mathbb{N}$ we have $n_j < \|c^{(r_j)}\|$, and $(\beta, \alpha; n_j)$ is a reduced triple for T, produced by the r_j-th step of the Rauzy-Veech iteration on T. More precisely:*

- *if (β, α) is of type A, we can take r_j to be the instants when $\gamma_T(r_j)$ ends with $\eta(\beta, \alpha)$ as previously defined. In this case $|I(\beta, \alpha; n_j)| = \lambda_\alpha^{(r_j)}$;*
- *if (β, α) is of type B, we can take r_j to be the instants when $\gamma_T(r_j)$ ends with $\eta(\beta, \alpha)$ as previously defined, and $\lambda_v^{(r_j)} < \lambda_q^{(r_j)}$. In this case $|I(\beta, \alpha; n_j)| = \lambda_v^{(r_j)}$.*

Proof. Let us consider the Zorich acceleration of the Rauzy-Veech renormalised algorithm defined in paragraph 3.2.B. According to Corollary 3.2.5, $\mathcal{R}^*_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ is an ergodic map with respect to a measure which is equivalent to the Lebesgue one. Let $E \subseteq \Delta(\mathcal{D})$ be a subset of positive measure (equivalently for the Lebesgue measure or for the invariant measure). Recall that almost every i.e.m. (on the unit interval) satisfies the Keane's property, so the map $\mathcal{R}^*_\Delta$ can be iterated infinitely many times; and, because of the ergodicity, for almost every i.e.m. T among these ones, the $\mathcal{R}^*_\Delta$-positive orbit of T visits infinitely many times the region E.

Now, if (β, α) is a couple of type A, we take $E = \Delta_{\eta(\beta,\alpha)}$. Every visit to the region means that an instant r_j exists such that $\gamma_T(r_j)$ ends with η. We conclude by applying Lemma 3.4.6, with $\gamma = \gamma_T(r_j)$.

Similarly, if (β, α) is a couple of type B, we take $E = \Delta_{\eta(\beta,\alpha)} \cap {}^T B_\eta^{-1}\{\lambda_v < \lambda_q\}$ and, arguing as before, Lemma 3.4.7 implies our statement. □

4 Khinchin theorem and i.e.m.s

A stronger property than topological transitivity (Definition 0.2.2) for dynamical systems could be that an orbit not only passes near a selected point infinitely many times; but that, at those passages, the selected point is approached at some selected speed: a *shrinking target property* (cfr. [Ath08]) is a formalisation of this situation.

A classical theorem by Khinchin is concerned with a Diophantine condition: starting from a real number, consider the fractional part of its integer multiples. Given a decreasing sequence $\phi(n)$, we ask whether there are infinitely many of those fractional parts that are less than the corresponding $\phi(n)$. Khinchin's answer is: yes if the sequence has divergent sum, no otherwise. The reason for this *dichotomy* is the Borel-Cantelli theorem: the numbers $\phi(n)$ are indeed 'proportional' to the probability of the event "the n-th fractional part is less than $\phi(n)$".

Anyway, this Diophantine condition can be interpreted as a property of the rotations of $\mathbb{T}$, that is i.e.m.s on 2 intervals. Luca Marchese proved that it is possible to generalise it to a property of *reduced triples* of general i.e.m.s, subject to the same dichotomy; and it may be regarded as a shrinking target property for singularities.

In this chapter we outline the proof of Marchese's theorem. As it is a common practice in the setting of i.e.m.s, we use dynamics in the parameter space to prove properties of single i.e.m.s. The convergent case is obtained from Borel-Cantelli after having proved that we are dealing with a sequence of events whose probabilities are proportional with $\phi(n)$. In so doing we will use the fact that reduced triples are detected (Proposition 3.4.3).

The divergent case, instead, is more complicated, and we will only give a partial proof. The statement of the theorem is implied by the production of reduced triples (Proposition 3.4.8) together with a shrinking target property for the Rauzy dynamics. This time it is not sufficient to show that the probability of the targets is proportional to $\phi(n)$: we need to prove explicitly that their supremum limit has probability 1.

The treatment of the Marchese's theorem in this chapter follows loosely the original work [Mar11]. The last section, which summarises [Mar], is about a theorem of Jon Chaika, according to which a shrinking target property involving generic points, rather than singularities, holds for i.e.m.s. A simple trick allows consideration of regular points as additional singularities, and therefore Marchese's result implies a weaker version of the one of Chaika.

4.1 Shrinking triples

§ 4.1.A Using Borel-Cantelli to check chaoticity We recall the statement of one of the most classical theorems in probability theory:

Theorem 4.1.1 (Borel-Cantelli). *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space; and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events. Then we have the following dichotomy:*

- *if $\sum_n \mathbf{P}(A_n) < +\infty$, then almost every $x \in \Omega$ belongs to only finitely many of those events;*
- *if $(A_n)_{n\in\mathbb{N}}$ is an independent family of events and $\sum_n \mathbf{P}(A_n) = +\infty$, then almost every $x \in \Omega$ belongs to infinitely many of those events.*

Suppose that we have a measurable and $\mathbf{P}$-invariant $f : \Omega \to \Omega$; we fix a sequence of events $(B_n)_{n\in\mathbb{N}}$, and then consider the events $A_n := f^{-n}(B_n)$. According to Borel-Cantelli, if we have $\sum_n \mathbf{P}(A_n) = \sum_n \mathbf{P}(B_n) < +\infty$, then for almost every $x \in \Omega$ there will exist only finitely many $n \in \mathbb{N}$ such that $x \in A_n$, i.e. $f^n(x) \in B_n$.

But if we have $\sum_n \mathbf{P}(B_n) = +\infty$, and at the same time $f^n(x) \in B_n$ happens infinitely often (for some subset of $x \in \Omega$), a look at the statement of Borel-Cantelli yields the intuitive conclusion that the events $f^{-n}(B_n)$ make up a 'weakly independent' family, that is iteration of f shows a random-like behaviour with respect to this sequence. This is particularly significant if the family of events (B_n) is decreasing (according to the partial ordering $\subseteq$) and such that $\mathbf{P}(B_n) \to 0$ for $n \to +\infty$. Such (B_n) is called a family of *shrinking targets* for the considered dynamical system, and the $x \in \Omega$ which belong to infinitely many $f^{-n}(B_n)$ are said to satisfy a *shrinking target property*.

§ 4.1.B Khinchin's dichotomy A classical theorem of Khinchin ([Khi97]) about a Diophantine condition shows, in fact, an example of shrinking target property:

Theorem 4.1.2 (Khinchin). *Let $\phi = (\phi(n))_{n\in\mathbb{N}}$ be a positive sequence; let us consider the solutions $n \in \mathbb{N}$ to the inequality*

$$\{n\vartheta\} < \phi(n) \tag{4.1}$$

where $\vartheta \in [0, 1)$ is a fixed number, and $\{\cdot\}$ denotes the fractional part of a real number. Then

- *if $\sum_n \phi(n) < +\infty$, then for almost any $\vartheta \in [0, 1)$ there are only finitely many solutions to inequality 4.1;*
- *if $(n\phi(n))_{n\in\mathbb{N}}$ is a decreasing sequence, and $\sum_n \phi(n) = +\infty$, then for almost any $\vartheta \in [0, 1)$ there are infinitely many solution to inequality 4.1.*

Even if this is not exactly the situation described above, it is quite clear that we are considering the dynamics induced by a rotation of angle $2\pi(1 - \vartheta)$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: we know that, for almost every ϑ (namely the irrational ones), this rotation is minimal; but the theorem says at what 'speeds' the point 0 is approached by its own orbit.

Equivalently, this is a theorem about i.e.m.s on $d = 2$ intervals, with length data $(\vartheta, 1 - \vartheta)$. Since the possible triples (in the sense of Definition 3.4.1) for the considered i.e.m. are associated with intervals whose length is $|\vartheta - \{n(1 - \vartheta)\}| = \{(n + 1)\vartheta\}$, the theorem is actually telling what conditions should the sequence ϕ satisfy, so that intervals generated by triples are able to shrink infinitely many times according to it. Equivalently, it says how should the targets $B_n = (\vartheta - \phi(n), \vartheta + \phi(n))$ shrink if we want the point $1 - \vartheta$ to satisfy the shrinking target property.

§ 4.1.C What about i.e.m.s? Khinchin's theorem has been generalised by Luca Marchese to an analogous result for general i.e.m.s:

Theorem 4.1.3 (Marchese). *Let π_0 be an admissible marked permutation belonging to a Rauzy class $\mathcal{C}$; and let $\phi = (\phi(n))_{n\in\mathbb{N}}$ be a positive sequence. Given an i.e.m. T, we consider the triples $(\beta, \alpha; n) \in \mathcal{A}^2 \times \mathbb{N}$ such that $\alpha \neq t_{\mathcal{C}}$ and $\beta \neq b_{\mathcal{C}}$ the leftmost letters of the two rows of π_0 (see Remark 2.1.8), which are solutions for the inequality*

$$|I(\beta, \alpha; n)| = |T^n u^b_\beta - u^t_\alpha| < \phi(n). \tag{4.2}$$

- *Suppose that ϕ is decreasing, and $\sum_n \phi(n) < +\infty$. Then, for almost any i.e.m. $T \in \Delta_{\pi_0}$, there are only finitely many triples $(\beta, \alpha; n)$ as above which are solutions to inequality 4.2.*
- *Suppose instead that $(n\phi(n))_{n\in\mathbb{N}}$ is decreasing, and $\sum_n \phi(n) = +\infty$. Then, for any fixed (β, α) as above, and almost any i.e.m. $T \in \Delta_{\pi_0}$, there are infinitely many $n \in \mathbb{N}$ such that $(\beta, \alpha; n)$ is a reduced triple for T, and a solution to inequality 4.2.*

This statement is about shrinking target properties for *singularities* of an i.e.m.: we already know from the Keane's Theorem 2.2.8 that an i.e.m. without connections is minimal; therefore the orbit of each bottom singularity approaches each of the top ones arbitrarily, even if the two points have to stay distinct. Here we get an answer to whether the orbit under T of a singularity for T^{-1} is able to pass nearer and nearer a selected singularity of T *at a selected 'speed'* — in other words, whether there is a sequence of integers that 'approximates' a connection between the two selected singularities as well as desired.

Remark 4.1.4. A restatement of the divergent case is that, for sequences ϕ as above, for almost any T, and for any (β, α), we have

$$\liminf_{n\to+\infty} \frac{|T^n u^b_\beta - u^t_\alpha|}{\phi(n)} \leq 1.$$

But, for any $\varepsilon \geq 0$, the sequence $\varepsilon\phi$ satisfies the same hypotheses of ϕ, so that inferior limit is $\leq \varepsilon$; that is, it is zero. ◇

In order to give a (partial) proof of this theorem, we will need to work in a more 'probabilistic' setting, so we will concentrate on normalised i.e.m.s.

Lemma 4.1.5. *Suppose a statement analogous to Theorem 4.1.3 is true, but replacing Δ_{π_0} with $\Delta^{(1)}_{\pi_0}$. Then Theorem 4.1.3 holds.*

Proof. For any i.e.m. $T = (\pi_0, \lambda) \in \Delta_{\pi_0}$ and any $\kappa \in \mathbb{R}_+$, set $T_\kappa := (\pi_0, \kappa\lambda)$; and set $\phi_\kappa(n) := \kappa\phi(n)$ for all $n \in \mathbb{N}$. It is easily checked that $(\beta, \alpha; n)$ is a solution to inequality 4.2 for the i.e.m. T if and only if it is a solution to inequality

$$|I_\kappa(\beta, \alpha; n)| < \phi_\kappa(n).$$

related to the i.e.m. T_κ. Moreover, obviously $\sum_{n\in\mathbb{N}} \phi(n) < +\infty$ if and only if $\sum_{n\in\mathbb{N}} \phi_\kappa(n) < +\infty$; and ϕ [resp. $(n\phi(n))$] is decreasing if and only if ϕ_κ [resp. $(n\phi_\kappa(n))$] is decreasing.

The map $K : \Delta^{(1)}_{\pi_0} \times \mathbb{R}_+ \to \Delta_{\pi_0}$ given by $(\lambda, \kappa) \mapsto \kappa\lambda$ is bijective and a subset of $\Delta^{(1)}_{\pi_0} \times \mathbb{R}$ has measure zero, or full measure (i.e. its complement has measure zero), if and only if the same holds for its image under K.

Let us suppose that ϕ is a decreasing sequence with finite sum. Then, according to what we are supposing, the sets $A_\kappa \subseteq \Delta^{(1)}_{\pi_0}$ of the normalised i.e.m.s T which have only finitely many solutions to

$$|I(\beta, \alpha; n)| < \phi_{1/\kappa}(n)$$

are full measure subsets; so, because of Fubini's theorem, $\bigsqcup_{\kappa\in\mathbb{R}_+} A_\kappa \times \{\kappa\}$ is subset of $\Delta^{(1)}_{\pi_0} \times \mathbb{R}_+$ with full measure, and also its image under K in Δ_{π_0} has full measure. But the latter is made up exactly of those i.e.m.s T which have finitely many solutions to 4.2.

In the other case, when $(n\phi(n))$ is decreasing and ϕ as infinite sum, we can fix any triple (β, α) and argue similarly as before. □

§ 4.1.D Measures on simplices Let us specify some matters and notation related to measures. First of all, we will put on every simplex $\Delta^{(1)}_\pi$ the standard Lebesgue measure (even if it is *not* the a.c.i.p., see paragraph 3.2.C), normalised so that its total mass is 1. So, we will often use the notation $\mathbf{P}$ for that measure — without any explicit reference to π, as it is irrelevant. If $\gamma \in \Pi_\pi(\mathcal{D})$ is a Rauzy path, we will also denote $\mathbf{p}(\gamma) := \mathbf{P}(\Delta^{(1)}_\gamma)$; and, if $\Gamma \subseteq \Pi_\pi(\mathcal{D})$, also $\mathbf{p}(\Gamma) := \mathbf{P}\left(\bigcup_{\gamma\in\Gamma} \Delta^{(1)}_\gamma\right)$. As a consequence of the formula for vertices seen in paragraph 3.1.C, we have

$$\mathbf{p}(\gamma) = \left(\prod_{x \in \mathcal{A}} c_x^\gamma\right)^{-1}. \tag{4.3}$$

If $\nu \in \Pi(\mathcal{D})$ is a path with r arrows, which starts at π and ends at π', we will also use the following notation: given a measurable $E \subseteq \Delta^{(1)}_{\pi'}$, call $\tilde{E} = \mathcal{R}_\Delta^{-r}(E) \cap \Delta^{(1)}_\nu = {}^T B_\nu(\text{cone over } E \text{ in } \mathbb{R}_+^{\mathcal{A}}) \cap \Delta^{(1)}_\pi \subseteq \Delta^{(1)}_\nu$. We set

$$\mathbf{P}_\nu(E) \coloneqq \mathbf{P}(\tilde{E})/\mathbf{P}(\Delta^{(1)}_\nu). \tag{4.4}$$

Similarly we can define $\mathbf{p}_\nu$ for (families of) paths starting at π'. In particular if $\gamma \in \Pi_{\pi'}(\mathcal{D})$ then

$$\mathbf{p}_\nu(\gamma) = \frac{\mathbf{p}(\nu\gamma)}{\mathbf{p}(\nu)} = \prod_{x \in \mathcal{A}} \frac{c_x^\nu}{c_x^{\nu\gamma}}. \tag{4.5}$$

4.2 Convergent case of Marchese's theorem

We fix an admissible marked permutation $\pi_0 \in \mathcal{C}$, and a decreasing sequence ϕ such that $\sum_n \phi(n) < +\infty$. This section is devoted to prove that, for almost any i.e.m.s $T \in \Delta^{(1)}_{\pi_0}$, there are only finitely many solutions to inequality 4.2.

§ 4.2.A Minimal detecting paths Let us fix a triple $(\beta, \alpha; n)$. We define $\Delta_{\pi_0}(\beta, \alpha; n)$ as the set of the i.e.m. $T \in \Delta^{(1)}_{\pi_0}$ which satisfy the Keane's property and such that $(\beta, \alpha; n)$ is a reduced triple for T.

We also define $\Gamma(\beta, \alpha; n) \subseteq \Pi_{\pi_0}(\mathcal{D})$ as the set of the *minimal* paths (with respect to the ordering $\prec$) which detect $(\beta, \alpha; n)$ according to all the possible choices of $\lambda \in \Delta_{\pi_0}(\beta, \alpha; n)$; so, $\Gamma(\beta, \alpha; n)$ is a disjoint family. For $\gamma \in \Gamma(\beta, \alpha; n)$, we set $\Delta^*_\gamma = \Delta_\gamma \cap \Delta_{\pi_0}(\beta, \alpha; n)$.

Let us summarise some easy-to-prove property of Rauzy paths $\gamma \in \Gamma(\beta, \alpha; n)$ that will be useful afterwards. Let r be the length of γ.

The last arrow of γ is either a top arrow with loser β, or a bottom arrow with loser α.

As γ is a minimal path that produces $(\beta, \alpha; n)$, we cannot have both $u_\beta^{(r-1),b} = u_\beta^{(r),b}$ and $u_\alpha^{(r-1),t} = u_\alpha^{(r),t}$; so the property above must be true. An immediate corollary is that:

Recalling Remark 2.1.7, let $l, h; l', h' \in \mathbb{N}$ be such that

$$u_\beta^{(r),b} = T^l u_\beta^b; \qquad u_\alpha^{(r),t} = T^{-h} u_\alpha^t; \qquad u_\beta^{(r-1),b} = T^{l'} u_\beta^b; \qquad u_\alpha^{(r-1),t} = T^{-h'} u_\alpha^t.$$

Let w be the winner of the last arrow of γ. Then either $l = l' + c_w^\gamma$, $h = h'$; or $l = l'$, $h = h' + c_w^\gamma$.

Indeed, if $u_\beta^{(r-1),b} \neq u_\beta^{(r),b}$ then $u_\beta^{(r),b} = T^{(r-1)} u_\beta^{(r-1),b} = T^{c_w^\gamma}(T^{l'} u_\beta^b)$ according to Proposition 3.1.1; and similarly if $u_\alpha^{(r-1),t} \neq u_\alpha^{(r),t}$.

Define l, h as above. Then $l + h = n$.

This would be true if the construction performed in the proof of Proposition 3.4.3 yielded the minimal detecting path; we call l_0 and h_0 the exponents it gives. In any case, the minimal detecting path is an initial subpath of the one found with this construction. This means that the associated exponents are some $l \leq l_0$ and $h \leq h_0$. In particular $l \leq n$, so $u_\beta^{(r),b} - u_\alpha^{(r),t} = T^n u_\beta^b - u_\alpha^t = T^l u_\beta^b - T^{l-n} u_\alpha^t = u_\beta^{(r),b} - T^{l-n} u_\alpha^t$; therefore $u_\alpha^{(r),t} = T^{l-n} u_\alpha^t$ and $l - n = -h$.

No singularities for $(T^{(r)})^{\pm 1}$ lie between $u_\alpha^{(r),t}$ and $u_\beta^{(r),b}$.

We have just seen that the interval between $u_\alpha^{(r),t}$ and $u_\beta^{(r),b}$ is exactly $T^{-h}I(\beta,\alpha;n)$. A singularity for $(T^{(r)})^{\pm 1}$ in this interval would either be the image under T^{-h} of a singularity for T lying in $I(\beta,\alpha;n)$; or the image under T^l of a singularity for T^{-1} lying in $T^{-n}I(\beta,\alpha;n)$. Both situations are impossible.

It holds $\max\{c_\alpha^\gamma, c_\beta^\gamma\} > n/2$*, so in particular* $\max c^\gamma > n/2$ *(this notation meaning the maximum entry of a vector).*

Since $l + h = n$, one between l and h must be $\geq n/2$; and according to Remark 3.1.2, in the first case $c_\beta^\gamma > n/2$; in the second one $c_\alpha^\gamma > n/2$.

Let w *be the winner of the last arrow of* γ*. Then* $c_w^\gamma \leq n$*.*

We have seen that $l + h = n$; moreover, above is stated that either $c_w^\gamma = h - h'$ or $c_w^\gamma = l - l'$. So, in both cases it is $\leq n$.

§ 4.2.B Reducing to an estimate on measures of simplices Now the proof of our statement may begin.

Step 1 – We only consider reduced triples: Let us fix an i.e.m. T: the finiteness of the number of triples for T which solve inequality 4.2 is equivalent to the finiteness of the number of *reduced* triples which do so.

Indeed, given a non-reduced triple $(\beta,\alpha;n)$ which is solutions, there are two cases:

- if $I(\beta,\alpha;n)$ contains one or more singularities for T, let $u_{\hat\alpha}^t$ be the closest to $T^n u_\beta^b$ and consider the new triple $(\hat\beta,\hat\alpha;\hat n) = (\beta,\hat\alpha;n)$;
- otherwise, let $0 \leq j \leq n$ be the least such that $T^{-j}(I(\beta,\alpha;n))$ contains one or more singularities for T^{-1} (being j the least possible, $T^{-j}(I(\beta,\alpha;n))$ is still an interval); among those singularities let $u_{\hat\beta}^b$ be the one which is closest to the endpoint $T^{n-j}u_\alpha^t$, and consider the triple $(\hat\beta,\hat\alpha;\hat n) = (\hat\beta,\alpha;n-j)$.

In both cases, $(\hat\beta,\hat\alpha;\hat n)$ is a triple such that

$$|I(\hat\beta,\hat\alpha;\hat n)| < |I(\beta,\alpha;n)| < \phi(n) \leq \phi(\hat n)$$

and reiterating this step up to n times, we obtain a reduced triple (β',α',n') that is still solution to inequality 4.2. If we had a sequence of distinct triples $(\beta_k,\alpha_k;n_k)_{k\in\mathbb{N}}$ which solve inequality 4.2, then the lengths of the corresponding intervals would go to zero; and the same would hold for the intervals corresponding to the reduced triples $(\beta_k',\alpha_k';n_k')$: this means that we would have infinitely many reduced solutions to 4.2.

Step 2 – Application of Borel-Cantelli: So, it suffices to prove that, for each fixed couple (β,α), and for almost any $T \in \Delta_{\pi_0}^{(1)}$, there are only finitely many $n \in \mathbb{N}$ such that $(\beta,\alpha;n)$ is a reduced triple which satisfies 4.2. And, according to the convergent case of the Borel-Cantelli Theorem (4.1.1), it is enough to show that the sets

$$A_n := \left\{\lambda \in \Delta_{\pi_0}^{(1)} \,\middle|\, (\beta,\alpha;n) \text{ is reduced for } T = (\pi_0,\lambda), \text{ and } |I(\beta,\alpha;n)| < \phi(n)\right\},$$

satisfy $\sum_{n=0}^{\infty} \mathbf{P}(A_n) < +\infty$. We immediately replace each A_n with its subset made up by the i.e.m.s satisfying the Keane's property, which has the same measure.

Since we know that reduced triples are detected, we divide A_n according to the minimal path which detects $(\beta,\alpha;n)$: $A_n = \bigsqcup_{\gamma\in\Gamma(\beta,\alpha;n)} \Delta_\gamma^{(1)} \cap A_n$. We would like to bound the measures of the sets in this union from above.

Let us fix a path $\gamma \in \Gamma(\beta,\alpha;n)$, with r its length and π its ending point. We set $\sigma_{\beta\alpha} := \sum_{\pi_b(x)<\pi_b(\beta)} e_x - \sum_{\pi_t(x)<\pi_t(\alpha)} e_x$ (the e_x's being the standard basis of $\mathbb{R}^{\mathcal{A}}$) and observe that, if $T \in \Delta_\gamma^*$, then

$$|I(\beta,\alpha;n)| = |u_\beta^{(r),b} - u_\alpha^{(r),t}| = |\langle \lambda^{(r)}, \sigma_{\beta\alpha}\rangle| = |\langle \lambda, B_\gamma^{-1}\sigma_{\beta\alpha}\rangle|.$$

Step 3 – A combinatorial property of π: We claim that two letters $\xi_+, \xi_- \in \mathcal{A}$ exist such that $\langle e_{\xi_\pm}, \sigma_{\beta\alpha}\rangle = \pm 1$. Indeed, we saw above that γ ends either with a top arrow with loser β, or a bottom one with loser α. We only consider the first case, the other one being similar. In this case we can set $\xi_+ = w(\gamma)$ the winner of the last arrow of γ: it will be at the rightmost position in π_t, and just before β in π_b.

The absence of a letter ξ_- as required would mean that, for each $\xi \in \mathcal{A}$ such that $\pi_t(\xi) < \pi_t(\alpha)$ one also has $\pi_b(\xi) < \pi_b(\beta)$. In particular $u_\alpha^{(r),t} < u_\beta^{(r),b}$. The position of w implies $u_\alpha^{(r),t} > u_w^{(r),b}$, because no singularity lies between $u_\alpha^{(r),t}$ and $u_\beta^{(r),b}$. And, since $\pi_t(w) = d > \pi_t(\alpha)$, then for each $\xi \in \mathcal{A}$ such that $\pi_t(\xi) < \pi_t(\alpha)$, we have $\pi_b(\xi) < \pi_b(w)$. But this contradicts $u_w^{(r),b} < u_\alpha^{(r),t}$: our claim is proved.

Step 4 – The sum is finite: If e_ξ is any vertex of $\Delta_\pi^{(1)}$, then one among e_ξ, $(e_\xi + e_{\xi_+})/2$, $(e_\xi + e_{\xi_-})/2$ belongs to $\Delta_\pi^{(1)} \cap \sigma_{\beta\alpha}^\perp$. So, $\Delta_\pi^{(1)} \cap \sigma_{\beta\alpha}^\perp$ is a $(d-2)$-simplex, and it is not contained in $\partial\Delta_\pi^{(1)}$.

Now, for $\lambda \in \Delta_\gamma^{(1)}$, we have $\langle \lambda, B_\gamma^{-1}\sigma_{\beta\alpha}\rangle = \langle {}^T B_\gamma^{-1}\lambda, \sigma_{\beta\alpha}\rangle$, thus $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1}\sigma_{\beta\alpha})^\perp$ also has dimension $d-2$, and is not contained in $\partial\Delta_\gamma^{(1)}$. Let us consider the vertex $v_w = (c_w^\gamma)^{-1}\,{}^T B_\gamma e_w$ of $\Delta_\gamma^{(1)}$: then $\langle v_w, B_\gamma^{-1}\sigma_{\beta\alpha}\rangle = \pm(c_w^\gamma)^{-1}$; in particular $v_w \notin (B_\gamma^{-1}\sigma_{\beta\alpha})^\perp$; we write $v_w = v'_w + v''_w$, where $v''_w \in \Delta_\gamma^{(1)} \cap (B_\gamma^{-1}\sigma_{\beta\alpha})^\perp$. This yields

$$\left\{T \in \Delta_\gamma^{(1)} \cap \Delta_\gamma^* \,\middle|\, |I(\beta,\alpha;n)| < \varepsilon\right\} = \left\{T \in \Delta_\gamma^{(1)} \cap \Delta_\gamma^* \,\middle|\, |\langle \lambda, B_\gamma^{-1}\sigma_{\beta\alpha}\rangle| < \varepsilon\right\} \subseteq$$
$$\Delta_\gamma^{(1)} \cap (B_\gamma^{-1}\sigma_{\beta\alpha})^\perp + (-\varepsilon c_w^\gamma, \varepsilon c_w^\gamma)v'_w.$$

Therefore the measure of the set on the left hand side is $< 2\varepsilon c_w^\gamma \mathbf{p}(\gamma)$ — up to a possible multiplicative constant, independent of ε and γ.

The key result to end the proof will be proven in the following paragraph:

Proposition 4.2.1. *A constant $C > 0$, only depending on $d = \#\mathcal{A}$, exists such that, for all positive integers N, the following inequality holds:*

$$\sum_{n=2^{N-1}}^{2^N-1} \sum_{\gamma\in\Gamma(\beta,\alpha;n)} c_{w(\gamma)}^\gamma \mathbf{p}(\gamma) \le C2^N.$$

According to this statement, we have

$$\sum_{n\in\mathbb{N}} \mathbf{P}(A_n) = \sum_{n\in\mathbb{N}} \sum_{\gamma\in\Gamma(\beta,\alpha;n)} \mathbf{P}(A_n \cap \Delta_\gamma^{(1)}) \le 2\sum_{n\in\mathbb{N}} \phi(n) \sum_{\gamma\in\Gamma(\beta,\alpha;n)} c_{w(\gamma)}^\gamma \mathbf{p}(\gamma) \text{ (because of the estimate}$$

$$\text{above)} \le \sum_{N>0} \phi(2^{N-1}) \sum_{n=2^{N-1}}^{2^N-1} \sum_{\gamma\in\Gamma(\beta,\alpha;n)} c_{w(\gamma)}^\gamma \mathbf{p}(\gamma) \le 2C \sum_{N>0} 2^{N-1}\phi(2^{N-1}) \text{ (because } c_{w(\gamma)}^\gamma \le n)$$

Since ϕ is a decreasing sequence, the fact that $\sum \phi(n) < +\infty$ is equivalent to the finiteness of the last summation above. □

§ 4.2.C Proof of the estimate

Step 1 – First grouping: Let's do some manipulations on the sum. First of all we restrict the inner sum only to those γ such that their last winning letter is a fixed one w; if the inequality is true after this modification, the original one, which is obtained by summing over w, is also true. Now we subdivide the set of the minimal detecting paths in

$$\Gamma_{n,k} := \{\gamma \in \Gamma(\beta,\alpha;n) |\, \gamma \text{ ends with an arrow whose winner is } w \text{, and } 2^k \le c_w^\gamma < 2^{k+1}\}.$$

As a consequence of the last property mentioned in paragraph 4.2.A is that $\Gamma_{n,k} = \emptyset$ for $k > \log n$ (where log is the logarithm in base 2). So,

$$\sum_{n=2^{N-1}}^{2^N-1} \sum_{\substack{\gamma\in\Gamma(\beta,\alpha;n) \\ \text{with last winner } w}} c_w^\gamma \mathbf{p}(\gamma) \leq \sum_{n=2^{N-1}}^{2^N-1} \sum_{0\leq k\leq \log n} 2^{k+1}\mathbf{p}(\Gamma_{n,k}) = \sum_{k=0}^{N-1} \sum_{n=2^{N-1}}^{2^N-1} 2^{k+1}\mathbf{p}(\Gamma_{n,k})$$

(indeed the inner sum should be performed for $\max\{2^k, 2^{N-1}\} \leq n < 2^N$, and the lower bound equals 2^{N-1}).

Step 2 – Second grouping: We now group together some of the sets $\Gamma_{n,k}$ varying the index n rather than k: we set $G_{k,i} := \bigsqcup_{n=2^{N-1}+i2^k}^{2^{N-1}+(i+1)2^k-1} \Gamma_{n,k}$ (for the i's such that it makes sense: $0 \leq i < 2^{N-k-1}$). To go on with the proof we need these new families of paths to be disjoint, in the sense of paragraph 2.1.D. That is, we need to show that for $\gamma_0 \in \Gamma_{n_0,k}$ and $\gamma_1 \in \Gamma_{n_1,k}$, with $2^{N-1} + i2^k \leq n_0 < n_1 < 2^{N-1} + (i+1)2^k$, we have $\gamma_0 \not\prec \gamma_1$. Indeed we cannot have $\gamma_1 \prec \gamma_0$: we saw in paragraph 4.2.A that, if r_0, r_1 are the lengths of the two paths, then $u_\beta^{(r_j),b} = T^{l_j}$ and $u_\alpha^{(r_j),t} = T^{-h_j}$ with $l_j + h_j = n_j$ for $j = 0, 1$; and clearly $\gamma_1 \prec \gamma_0 \Rightarrow l_1 \leq l_0, h_1 \leq h_0 \Rightarrow n_1 \leq n_0$.

But if $\gamma_0 \prec \gamma_1$, we use again the results in paragraph 4.2.A: first of all, we have either $l_1 = l_1' + c_w^{\gamma_1}$, $h_1 = h_1'$, or $l_1 = l_1'$, $h_1 = h_1' + c_w^{\gamma_1}$ (where l_1' and h_1' are the exponents associated to «γ_1 deprived of its last arrow»). In both cases, $n_1 = l_1 + h_1 = l_1' + h_1' + c_w^{\gamma_1} \geq l_0 + h_0 + c_w^{\gamma_0} \geq 2^{N-1} + i2^k + 2^k$, a contradiction.

Thus, $G_{k,i}$ is a disjoint family of paths, therefore $\mathbf{p}(G_{k,i}) = \sum_{n=2^{N-1}+i2^k}^{2^{N-1}+(i+1)2^k-1} \mathbf{p}(\Gamma_{n,k})$ and we can rewrite the last double summation above:

$$\sum_{k=0}^{N-1} \sum_{n=2^{N-1}}^{2^N-1} 2^{k+1}\mathbf{p}(\Gamma_{n,k}) = \sum_{k=0}^{N-1} \sum_{i=0}^{2^{N-k-1}-1} 2^{k+1}\mathbf{p}(G_{k,i}).$$

Step 3 – Effective estimate: To complete the proof, we will lean on an estimate by Avila, Gouëzel and Yoccoz ([AGY06]):

Theorem 4.2.2. *There exist two constants $C, \theta > 0$, only depending on $d = \#\mathcal{A}$, with the following property. Let $\mathcal{A}' \subset \mathcal{A}$ be a nonempty proper subset, $0 \leq m \leq M$ be integers, $\nu \in \pi(\mathcal{D})$ be a path starting from π_0 and ending in π_1. Then the following inequality holds:*

$$\mathbf{p}_\nu \left\{ \gamma \in \Pi_{\pi_1}(\mathcal{D}) \,\middle|\, \max B_\gamma c^\nu > 2^M \max c^\nu;\ \max_{\mathcal{A}'} B_\gamma c^\nu < 2^{M-m} \max c^\nu \right\} \leq C\frac{(m+1)^\theta}{2^m}$$

where we denote $\max_{\mathcal{A}'} c = \max_{\xi\in\mathcal{A}'} c_\xi$ *and* $\max c = \max_{\xi\in\mathcal{A}} c_\xi$.

In our case, we take $\mathcal{A}' = \{w\}$ and ν the null path, so that c^ν is the vector with all entries equal to 1, and $B_\gamma c^\nu = c^\gamma$. For any $\gamma \in \Gamma_{n,k}$ (where $2^{N-1} \leq n < 2^N$ is variable, whereas we suppose k fixed) we have (according to paragraph 4.2.A) $\max c^\gamma > n/2 \geq 2^{N-2}$, while $c_w^\gamma < 2^{k+1}$. Therefore, if we take $M = N - 2$ and $m = N - k - 3$, then for all indices i considered above, $G_{k,i}$ is a family of paths contained in the one considered in the estimate by Avila, Gouëzel and Yoccoz: this means that

$$\mathbf{p}(G_{k,i}) \leq C\frac{(N-k-3)^\theta}{2^{N-k-3}}$$

or, at least, this is true when $k \leq N - 3$; but no problems arise if we trivially bound $\mathbf{p}(G_{N-2,i})$,

$\mathbf{p}(G_{N-1,i}) \leq 1$. We can conclude our estimate:

$$\sum_{k=0}^{N-1} \sum_{i=0}^{2^{N-k-1}-1} 2^{k+1}\mathbf{p}(G_{k,i}) \leq C \sum_{k=0}^{N-3} 2^{k+1} \sum_{i=0}^{2^{N-k-1}-1} \frac{(N-k-3)^\theta}{2^{N-k-3}} + 2^{N-1} + 2^N =$$

$$= C2^N \sum_{k=0}^{N-3} \frac{(N-k-3)^\theta}{2^{N-k-3}} + 2^{N-1} + 2^N \leq (CK+2)2^N$$

where $K = \sum_{m\geq 0} m^\theta/2^m < +\infty$ only depends on θ, which in turn only depends on d. □

4.3 Divergent case of Marchese's theorem

Our proof of the divergent case of Theorem 4.1.3 will be only partial: we will restate it as a new shrinking target property, and we will not show that «the targets do not shrink too quickly». We fix an admissible marked permutation $\pi_0 \in \mathcal{C}$; and a positive sequence $\phi = (\phi(n))_{n\in\mathbb{N}}$ such that $(n\phi(n))_{n\in\mathbb{N}}$ is decreasing and $\sum \phi(n) = +\infty$. Our claim is: for each couple of letters $\alpha, \beta \in \mathcal{A}$ letters with $\alpha \neq t_\mathcal{C}$, $\beta \neq b_\mathcal{C}$, and for almost any i.e.m. $T \in \Delta^{(1)}_{\pi_0}$, there exist infinitely many $n \in \mathbb{N}$ such that $(\beta, \alpha; n)$ is a reduced triple for T such that $|I(\beta, \alpha; n)| = |T^n u^b_\beta - u^t_\alpha| < \phi(n)$.

§ 4.3.A Neat, positive, and reference paths

The proof of the statement above is based on the production of reduced triples, with particular requests. We begin by requiring something more than before to the paths $\eta(\beta, \alpha)$ defined in paragraph 3.4.C.

Definition 4.3.1. A path $\gamma \in \Gamma(\mathcal{D})$ is called *neat* if none of its initial sub-paths equals its ending, i.e. each time we have an equality $\gamma = \gamma_1\gamma_2 = \gamma_3\gamma_1$, either γ_1 is trivial or γ_2 and γ_3 are. The path γ is called *positive* if the entries of the matrix B_γ all are positive integers.

A path γ is neat if and only if $\Delta^{(1)}_\gamma \cap \mathcal{R}^j_\Delta \Delta^{(1)}_\gamma = \varnothing$ for each $0 < j <$ length of γ. Indeed, a nonempty intersection for some j exactly means that, when we eliminate from γ its first j arrows, we obtain again the beginning of γ.

Definition 4.3.2. Let (β, α) be a couple with $\alpha \neq t_\mathcal{C}$ and $\beta \neq b_\mathcal{C}$. A path $\eta = \eta(\beta, \alpha)$, whose ending part is as described in paragraph 3.4.C, is called a *reference path* for the couple if it satisfies the further requests below:
- its starting point is π_0;
- it is neat and positive;
- in case (β, α) is of type A, it contains at least 2 arrows with winner α;
- in case (β, α) is of type B, it contains at least d arrows with winner v.

A reference path always exists: the conditions in paragraph 3.4.C only specify its ending part, and the other ones are satisfied if we choose its initial arrows (even a large number of them) in a careful way. Then we connect the head with the tail in any way. We now fix a couple (β, α) and a reference path $\eta = \eta(\beta, \alpha)$; and we call π_1 its ending point.

According to Theorem 3.2.4, the map $\mathcal{R}^*_\Delta : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ is ergodic with respect to an a.c.i.p.. Since $\Delta^{(1)}_\eta$ and $\Delta^{(1)}_{\pi_1}$ are two subsets of $\Delta(\mathcal{D})$ with positive measure (equivalently with respect to the measure specified in paragraph 4.1.D, or to the a.c.i.p.), almost every $T \in \Delta^{(1)}_{\pi_1}$ has an orbit under $\mathcal{R}^*_\Delta$ that eventually enters $\Delta^{(1)}_\eta$; so, this must also be true if we consider the orbits under $\mathcal{R}_\Delta$. So, we define the *first entering map* $E : \Delta^{(1)}_{\pi_1} \to \Delta^{(1)}_\eta$ which associates, to almost every λ, the first point of $\Delta^{(1)}_\eta$ that appears in its $\mathcal{R}_\Delta$-orbit. If ℓ is the length of η, we also have a bijection

$\mathcal{R}^{\ell}_{\Delta} : \Delta^{(1)}_{\eta} \to \Delta^{(1)}_{\pi_1}$. So, the map

$$\mathcal{F}_{\eta} := \mathcal{R}^{\ell}_{\Delta} \circ E : \Delta^{(1)}_{\pi_1} \to \Delta^{(1)}_{\pi_1}$$

is defined for almost every point. If we denote

$$MP_{\eta} := \{\gamma \in \Pi(\mathcal{D}) | \, \gamma \text{ starts in } \pi_1, \text{ ends with } \eta, \text{ and contains no other copy of } \eta\},$$

then the connected components of the domain of $\mathcal{F}_{\eta}$ are exactly the $\Delta^{(1)}_{\gamma}$, varying $\gamma \in MP_{\eta}$. Similarly, if MP^k_{η} is the set of the paths in $\mathcal{D}$ which are concatenation of k paths of the family MP_{η}, then the connected components of the domain of $\mathcal{F}^k_{\eta}$ are the $\Delta^{(1)}_{\gamma}$ varying $\gamma \in MP^k_{\eta}$.

§ 4.3.B A stronger statement The proof of the divergent case of the Marchese's theorem is based on a stronger result. To state it we need:

Lemma 4.3.3. *Let $T \in \Delta^{(1)}_{\pi_0}$ be an i.e.m. which satisfies the Keane's property, and let r_k be the (increasing) sequence of instants such that the Rauzy path $\gamma_T(r_k)$ ends with η. Then a constant $\theta > 1$ exists (independent of T) such that $\|c^{\gamma_T(r_k)}\| < \theta^k$ for all k big enough.*

Proof. If we take a generic T, we can suppose that the Rauzy algorithm on it can be iterated infinitely many times, with infinitely many returns to $\Delta^{(1)}_{\eta}$. For each r_k, we call $r'_k \geq r_k$ the integer such that the arrows of γ_T from the r_k-th to the r'_k-th are all of the same type, while the $(r'_k + 1)$-th is of the opposite type. That is, $\gamma_T(r'_k)$ is the minimal path containing $\gamma_T(r_k)$ which can be obtained by Zorich-accelerating the Rauzy algorithm on T. Observe that $\gamma_T(r'_k)$ still contains exactly k copies of η, because the arrows composing the latter do not all have the same winner.

There is an unique decomposition $\gamma_T = \tilde{\gamma}_1 \cdots \tilde{\gamma}_j \cdots$ where the $\tilde{\gamma}_j$'s are maximal concatenations of arrows of the same type; let us suppose that $\gamma_T(r_k)$ is the concatenation of the first $N = N(k)$ ones. Then

$$\|c^{(r_k)}\| \leq \|c^{(r'_k)}\| \leq \|B_{\tilde{\gamma}_N} \cdots B_{\tilde{\gamma}_1}\| \cdot \|\vec{1}\|$$

and if ν is the biggest Lyapounov exponent of the discrete Kontsevich-Zorich cocycle (see paragraph 3.3.C), then for $N = N(k)$ sufficiently big a number $\varepsilon > 0$ exists such that $\|B_{\tilde{\gamma}_N} \cdots B_{\tilde{\gamma}_1}\| \leq e^{(\nu+\varepsilon)N}$.

Now, if $N(k)$ grows at most linearly in k, we are done. Let χ be the characteristic function of $\Delta^{(1)}_{\eta}$, and $\tilde{S}_N$ denote the N-th Birkhoff sum for the iteration of $\mathcal{R}^*_{\Delta}$; because of Birkhoff ergodic Theorem 0.2.6, for N big enough and for a generic T a (small) $\delta > 0$ exists (independent of T) such that $\tilde{S}_N \chi(T)/N \geq \mu(\Delta^{(1)}_{\eta}) - \delta$. On the other hand, as η is a neat path, we know that $\sum_{j=0}^{r'_k - 1} \chi(\mathcal{R}^j_{\Delta} T) = k$, so $\tilde{S}_N \chi(T) \leq k$ and $N \leq k / \left(\mu(\Delta^{(1)}_{\eta}) - \delta\right)$. □

The divergent case of Marchese's theorem is a consequence of the following «shrinking target style» statement. We extend our sequence ϕ to a function $\phi : [1, +\infty) \to (0, +\infty)$ such that $t\phi(t)$ is decreasing.

Proposition 4.3.4. *Let $M := \|B_{\eta}\|$, θ be as above, and for each $k \in \mathbb{N}$ set*

$$\psi_k := \frac{\theta^k \phi(\theta^k)}{dM}.$$

Then, for almost every $T \in \Delta^{(1)}_{\pi_1}$, there exist infinitely many $k \in \mathbb{N}$ such that:

- $\mathcal{F}^k_{\eta} T \in \left\{\lambda \in \Delta^{(1)}_{\pi_1} | \lambda_{\alpha} < \psi_k\right\}$*, if (β, α) is of type A;*
- $\mathcal{F}^k_{\eta} T \in \left\{\lambda \in \Delta^{(1)}_{\pi_1} | \lambda_{v} < \min\{\psi_k, \lambda_q\}\right\}$*, if (β, α) is of type B.*

We prove that this proposition implies the original statement only for (β, α) of type A, the other case being totally similar.

Step 1 – Recalling the production of triples: Take $T_0 = (\pi_0, \lambda) \in \Delta^{(1)}_{\pi_0}$, and let r_j be the (increasing and, for a generic T_0, infinite) sequence of instants such that $\gamma_{T_0}(r_j)$ ends with η; then, according to Proposition 3.4.8, fro each j a nonnegative integer $n = n(j) < \|c^{(r_j)}\|$ exists such that $(\beta, \alpha; n)$ is a reduced triple for T_0, and $|T_0^n u^b_\beta - u^t_\alpha| = \lambda^{(r_j)}_\alpha$. But, since $T_1 := \mathcal{R}^{r_1}_\Delta T_0 \in \Delta^{(1)}_{\pi_1}$, it holds that $\mathcal{R}^{r_j}_\Delta T_0 = \mathcal{F}^{j-1}_\eta(T_1)$. So, according to the proposition above, for a generic T_1 (and a generic T_0), there exist infinitely many $j \in \mathbb{N}$ such that $(\beta, \alpha; n)$ is a reduced triple and $\mathcal{R}^{r_j}_\Delta T_0 \in \left\{\lambda \in \Delta^{(1)}_{\pi_1} | \lambda_\alpha < \psi_j\right\}$, therefore such that $|T_0^n u^b_\beta - u^t_\alpha| = \lambda^{(r_j)}_\alpha < \|\lambda^{(r_j)}\|\psi_j$.

When (β, α) is of type B, the same argument holds, except for some minor modification needed because the r_j's are defined so to take into account the further condition $\lambda_v < \lambda_q$.

Step 2 – Lengths shrink as claimed: Now we only have to check that the lengths of our intervals satisfy the required bounds. We start a chain of inequalities:

$$|T_0^n u^b_\beta - u^t_\alpha| < \|\lambda^{(r_j)}\| \frac{\theta^j \phi(\theta^j)}{dM} \leq \|\lambda^{(r_j)}\| \frac{\|c^{(r_j)}\|}{dM} \phi(\|c^{(r_j)}\|)$$

(the last one holds because of Lemma 4.3.3, and because $t\phi(t)$ is decreasing).

REMARK 4.3.5. Let γ be any path ending with η, so we write $\gamma = \gamma'\eta$. Since the entries of B_η are all positive integers, each component of c^γ is $\geq \sum_x c^{\gamma'}_x = \|c^{\gamma'}\|$; but, on the other hand, $\|c^\gamma\| \leq M\|c^{\gamma'}\|$ and the same upper bound holds for every component of c^γ. So, two components of c^γ always satisfy $c^\gamma_\xi \leq M\|c^{\gamma'}\| \leq Mc^\gamma_x$; and summing over ξ, $\|c^\gamma\|/(dM) \leq c^\gamma_x$. $\diamond$

Recalling that $n = n(j) < \|c^{(r_j)}\|$:

$$\|\lambda^{(r_j)}\| \frac{\|c^{(r_j)}\|}{dM} \phi(\|c^{(r_j)}\|) \leq \sum_x \lambda^{(r_j)}_x \frac{\|c^{(r_j)}\|}{dM} \phi(n) \leq \sum_x \lambda^{(r_j)}_x c^{(r_j)}_x \phi(n) =$$
$$= \langle \lambda^{(r_j)}, c^{(r_j)} \rangle \phi(n) = \langle {}^T B^{-1}_{\gamma_T(r_k)} \lambda, B_{\gamma_T(r_k)} \vec{1} \rangle \phi(n) = \langle \lambda, \vec{1} \rangle \phi(n) = 1 \cdot \phi(n)$$

and the implication is proved. □

§ 4.3.C A controlled shrinking is sufficient

Let us define, for each $\varepsilon > 0$, the set

$$E_\varepsilon := \begin{cases} \left\{\lambda \in \Delta^{(1)}_{\pi_1} | \lambda_\alpha < \varepsilon\right\} & \text{if } (\beta, \alpha) \text{ is of type A;} \\ \left\{\lambda \in \Delta^{(1)}_{\pi_1} | \lambda_v < \min\{\varepsilon, \lambda_q\}\right\} & \text{if } (\beta, \alpha) \text{ is of type B.} \end{cases}$$

By means of rather technical arguments (see [Mar11]), one can prove that

Proposition 4.3.6. *A constant $C > 0$, only depending on $d = \#\mathcal{A}$, exists such that for every $\varepsilon > 0$ it holds that*

$$\mathbf{P}(E_\varepsilon) > C\varepsilon.$$

It is in the proof of this result that the last request of Definition 4.3.2 becomes necessary. We now explain how the proposition above is needed in proving Proposition 4.3.4.

Step 1 – An explicit formula: If we denote $E_m := E_{\psi_m}$, the claim of Proposition 4.3.4 is that

$$\mathbf{P}\left(\limsup_{m \to +\infty} \mathcal{F}^{-m}_\eta(E_m)\right) = \mathbf{P}\left(\bigcap_{m \in \mathbb{N}} \bigcup_{j \geq m} \mathcal{F}^{-j}_\eta(E_j)\right) = 1$$

or, setting $N_k := \Delta^{(1)}_{\pi_1} \setminus E_k$, that for every $k \in \mathbb{N}$ we have

$$\mathbf{P}\left(\bigcap_{j \geq m} \mathcal{F}^{-j}_\eta(N_j)\right) = 0.$$

We begin by making estimates for a finite intersection $\bigcap_{j=m}^{n} \mathcal{F}_\eta^{-j}(N_j)$. An i.e.m. $T \in \Delta_{\pi_1}^{(1)}$ (which we assume to satisfy the Keane's property) appears in this intersection if and only if there exist $\gamma_m \in MP_\eta^m$, and $\gamma(j) \in MP_\eta$ for each $m < j \leq n$, such that, if we set $\gamma_j = \gamma_m \gamma(m+1) \cdots \gamma(j)$ and call r_j the length of γ_j, then

- $T \in \Delta_{\gamma_m}$;
- $\mathcal{R}_\Delta^{r_j} T \in N_j \cap \Delta_{\gamma(j+1)}$ for all $m \leq j < n$;
- $\mathcal{R}_\Delta^{r_n} T \in N_n$;

indeed this means exactly that

- γ_T begins with γ_n;
- «at the end of each sub-path», Rauzy iteration gives a $\mathcal{R}_\Delta^{r_j} T \in N_j$.

Therefore, the probability of this intersection is, recalling equation 4.4:

$$p(m,n) := \sum \mathbf{P}(\Delta_{\gamma_m}) \mathbf{P}_{\gamma_m}(N_m \cap \Delta_{\gamma(m+1)}^{(1)}) \cdots \mathbf{P}_{\gamma_{n-1}}(N_{n-1} \cap \Delta_{\gamma(n)}^{(1)}) \mathbf{P}_{\gamma_n}(N_n),$$

where the sum is meant over all the possible paths $\gamma_m, \gamma(m+1), \ldots, \gamma(n)$.

Step 2 – Measure distortions are bounded: Now, our aim is to prove that $p(m,n) \leq \prod_{j=m}^{n}(1 - C'\psi_j)$ (where C' is a suitable constant). We already know that $\mathbf{P}(N_j) < 1 - C\psi_j$, but now we need an estimate for the 'relative' probabilities $\mathbf{P}_{\gamma_j}$.

Let $\nu \in \Pi_{\pi_1}(\mathcal{D})$ be any path, and let γ be any path ending with η. According to the equality 4.5, we have $\mathbf{P}_\gamma(\Delta_\nu^{(1)}) = \left(\prod_\xi c_\xi^\gamma\right) \Big/ \left(\prod_\xi c_\xi^{\gamma\nu}\right)$. Now, recalling Remark 4.3.5, we have

$$c_\xi^{\gamma\nu} = \sum_x (B_\nu)_{\xi x} c_x^\gamma \leq \left(\sum_x (B_\nu)_{\xi x}\right) \max c^\gamma \leq c_\xi^\nu \cdot M c_\xi^\gamma,$$

thus $\mathbf{P}_\gamma(\Delta_\nu^{(1)}) \geq \prod_\xi (c_\xi^\nu / M) = M^{-d} \mathbf{P}(\Delta_\nu^{(1)})$. Since it can be seen that the simplices $\Delta_\nu^{(1)}$ are a basis for the Borel σ-algebra of $\Delta_{\pi_1}^{(1)}$, the same inequality holds for the targets E_ε: so, a corollary of Proposition 4.3.6 is that a constant $C' = M^{-d}C$, only depending on the matrix B_η, exists such that $\mathbf{P}_\gamma(N_j) < 1 - C'\psi_j$: in particular it does not depend on γ.

Step 3 – Upper bound for finite intersection: We can now prove the estimation stated at the beginning of the previous step: we proceed by induction on $n - m$. If $n - m = 0$ then $p(m,m) = \sum \mathbf{P}(\Delta_{\gamma_m}^{(1)}) \mathbf{P}_{\gamma_m}(N_m)$; but $\bigsqcup_{\gamma_m \in MP_\eta^m} \Delta_{\gamma_m}^{(1)} = \Delta_{\pi_1}^{(1)}$ (up to subsets of measure zero), so evidently $p(m,m) \leq 1 - C'\psi_m$.

Now suppose we have proven the desired inequality for $p(m, n-1)$; then

$$p(m,n) \leq \sum \mathbf{P}(\Delta_{\gamma_m}) \mathbf{P}_{\gamma_m}(N_m \cap \Delta_{\gamma(m+1)}^{(1)}) \cdots \mathbf{P}_{\gamma_{n-1}}(N_{n-1} \cap \Delta_{\gamma(n)}^{(1)})(1 - C'\psi_n)$$

and $\bigsqcup_{\gamma(n) \in MP_\eta}(N_{n-1} \cap \Delta_{\gamma(n)}^{(1)}) = N_{n-1}$ (up to subsets of measure zero), so the right hand side is

$$\leq (1 - C'\psi_n) \sum \mathbf{P}(\Delta_{\gamma_m}) \mathbf{P}_{\gamma_m}(N_m \cap \Delta_{\gamma(m+1)}^{(1)}) \cdots \mathbf{P}_{\gamma_{n-1}}(N_{n-1}) = (1 - C'\psi_n) p(m, n-1)$$

and we apply the inductive hypothesis.

Step 4 – Conclusion: To conclude, it is sufficient to note that ψ is a decreasing sequence with $\sum_{j=m}^{+\infty} \psi_j = +\infty$, and this is equivalent to $\prod_{j=m}^{+\infty}(1 - C'\psi_j) = 0$.

The sequence of the finite intersections $\left(\bigcap_{j=m}^{n} \mathcal{F}_\eta^{-j}(N_j)\right)_{n \geq m}$ is decreasing with respect to inclusion, so the limit of their probabilities, which is evidently 0, is the probability of the intersection $\bigcap_{j \geq m} \mathcal{F}_\eta^{-j}(N_j)$ (for all $m \in \mathbb{N}$). Proposition 4.3.4 is proved, and so is the divergent case of Marchese's theorem. □

4.4 A theorem of Chaika

§ 4.4.A Singularities vs. generic points The study of shrinking target properties for i.e.m.s also led to other kinds of results. An approach different than Marchese's one has been followed at the same time by Boshernitzan and Chaika [Cha11]:

Theorem 4.4.1 (Chaika). *Let $\phi = \left(\phi(n)\right)_{n\in\mathbb{N}}$ be a decreasing positive sequence with $\sum_{n\in\mathbb{N}} \phi(n) = +\infty$; and let π_0 be an admissible marked permutation. Then, for almost every i.e.m. $T \in \Delta_{\pi_0}$, and any $y \in I$ the interval where T acts, the set*

$$\limsup_{n\to+\infty} T^{-n}\left(y - \phi(n), y + \phi(n)\right)$$

is a subset of I with full Lebesgue measure.

This statement is a shrinking target property for generic points of I, whereas we noted that (the divergent case of) Theorem 4.1.3 gives a sort of shrinking target property for singularities: the two theorems appear to be complementary to each other. But it is easy to obtain a slightly weaker version of Chaika's theorem from Marchese's one:

Corollary 4.4.2. *Let $\phi = \left(\phi(n)\right)_{n\in\mathbb{N}}$ be a positive sequence such that $(n\phi(n))_{n\in\mathbb{N}}$ is decreasing, and $\sum_{n\in\mathbb{N}} \phi(n) = +\infty$; and let π_0 be an admissible marked permutation. Then, for almost every i.e.m. $T \in \Delta_{\pi_0}$, and almost every pair of points $x, y \in I$, there exist infinitely many $n \in \mathbb{N}$ such that*

$$|T^n x - y| < \phi(n).$$

Similarly, for almost every $x \in I$ there exist infinitely many $n \in \mathbb{N}$ such that

$$|T^n x - x| < \phi(n).$$

According to Remark 4.1.4, the first statement is equivalent to

$$\liminf_{n\to+\infty} \frac{|T^n x - y|}{\phi(n)} = 0$$

and similarly for the second one.

§ 4.4.B The proof The main idea to prove Corollary 4.4.2 is a *principle of virtual singularities*: that is, we will mark two points x and y of I as if they were singularities, and this will lead to an augmented parameter space; applying Marchese's theorem here we will get our claim.

Step 1 – The augmented space: We begin the proof of the first statement. We fix two letters $\beta, \alpha \in \mathcal{A}$, and would like to take $y \in I^b_\beta$, $x \in I^t_\alpha$. To be simple, we suppose for now $\alpha \neq \beta$. We denote

$$\Delta^{\times\times}_{\pi_0;\beta,\alpha} := \bigcup_{\lambda\in\Delta_{\pi_0}} \{\lambda\} \times I^b_\beta \times I^t_\alpha$$

where the intervals I^t_α and I^b_β are meant with respect to the length data λ. Now, if we consider x and $T^{-1}y$ as two additional singularities for T (and therefore Tx and y as two additional singularities for T^{-1}), we can regard it as an i.e.m. on $d+2$ sub-intervals, indexed by the alphabet $\mathcal{A}' = \mathcal{A} \sqcup \{\alpha', \beta'\}$.

The associated marked permutation will be an (admissible) π'_0 which is obtained from π_0 by replacing α with the couple $\alpha\,\alpha'$ and β with the couple $\beta\,\beta'$, both in its top row and in its bottom one. The length data λ' are obtained from λ by splitting the sub-interval I^t_β at the point $T^{-1}y$, and the sub-interval I^t_α at the point x. This leads to a measurable bijection $\Delta^{\times\times}_{\pi_0;\beta,\alpha} \to \Delta_{\pi'_0}$, which makes the subsets of measure zero on the two sides correspond.

If we have $\alpha = \beta$, some more care is required, as we have to distinguish the two cases $x < T^{-1}y$, $x > T^{-1}y$, i.e. we define two distinct spaces $\Delta^{\times\times}_{\pi_0;\alpha+}$ and $\Delta^{\times\times}_{\pi_0;\alpha-}$. In both cases we obtain a bijection as above with some π'_0 marked permutation on the same alphabet $\mathcal{A}' = \mathcal{A} \sqcup \{\alpha', \beta'\}$ as before (we name the new sub-intervals so that the left endpoint of $I^t_{\alpha'}$ is x and the left endpoint of $I^b_{\beta'}$ is y).

Step 2 – Application of Fubini's theorem: However π'_0 has been defined, it is an admissible permutation; therefore, according to the divergent case of Theorem 4.1.3, for almost any length data $\lambda' \in \Delta_{\pi'_0}$ there exist infinitely many $n \in \mathbb{N}$ such that

$$|T^n u^b_{\beta'} - u^t_{\alpha'}| < \phi(n)$$

but, by construction, $u^b_{\beta'} = y$ and $u^t_{\alpha'} = x$. Because of the bijection above described, the triples (λ, y, x) which satisfy our claim make up a full measure subset of $\Delta^{\times\times}_{\pi_0;\beta,\alpha}$ (or of $\Delta^{\times\times}_{\pi_0;\alpha\pm}$). Using Fubini's theorem we deduce that, for almost any λ, the claim must be true for almost any (x, y). The first statement of the Corollary follows by varying $\beta, \alpha \in \mathcal{A}$.

Step 3 – The second statement: The proof of the second statement goes similarly as the previous one. This time we only have to choose $\alpha \in \mathcal{A}$, and take $x \in I^t_\alpha$. The augmented space is

$$\Delta^{\times}_{\pi_0;\alpha} := \bigcup_{\lambda \in \Delta_{\pi_0}} \{\lambda\} \times I^t_\alpha,$$

and regarding x as a singularity of T we obtain a marked permutation π'_0 on the alphabet $\mathcal{A}' = \mathcal{A} \sqcup \{\alpha'\}$. Theorem 4.1.3 applied to the marked permutation π'_0, to the couple (α', α') and to the sequence $\tilde{\phi}(n) := \phi(n+1)$ yields that for almost any $T \in \Delta_{\pi'_0} \cong \Delta^{\times}_{\pi_0;\alpha}$ there exist infinitely many $n \in \mathbb{N}$ such that

$$|T^n u^b_{\alpha'} - u^t_{\alpha'}| < \tilde{\phi}(n), \text{ that is } |T^{n+1}x - x| < \phi(n+1)$$

and the statement follows again by application of the Fubini's theorem. □

5 Khinchin theorem and translation surfaces

Marchese's Theorem 4.1.3 for i.e.m.s was concerned with reduced triples. When we relate i.e.m.s with translation surfaces by means of the Veech construction, a saddle connection on the surface projects to the interval associated with a triple $(\beta, \alpha; n)$; and the length of this interval is exactly the real part of the *holonomy* (Definition 1.2.8) of the saddle connection, that is «the connection itself, seen as a vector in $\mathbb{C}$».

This correspondence suggests that the Diophantine condition whose solutions were the concern of Marchese's theorem can be transformed to fit the setting of translation surfaces. More precisely, this becomes a condition about holonomies: $\left|\Re\left(\mathrm{Hol}(\gamma)\right)\right| < \phi\left(|\mathrm{Hol}(\gamma)|\right)$. The solutions to this condition are finitely, or infinitely many, according to the same dichotomy seen in the previous chapter. The first two sections of this chapter follow [Mar10] and are dedicated to this result. The convergent case will be again an application of Borel-Cantelli, whereas the proof of the divergent case will require us to exhibit a saddle connection for every reduced triple.

The third section states a version of Chaika's theorem that appears as a shrinking target property for the vertical flow on translation surfaces. Following [Mar], we will use — similarly as in the previous chapter — a «principle of virtual singularities» to obtain a weaker version of Chaika's theorem from Marchese's result for saddle connections.

The last section, again inspired by [Mar10], passes to the upper level, namely uses the theorem for saddle connections to derive a new dichotomy, as well as a *logarithm law*, for the Teichmüller flow. One defines the *systole* of a translation surface as the minimum length of a saddle connection; the decrease of the systoles for a sequence of translation structures can be interpreted as a measure of how much is this sequence going far towards the infinity. Recurrence of the Teichmüller flow in a moduli stratum implies that it cyclically makes a wandering towards a cusp and then gets back, and the theorem uses systoles to quantify this phenomenon. Its proof is not difficult and involves, above all, investigation of the trends followed by holonomies under the Teichmüller flow.

5.1 A change in the language

In this section we introduce the most natural re-statement of Theorem 4.1.3 in the setting of translation surfaces; and prove its easy half. Before doing this, we have to spend some words about *frames*.

§ 5.1.A Framed translation surfaces In paragraph 2.3.D we defined marked translation surfaces and marked strata, which are a finite covering of 'standard' strata. We would like now to repeat exactly the same construction, except that we mark a separatrix for *each* point of Σ. So, a *framed translation structure* on a surface is specified by a translation structure and by the choice, for each point $p_j \in \Sigma$, of a horizontal separatrix H_j outgoing from p_j. We denote $\widehat{\mathrm{Flat}}(S, \Sigma, 2h, 1)$ the set of the framed translation structures on S with singularities in Σ and conical angles $2\pi h_j$ wide; and we set

$$\hat{\mathcal{T}}(S,\Sigma,h) := \widehat{\mathrm{Flat}}(S,\Sigma,2h,1)\big/_{\mathrm{Diff}_0(S,\Sigma)}, \quad \text{and}$$
$$\hat{\mathcal{H}}(S,\Sigma,h) := \widehat{\mathrm{Flat}}(S,\Sigma,2h,1)\big/_{\mathrm{Diff}^+(S,\Sigma)} = \hat{\mathcal{T}}(S,\Sigma,h)\big/_{\mathrm{Mod}(S,\Sigma)}.$$

We call them *framed strata* in Teichmüller and moduli space, respectively.

As usual, there are also the unit area versions $\hat{\mathcal{T}}^{(1)}(S,\Sigma,h)$ and $\hat{\mathcal{H}}^{(1)}(S,\Sigma,h)$. Similarly as the case of marked strata, the obvious maps from framed strata to standard ones are coverings of degree $h_1\cdots h_s$; so, a canonical measure and a volume form on each of these new spaces can be defined via pullback.

If $\hat{X}$ is (a representative of) an element of $\hat{\mathcal{H}}(S,\Sigma,h)$, we can evaluate slopes of segments at a point $p_j \in \Sigma$ from the selected separatrix H_j counterclockwise. We call *bundles* of saddle connections the sets $\mathcal{B}^{p_i,l}_{p_j,m}(\hat{X})$ (where $p_j, p_i \in \Sigma$; $1 \le m \le h_j$; $1 \le l \le h_i$) whose elements are the saddle connections on $\hat{X}$ which start at the point p_j with an angle $\varphi_0 \in [2\pi(m-1), 2\pi m)$, and end at the point p_i with an angle $\varphi_1 \in [2\pi(l-1), 2\pi l)$. We also denote

$$\mathrm{Hol}^{p_i,l}_{p_j,m}(\hat{X}) = \left\{\mathrm{Hol}(\gamma)\,\middle|\,\gamma \in \mathcal{B}^{p_i,l}_{p_j,m}(\mathcal{X})\right\}$$

where $\mathrm{Hol}(\gamma)$ is the holonomy of γ (Definition 1.2.8); and similarly we call $\mathrm{Hol}(X)$ the set of all holonomies of saddle connections, regardless to the bundle (this also makes sense when X is a [class of] non-framed surface). All these sets are at most countable.

Remark 5.1.1. The Teichmüller flow g^t on a framed stratum preserves each bundle, in the sense that for each $t \in \mathbb{R}$, and $\hat{X} \in \hat{\mathcal{H}}(S,\Sigma,h)$, one has $g^t\left(\mathcal{B}^{p_i,l}_{p_j,m}(\hat{X})\right) = \mathcal{B}^{p_i,l}_{p_j,m}(g^t\hat{X})$. Indeed horizontal segments on a translation surface do not change their direction under action of the Teichmüller flow; and any saddle connection either is an horizontal segment, or has both endpoints lying between two fixed horizontal separatrices: both properties are preserved by the flow. ◇

§ 5.1.B The theorem We are now ready to state Marchese's generalisation of Khinchin-like dichotomy for saddle connections:

Theorem 5.1.2 (Marchese). *Let S be a topological surface, with $S \supset \Sigma = \{p_1,\dots,p_s\}$ and $h \in \mathbb{N}^s_*$; moreover let $\phi : [0,+\infty) \to (0,+\infty)$ be a function. We consider the solutions to the inequality*

$$\left|\Re(z)\right| < \phi\left(|z|\right). \tag{5.1}$$

- *Suppose that ϕ is a decreasing function, and $\int_0^{+\infty}\phi(t)\mathrm{d}t < +\infty$. Then, for almost any $X \in \mathcal{H}(S,\Sigma,h)$, there are only finitely many $z \in \mathrm{Hol}(X)$ which are solutions to inequality 5.1.*
- *Suppose instead that $t\phi(t)$ is a decreasing function and $\int_0^{+\infty}\phi(t)\mathrm{d}t = +\infty$. Then, for almost any $\hat{X} \in \hat{\mathcal{H}}(S,\Sigma,h)$, and for any $p_j, p_i \in \Sigma$, $1 \le m \le h_j$, $1 \le l \le h_i$, there are infinitely many $z \in \mathrm{Hol}^{p_i,l}_{p_j,m}(\hat{X})$ which are solutions to inequality 5.1.*

The statement of this theorem totally looks alike Theorem 4.1.3; of course now ϕ has to be a function rather than a sequence. When we stated Marchese's theorem for i.e.m.s, we remarked that its statement concerned approximations of connections for i.e.m.s. But, when one obtains an i.e.m. as a return map of the vertical flow of a translation surface, connections for it come from vertical saddle connections for the translation surface: it is therefore natural that the theorem above mentioned concerns sequences of saddle connections between two fixed singularities, whose horizontal projections go to zero. In this correspondence, the choice of a bundle has the same role of the choice of (β,α) in the previous chapter (we will make it precise in paragraph 5.2.A).

Proof (of the convergent case). If $X \in \mathcal{H}(S, \Sigma, h)$ and $\vartheta \in \mathbb{T}$, we call $R_\vartheta \in SL(2, \mathbb{R})$ the rotation of angle ϑ; so, $R_\vartheta X \in \mathcal{H}(S, \Sigma, h)$ is the (class of) the translation structure obtained from X according to the action described in paragraph 1.3.A. We claim that for every fixed $X \in \mathcal{H}(S, \Sigma, h)$, for almost every $\vartheta \in \mathbb{T}$ (according to the Haar measure $\mathbf{P}$), the set $\mathrm{Hol}(R_\vartheta X) = e^{2\pi i \vartheta} \cdot \mathrm{Hol}(X)$ has only finitely many solution to inequality 5.1. The statement of the convergent case will follow from application of Fubini's theorem.

For every $z \in \mathrm{Hol}(X)$, we call $I(z) := \left\{\vartheta \in \mathbb{T} \,\middle|\, |\Re(e^{2\pi i \vartheta} z)| < \phi(|z|)\right\}$. If we are able to show that $\sum_{z \in \mathrm{Hol}(X)} \mathbf{P}\left(I(z)\right) < +\infty$, then the convergent case of the Borel-Cantelli Theorem (4.1.1) implies our claim.

We observe that $\Re(e^{2\pi i \vartheta} z)/|z| = \cos(\vartheta + \arg(z))$, so $I(z) = \left\{\vartheta \in \mathbb{T} \,\middle|\, |\cos(\vartheta + \arg(z))| < \phi(|z|)/|z|\right\}$. Since the function $\phi(|z|)$ is bounded for $|z| \to +\infty$, then for $|z|$ big enough a constant $\varepsilon > 0$ (independent of $|z|$) exist such that $\mathbf{P}(I(z)) < (2 + \varepsilon)\phi(|z|)/|z|$.

Now we need a result of Masur (see [Mas88]):

For any translation surface X, let $N_X(L)$ be the number of saddle connections on X whose length is at most L. Then two constants $0 < c_X < C_X$ exist such that $c_X L^2 \leq N_X(L) \leq C_X L^2$ for any L big enough.

We split the sum into two parts:

$$\sum_{z \in \mathrm{Hol}(X)} \mathbf{P}\left(I(z)\right) = \sum_{\substack{z \in \mathrm{Hol}(X) \\ |z| < 2^{k_0}}} \mathbf{P}\left(I(z)\right) + \sum_{k \geq k_0} \sum_{\substack{z \in \mathrm{Hol}(X) \\ 2^k \leq |z| < 2^{k+1}}} \mathbf{P}\left(I(z)\right).$$

The first sum is finite, and we choose k_0 sufficiently big for the following arguments to hold. For $2^k \leq |z| < 2^{k+1}$, we have $\mathbf{P}(I(z)) < (2 + \varepsilon)\phi(2^{k+1})/2^k$; according to Masur's result stated above, there are at most $C_X 2^{2(k+1)}$ such values of $z \in \mathrm{Hol}(X)$. So, the second sum is bounded by $C'_X \sum_k 2^{k+1}\phi(2^{k+1})$; and, ϕ being a decreasing function, finiteness of $\int_0^{+\infty} \phi(t)\mathrm{d}t$ is equivalent to finiteness of that sum. □

The next section is concerned with proving the divergent case using the divergent case of Theorem 4.1.3.

5.2 Paraphrasing the divergent case

§ **5.2.A Bundles in the Veech construction** Let X be a translation surface obtained with the Veech construction from an admissible marked permutation π on an alphabet $\mathcal{A}$ and belonging to a Rauzy class $\mathcal{C}$, length data $\lambda \in \Delta_\pi$ and suspension data $\tau \in \Theta_\pi$.

If $\Sigma = \{p_1, \ldots, p_s\}$ is the set of the singularities for X and h is the associated vector of indices of the singularities, then each singularity $p_j \in \Sigma$ possesses exactly h_j outgoing horizontal separatrices. So, according to Lemma 2.2.2, the total number of horizontal separatrices is $d - 1$.

Recall the notations we introduced in paragraphs 2.1.A and 2.2.A. We will use also another convention: we will see the vectors $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$, $\eta = (\eta)_{\alpha \in \mathcal{A}}$ and $\theta = (\theta)_{\alpha \in \mathcal{A}}$ as functions on the interval I that are constant on each sub-interval I^t_α, i.e. $x \in I^t_\alpha \Rightarrow \theta(x) := \theta_\alpha$ (and similar formulae). Therefore their Birkhoff sums under T will make sense as well.

If, for each $\alpha \in \mathcal{A} \setminus \{t_\mathcal{C}\}$, we call H^t_α the horizontal half-line in $\mathbb{C}$ which goes from ξ^t_α rightwards, we obtain $d - 1$ lines, each of which projects to an outgoing separatrix of the horizontal flow on X. No two of them can project to the same, since their initial segments are contained in the interiors of pairwise different top rectangles in the Veech construction, which are a fundamental domain for the projection.

Since the horizontal separatrices on X are as many as these lines, a natural correspondence between letters of $\mathcal{A} \setminus \{t_{\mathcal{C}}\}$ and horizontal separatrices of X is set up. In the same way we define the half-lines H^b_β for $\beta \in \mathcal{A} \setminus \{b_{\mathcal{C}}\}$, and we establish a correspondence between this set and the horizontal separatrices of X.

For any pair $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq t_{\mathcal{C}}$ and $\beta \neq b_{\mathcal{C}}$, we call $\mathcal{B}^\alpha_\beta(X)$ the set of the saddle connections that start at the singularity corresponding to ξ^b_β, making an angle $\varphi_\beta \in [0, 2\pi)$ with H^b_β *counter-clockwise*; and end at the singularity corresponding to ξ^t_α, making an angle $\varphi_\alpha \in [0, 2\pi)$ with H^t_α *clockwise*.

If we fix a frame for X, each of the bundles $\mathcal{B}^{p_i,l}_{p_j,m}(X)$ coincides with some of the $\mathcal{B}^\alpha_\beta(X)$, establishing a one-to-one correspondence.

§ 5.2.B Reduced triples correspond to saddle connections We now prove that reduced triples for the i.e.m. $T = (\pi, \lambda)$ are 'horizontal projections' of some saddle connection (at least, except for some particular cases).

Proposition 5.2.1. *Suppose that $T = (\pi, \lambda)$ satisfies the Keane's property; let X be the surface obtained from the Veech construction on T and some suspension data $\tau \in \Theta_\pi$, and let $\alpha \neq t_{\mathcal{C}}$, $\beta \neq b_{\mathcal{C}}$ be letters of $\mathcal{A}$. Then a number $n_0 = n_0(X) \in \mathbb{N}$ exists such that, for any $n \geq n_0$ such that the triple $(\beta, \alpha; n)$ is reduced, a saddle connection $\gamma \in \mathcal{B}^\alpha_\beta(X)$ exists such that*

$$\mathrm{Hol}(\gamma) = \xi^t_\alpha - \xi^b_\beta - S_n\theta(u^b_\beta);$$

in particular $\Re(\mathrm{Hol}(\gamma)) = u^t_\alpha - T^n u^b_\beta$.

Proof. We denote $z := \xi^t_\alpha - \xi^b_\beta - S_n\theta(u^b_\beta) \in \mathbb{C}$ and note that, for all n, $\Re(z) = u^t_\alpha - u^b_\beta - S_n\delta(u^b_\beta) = u^t_\alpha - T^n u^b_\beta$: this equality is easily obtained by induction on n.

Now let us suppose that $(\beta, \alpha; n)$ is reduced; to be simple, we will also suppose $\Re(z) > 0$, that is $u^t_\alpha > T^n u^b_\beta$; and we can also suppose $n > 0$. Consider the configuration in $\mathbb{C}$ which gives X by means of the Veech's construction and let $\tilde{\gamma}$ be the half-line in $\mathbb{C}$ given by $t \mapsto \xi^b_\beta + tz$, for $t \geq 0$. The initial segment of this line lies in R^b_β, thus it projects to the initial segment of a geodesic γ in X. We claim that this geodesic is defined till to time 1, and at this instant it reaches the singularity of X which is projection of ξ^t_α, making an angle with the separatrix H^t_α as desired.

In order to analyse the behaviour of γ, we study how do $\tilde{\gamma}$, and the other curves which project to pieces of γ, move in $\mathbb{C}$. As said before, the initial part of $\tilde{\gamma}$ lies in the rectangle $R_0 = R^b_\beta$. As $\Re(\tilde{\gamma}(1) - \tilde{\gamma}(0)) = u^t_\alpha - T^n u^b_\beta$ and $(\beta, \alpha; n)$ is a reduced triple, the orthogonal projection of $\tilde{\gamma}((0, 1))$ over I coincides with $T^{-n}I(\beta, \alpha; n)$ and is therefore strictly contained in I^b_β. Note, too, that the y coordinate of $\tilde{\gamma}$ increases with t; $\tilde{\gamma}$ leaves R_0 at an instant $t_0 < 1$, because the explicit formula for $\tilde{\gamma}$ implies that $\tilde{\gamma}([0, 1])$ covers a vertical gap bigger than σ_α; and when it does, it passes through the interior of its upper edge.

Then $\tilde{\gamma}$, without having encountered any pre-image of the singular points, passes to some upper rectangle R^t_x; but the Veech's construction identifies R^t_x with $R^b_x =: R_1$, so we are allowed to see γ as the projection of the line $\tilde{\gamma}_1 := \tilde{\gamma} + \theta_x = \tilde{\gamma} + S_1\theta(u^b_\beta)$. The orthogonal projection of $\tilde{\gamma}_1\,((0, 1))$ over I is $T^{-n}(\beta, \alpha; n) + S_1\delta(u^b_\beta) = T^{-(n-1)}(\beta, \alpha; n)$.

Because of reducedness, the latter is contained in the interior of I^b_x, which is the upper edge of R_1; $\tilde{\gamma}_1$ leaves that rectangle at an instant $t_1 < 1$, because the explicit formula for $\tilde{\gamma}$ implies that $\tilde{\gamma}([t_0, 1])$ covers a vertical gap bigger than the height of R_1; and again, it leaves R_1 passing through its top edge.

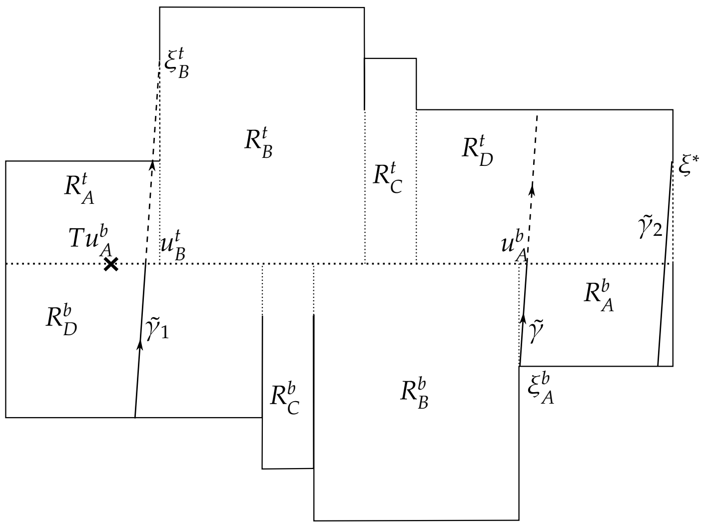

Figure 5.1: In this picture of a Veech construction the triple $(A, B; 1)$ is reduced, and Proposition 5.2.1 associates to it the saddle connection obtained by projection of the segments $\tilde{\gamma}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$. In particular the data we used made it necessary to define the segment $\tilde{\gamma}_2$, because ξ^t_B is not contained in ∂R^t_A.

We iterate this argument until it remains valid, that is until the n-th step: for $j = 1, \dots, n$ we have a line $\tilde{\gamma}_j = S_j\theta(u^b_\beta) + \tilde{\gamma}$ which projects to γ by means of Veech's construction; the segment $\tilde{\gamma}_j([t_{j-1}, t_j])$ is contained in a lower rectangle R_j of the Veech's construction; it enters R_j from its bottom edge, and leaves it through its top edge. All these segment project to X and join together to give an initial segment of the geodesic γ: indeed, at this stage, no singularities of X have been encountered yet.

What happens after $\tilde{\gamma}_n$ has left the last rectangle R_n (at the time $t_n < 1$)? Of course it will enter an upper rectangle R^t_χ and, again because of the reducedness of $(\beta, \alpha; n)$, the orthogonal projection of $\tilde{\gamma}_n((0, 1))$ over I is contained in I^t_χ; to better say, it is the rightmost part of this interval, because $\tilde{\gamma}_n(1) = \xi^t_\alpha$. In particular, χ is the letter of $\mathcal{A}$ which appears just before α in the top row of π. Now, we have two cases, according to Remark 2.2.4:

- $\eta_\chi \geq \sigma^t_\alpha$. In this case γ_n, after having entered R^t_χ, does not leave its interior until it reaches ξ^t_α. The geodesic γ has all the required properties.
- $\eta_\chi < \sigma^t_\alpha$, therefore χ is the rightmost letter of the bottom row of π, and $\sum_x \tau_x > 0$ (see Figure 5.1). We define $\tilde{\gamma}_{n+1} := \tilde{\gamma}_n + \theta_\chi = \tilde{\gamma} + S_{n+1}\theta(u^b_\beta)$, and call v the rightmost letter of the top row of π. We have $\tilde{\gamma}_{n+1}(1) = \xi^*$, and a segment $\tilde{\gamma}_{n+1}((1 - \varepsilon, 1))$ is contained in R^t_v. If the slope of $\tilde{\gamma}_{n+1}$, which is $\mathfrak{I}(z)/\mathfrak{R}(z)$, is sufficiently high, the segment of $\tilde{\gamma}_{n+1}$ which immediately precedes this one lies in R^b_χ. Since R^b_χ gets identified with R^t_χ by the Veech construction, this means that we are able to define the geodesic γ up to time 1 without encountering singularities in the meanwhile.
  The point $\gamma(1)$ is the singularity of X obtained by projection of ξ^*. The identifications performed in the case $\sum_x \tau_x > 0$ imply that projection of ξ^t_α gives the same singularity; and that the angles are the ones we desired.
  The condition «slope high enough» is satisfied for all n big enough: indeed the real part of z as a function of n is bounded (in modulus) by $|I|$, whereas the imaginary part contains the summand $S_n\eta(u^b_\beta)$ that grows to infinity.

To complete the proof, we just remark that in the case $\Re(z) < 0$ all the arguments before can be adapted if we define the geodesic 'backwards', starting from ξ^t_α and going towards ξ^b_β. □

§ 5.2.C Proof of the divergent case Now that we are able to associate a saddle connection to (nearly) each reduced triple, we can proceed with the proof of the divergent case of Theorem 5.1.2.

Step 1 – Restriction to the domain of a Veech chart: Let ϕ be a function $[0, +\infty) \to (0, +\infty)$ such that $t\phi(t)$ is decreasing and $\int_0^{+\infty} \phi(t)\mathrm{d}t = +\infty$. The divergent case of Theorem 5.1.2 will be proved if we do it 'locally' in a framed moduli stratum. Recall (Proposition 2.4.1 and below) that almost every $\hat{X}_0 \in \hat{\mathcal{H}}(S, \Sigma, h)$ (or, to better say, almost every non-framed translated surface X_0) is obtained by means of the Veech construction from some data $(\pi_0, \lambda_0, \tau_0)$, with π_0 an admissible marked permutation on $d = 2g + s - 1$ letters, $\lambda_0 \in \Delta_{\pi_0}$, $\tau_0 \in \Theta_{\pi_0}$.

Since $\mathcal{I}_{\pi_0} : \Omega_{\pi_0} \to \tilde{\mathcal{H}}(S, \Sigma, h)$ is a local homeomorphism whose domain is an open and contractible set, it is possible to lift it to a local homeomorphism $\hat{\mathcal{I}}_{\pi_0} : \Omega_{\pi_0} \to \hat{\mathcal{H}}(S, \Sigma, h)$ which includes in its image the point $\hat{X}_0$. Suppose we are able to prove that the theorem is true for the (classes of) framed translation structures belonging to a subset of $\hat{\mathcal{I}}_{\pi_0}(U \times V)$ with full measure, where $U \subset \Delta_{\pi_0}$ is an open and bounded neighbourhood of λ_0, and $V \subset \Theta_{\pi_0}$ is an open and bounded neighbourhood of τ_0. Then, by varying $\hat{X}_0$ among the framings of surfaces obtained by Veech construction, we conclude.

Step 2 – Identification of the right i.e.m.s: Given any function ϕ as above, we will say that an i.e.m. $T = (\pi_0, \lambda)$ is ϕ-*good* if T has the Keane's property, is uniquely ergodic and satisfies the statement of the divergent case of the Marchese's Theorem 4.1.3 for triples, with respect to the considered ϕ. For a fixed π_0, ϕ-good i.e.m.s are a full measure subset of Δ_{π_0}.

Now let us select a suspension data $\tau \in \Theta_{\pi_0}$ and call $\hat{X} = X(\pi_0, \lambda, \tau)$ with a selected frame; we also select a bundle $\mathcal{B}^{p_i,l}_{p_j,m}(\hat{X}) = \mathcal{B}^\alpha_\beta(\hat{X})$. Recall Proposition 5.2.1: if n is big enough and $(\beta, \alpha; n)$ is a reduced triple, then a saddle connection $\gamma_n \in \mathcal{B}^\alpha_\beta(\hat{X})$ is associated to it; in particular we have $\Im\left(\mathrm{Hol}(\gamma_n)\right) = S_n\eta(u^b_\beta) + \sigma^t_\alpha - \sigma^b_\beta$. Therefore

$$|\mathrm{Hol}(\gamma_n)| = \sqrt{\Re(\mathrm{Hol}(\gamma_n))^2 + \Im(\mathrm{Hol}(\gamma_n))^2} \le$$

$$\sqrt{(\lambda^*)^2 + (S_n\eta(u^b_\beta) + \sigma^t_\alpha - \sigma^b_\beta)^2} = S_n\eta(u^b_\beta)\sqrt{\frac{(\lambda^*)^2 + (S_n\eta(u^b_\beta) + \sigma^t_\alpha - \sigma^b_\beta)^2}{S_n\eta(u^b_\beta)^2}}$$

and, since $\lim_{n\to+\infty} S_n\eta(u^b_\beta) = +\infty$, the expression under square root goes to 1. We fix a (small) $\varepsilon > 0$: we will have $S_nh(u^b_\beta) > (1 - \varepsilon)|\mathrm{Hol}(\gamma_n)|$ for n big enough. On the other hand, since T is uniquely ergodic, we apply the stronger version of Birkhoff ergodic Theorem 0.2.8 on the function η (it is not continuous, but we can adapt the theorem to this particular case): this yields $\lim_{n\to+\infty} S_n\eta(u^b_\beta)/n = \int_I \eta = \mathrm{Area}(X)$ and in particular $S_n\eta(u^b_\beta) \le (1 + \varepsilon)n\mathrm{Area}(X)$ for n big enough. Thus

$$n \ge \frac{S_n\eta(u^b_\beta)}{(1 + \varepsilon)\mathrm{Area}(X)} \ge \frac{1 - \varepsilon}{1 + \varepsilon}\frac{|\mathrm{Hol}(\gamma)|}{\mathrm{Area}(X)}$$

and combining Proposition 5.2.1 with compliance of Marchese's statement, there are infinitely many $n \in \mathbb{N}$ such that

$$\left|\Re(\mathrm{Hol}(\gamma_n))\right| = |I(\beta, \alpha; n)| \le \phi(n) \le \phi\left(\frac{1 - \varepsilon}{1 + \varepsilon}\frac{|\mathrm{Hol}(\gamma_n)|}{\mathrm{Area}(X)}\right), \tag{5.2}$$

and in particular the set of the $z = \mathrm{Hol}(\gamma_n) \in \mathrm{Hol}^{p_i,l}_{p_j,m}(\hat{X})$ satisfying the inequalities above is infinite.

Step 3 – Selection of the neighbourhood: We are now ready for a 'local' proof of our theorem. Recalling the conclusions of Step 1, we select the open sets U and V in order to have $A_0/2 < \mathrm{Area}(X) < 2A_0$, where $A_0 = \mathrm{Area}(X_0)$, for each $X \in \mathcal{I}_{\pi_0}(U \times V)$. Then we set

$$\phi_0(t) := \phi\left(\frac{1+\varepsilon}{1-\varepsilon} 2A_0 t\right);$$

$\phi_0 : [0, +\infty) \to (0, +\infty)$ is still a function with infinite integral, and $t\phi_0(t)$ is decreasing. According to Step 2, the set U has a full measure subset U' such that the resulting i.e.m.s are ϕ_0-good; and for each $\hat{X} \in \hat{\mathcal{I}}_{\pi_0}(U' \times V)$ there are infinitely many $z \in \mathrm{Hol}^{p_i,l}_{p_j,m}(\hat{X})$ satisfying

$$\left|\Re(z)\right| \leq \phi_0\left(\frac{1-\varepsilon}{1+\varepsilon} \frac{|z|}{\mathrm{Area(X)}}\right) = \phi\left(\frac{2A_0}{\mathrm{Area(X)}}|z|\right) \leq \phi(|z|).$$

The proof is complete. □

Remark 5.2.2. Note that the proof of the divergent case exploited in an essential way the Veech construction, and in particular Proposition 2.3.1: this means that the set of full measure that we have found does not contain any translation structure with vertical saddle connections. Therefore, we actually find infinitely many solutions to $0 < \left|\Re(z)\right| < \phi\left(|z|\right)$. ◇

5.3 Chaika's theorem for translation surfaces

Given X a translation surface, we denote Φ^t the flow of its vertical vectorfield at the time t; $R_\vartheta X$ will be the translation surface rotated by angle ϑ, as before; and $B(p, r)$ will denote the ball for the flat metric on X with centre p and radius r.

The flow Φ^t is obviously recurrent, since X is compact; but what shrinking target properties can we expect it to satisfy? In [Cha11] Chaika, together with Theorem 4.4.1 for i.e.m.s, also proves a version for translation surfaces:

Theorem 5.3.1 (Chaika). *Let $\phi : [0, +\infty) \to (0, +\infty)$ be a decreasing function, such that $\int_0^{+\infty} \phi(t)\mathrm{d}t = +\infty$. Then, for any translation surface X, for almost any $\vartheta \in \mathbb{T}$, for any $p \in X$, if Φ^t is the vertical flow on $R_\vartheta X$ then the set*

$$\limsup_{t\to+\infty} \Phi^{-t} B(p, \phi(t))$$

has full Lebesgue measure in X.

In section 4.4 we saw how introduction of virtual singularities in an i.e.m. makes it possible to prove a weaker version of Chaika's theorem for i.e.m.s from Marchese's one. This time we do quite the same thing: we mark non-singular points of a translation surface as new singularities where the conical angle is 2π; and using Theorem 5.1.2 for holonomies of saddle connections, we have the following

Corollary 5.3.2. *Let $\phi : [0, +\infty) \to (0, +\infty)$ be a function, such that $t\phi(t)$ is decreasing and $\int_0^{+\infty} \phi(t)\mathrm{d}t = +\infty$; and let $\mathcal{H}(S, \Sigma, h)$ be a stratum. Consider on S any measure that is locally equivalent to the Lebesgue one.*
Then, for almost any $p, p' \in S$, and for almost any $X \in \mathcal{H}(S, \Sigma, h)$, if Φ^t is the vertical flow on (any representative of the class) X, there exists an increasing sequence $t_n \to +\infty$ such that

$$\mathrm{dist}_X\left(\Phi^{t_n}(p'), p\right) < \phi(t_n).$$

Similarly, for almost any $p \in X$ there exists an increasing sequence $t_n \to +\infty$ such that

$$\mathrm{dist}_X\left(\Phi^{t_n}(p), p\right) < \phi(t_n).$$

If we argue similarly as in Remark 4.1.4, the first statement is equivalent to

$$\liminf_{t\to+\infty} \frac{\mathrm{dist}_X\left(\Phi^{t_n}(p'),p\right)}{\phi(t)} = 0$$

and the same for the second one.

Proof. We begin from the first statement. We mark p and p' as new singularities, that is we set $\Sigma' = \Sigma \sqcup \{p,p'\}$ and define h' as the vector of indices obtained from h by adding two new entries 1, corresponding to the new singularities p and p'. This gives a natural map $\mathcal{H}(S,\Sigma',h') \to \mathcal{H}(S,\Sigma,h)$ which is obviously onto.

According to (the proof of) the divergent case of Theorem 5.1.2, for almost any $\hat{X} \in \hat{\mathcal{H}}(S,\Sigma',h')$ there is in infinite set of $n \in \mathbb{N}$ such that the saddle connections γ_n in the bundle $\mathcal{B}^{p,1}_{p',1}(\hat{X})$, found according to Proposition 5.2.1, satisfy $|\Re(\mathrm{Hol}(\gamma_n))| < \phi(|\mathrm{Hol}(\gamma_n)|)$.

Set $t_n := \Im(\mathrm{Hol}(\gamma_n)) \to +\infty$. Then, following what happens in the Veech rectangles in a way totally similar to what we did to prove Proposition 5.2.1, one checks the vertical flow of p' enters exactly the same rectangles as the geodesic γ_n, and that at the time t_n the two points $\Phi^{t_n}(p')$ and p lie in the same rectagle, on a horizontal segment. Therefore

$$\mathrm{dist}_X(\Phi^{t_n}(p'),p) = |\Re(\mathrm{Hol}(\gamma_n))| < \phi(|\mathrm{Hol}(\gamma)|) \le \phi(|\Im(\mathrm{Hol}(\gamma_n))|) = \phi(t_n).$$

The proof of the second statement is the same, except that we have to add only one virtual singularity. □

5.4 Wanderings towards infinity for the Teichmüller flow

§ 5.4.A Logarithm laws The Teichmüller flow on each connected component of a stratum of moduli spaces is, in general, an ergodic flow on a non-compact space whose diameter is infinite. Therefore we have to expect that a generic orbit of this flow makes excursions «towards infinity» and then goes back near their initial point, infinitely many times. Such a situation appears in various different settings, and a natural question is: with the passing of time, «how farther towards infinity» do these orbit go? That is, if we take a sequence of points on the orbit which goes to infinity, how rapidly does it diverge?

The answer is usually a *logarithm law*, namely fixed a point x_0 in the space, for a generic orbit $\Phi^t x$ of the flow, we have that something like

$$\limsup_{t\to+\infty} \frac{\mathrm{dist}(\Phi^t x, x_0)}{\log t}$$

is finite and nonzero. In several settings one can relate logarithm laws with shrinking target properties: and this is actually what we are going to do.

For the Teichmüller flow in moduli spaces, the following result holds ([Mas93]):

Theorem 5.4.1 (Masur). *Let S be a surface; fix $X \in \mathrm{M}(S)$, and a $q \in \mathrm{QD}(X)$ with unit area. Let R_ϑ denote the rotation of angle ϑ, and $g^t_\vartheta X$ denote the point of the moduli space reached by the Teichmüller flow starting from X in direction $R_\vartheta q$ at the time t. Then, for almost every $\vartheta \in \mathbb{T}$, it holds that*

$$\limsup_{t\to+\infty} \frac{d(g^t_\vartheta X, X)}{\log t} = 1.$$

§ 5.4.B Systole function Our result regards the Teichmüller flow in strata of (framed) translation structures and is a corollary of yet another theorem which states a dichotomy. We need first of all a definition:

Definition 5.4.2. Given a translation surface X, its *systole* is the length of the shortest saddle connection of X:

$$\operatorname{Sys}(X) := \min \{|z| \mid z \in \operatorname{Hol}(X)\}.$$

If $\hat{X}$ is a framed translation surface, we may also define a systole for each bundle:

$$\operatorname{Sys}^{p_i,l}_{p_j,m}(X) := \min \left\{|z| \,\middle|\, z \in \operatorname{Hol}^{p_i,l}_{p_j,m}(X)\right\}.$$

Of course, the systole function (of a bundle) is well defined on each (framed) stratum of moduli spaces. *Malher's criterion* states that it is the analogue for strata of the function ℓ introduced in paragraph 1.3.D for moduli spaces:

A sequence $(X_n)_{n\in\mathbb{N}}$ in a connected component of $\mathcal{H}(S,\Sigma,h)$ goes to infinity (that is, it eventually leaves any fixed compact subset) if and only if $\operatorname{Sys}(X_n) \to 0$.

Another similarity between classic moduli spaces and strata is the following (see [Boi10]):

Theorem 5.4.3 (Boissy). *Each connected component of $\mathcal{H}(S,\Sigma,h)$ has one topological end (Definition 1.3.9).*

However, there is a way to subdivide the end of a connected component, namely specification of what saddle connections have small holonomy (see [EMZ02]). Thus, if we know what are the bundles such that $\operatorname{Sys}^{p_i,l}_{p_j,m}(X_n) \to 0$, we have some information about 'where' is the sequence diverging to.

REMARK 5.4.4. If $X \in \mathcal{H}(S,\Sigma,h)$ has a vertical saddle connection, the latter is exponentially shrunk by the Teichmüller flow: therefore $\operatorname{Sys}(g^t X) \to 0$ for $t \to +\infty$ and, according to Mahler's criterion, its orbit diverges.

Otherwise, if X has no vertical connections, Proposition 2.3.1 states that X is obtained my means of the Veech construction, starting from an i.e.m. with the Keane's property. In particular the orbit of X cannot diverge: a possible reason is that the orbit of X takes place within the Veech boxes construction (paragraph 2.4.C), therefore its returns to $\Omega^*(\mathcal{D})$ make up a recurrent map. ◇

§ **5.4.C The theorems** We are going to prove the following dichotomy, which is a consequence of Theorem 5.1.2:

Theorem 5.4.5 (Marchese). *Let $\psi : [0,+\infty) \to (0,+\infty)$ be a monotone decreasing function; moreover, let S be a surface with $\Sigma \subset S$ a finite set and h be a vector of indices.*

- *Suppose $\int_0^{+\infty} \psi(t)\mathrm{d}t < +\infty$. Then almost every $X \in \mathcal{H}^{(1)}(S,\Sigma,h)$ is the starting point of a Teichmüller orbit which satisfies*

$$\lim_{t\to+\infty} \frac{\operatorname{Sys}(g^t X)}{\sqrt{\psi(t)}} = +\infty.$$

- *Suppose $\int_0^{+\infty} \psi(t)\mathrm{d}t = +\infty$. Then almost every $\hat{X} \in \hat{\mathcal{H}}^{(1)}(S,\Sigma,h)$ is the starting point of a Teichmüller orbit which satisfies, for every bundle of saddle connections $\mathcal{B}^{p_i,l}_{p_j,m}(\hat{X})$,*

$$\liminf_{t\to+\infty} \frac{\operatorname{Sys}^{p_i,l}_{p_j,m}(g^t \hat{X})}{\sqrt{\psi(t)}} = 0.$$

In particular the second statement gives, for a generic orbit, a control about its excursions towards *each* of the 'cusps'. The theorem implies the following version of logarithm law:

Corollary 5.4.6. *For almost every $\hat{X} \in \hat{\mathcal{H}}^{(1)}(S, \Sigma, h)$, and every bundle of saddle connections $\mathcal{B}^{p_i,l}_{p_j,m}(\hat{X})$, it holds that*

$$\limsup_{t\to+\infty} \frac{-\log \mathrm{Sys}^{p_i,l}_{p_j,m}(g^t\hat{X})}{\log t} = \frac{1}{2}.$$

Proof. We prove the statement of the corollary holds for (classes of) framed translation surfaces $\hat{X}$ that satisfy

- the divergent statement of the theorem above for the function $\psi(t) := \min\{1, t^{-1}\}$;
- the convergent statement of that theorem for all the functions $\psi_a(t) := \min\{1, t^{-a}\}$ with $a \in \mathbb{Q} \cap (1, +\infty)$.

The set of such classes is the intersection of countably many subsets of $\hat{\mathcal{H}}^{(1)}(S, \Sigma, h)$ with full measure, so it also has full measure.

To be simple we denote $s(t) := \mathrm{Sys}^{p_i,l}_{p_j,m}(g^t\hat{X})$. The hypotheses on $\hat{X}$ imply that a sequence $t_n \to +\infty$ exists such that

$$\frac{s(t_n)}{t_n^{-1/2}} \le e^{-n} \quad \text{for all } n \in \mathbb{N},$$

but on the other hand, it must also hold, for any $a \in \mathbb{Q} \cap (1, +\infty)$, that

$$\lim_{n\to+\infty} \frac{s(t_n)}{t_n^{-a/2}} = +\infty. \tag{5.3}$$

Taking the logarithms of both expressions,

$$\log s(t_n) + \frac{1}{2}\log t_n \le -n; \quad \log s(t_n) + \frac{a}{2}\log t_n \to +\infty.$$

For big values of n, we have $\log t_n > 0$; we divide by it and get, in particular, that

$$\frac{\log s(t_n)}{\log t_n} + \frac{1}{2} < 0; \quad \frac{\log s(t_n)}{\log t_n} + \frac{a}{2} > 0,$$

that is, $0 > \frac{\log s(t_n)}{\log t_n} + \frac{1}{2} > -\frac{a-1}{2}$ for all $a \in \mathbb{Q} \cap (1, +\infty)$, therefore $\frac{\log s(t_n)}{\log t_n} \to -\frac{1}{2}$.

Moreover, if another sequence t_n gave

$$\frac{\log s(t_n)}{\log t_n} \to b < b' < -\frac{1}{2}$$

then we would have $s(t_n) < t_n^{b'}$ for n big enough, so equality 5.3 would be false for some a. The corollary is proved. □

Remark 5.4.7. When we were developing Marchese's Theorem 4.1.3 for i.e.m.s, we used Lemma 4.1.5 to verify that one could talk indifferently of almost every i.e.m. or of almost every i.e.m. on unit length intervals. In the same way, here and in Theorem 5.1.2 we can indifferently talk of almost every $X \in \mathcal{H}(S, \Sigma, h)$ or almost every $X \in \mathcal{H}^{(1)}(S, \Sigma, h)$. ◇

§ 5.4.D Estimates for holonomies under the Teichmüller flow We already remarked that the Teichmüller flow preserves saddle connections, as well as bundles (Remark 5.1.1). If γ is a saddle connection on a (framed) translation surface X, then $g^t\gamma$ is a saddle connection of the translation surface g^tX; we will denote $\mathrm{Hol}_t(\gamma) = \mathrm{Hol}(g^t\gamma)$. The explicit formula of the Teichmüller flow yields $\mathrm{Hol}(\gamma) = a + ib \Rightarrow \mathrm{Hol}_t(\gamma) = e^t a + ie^{-t}b$. We will call

$$A(\gamma) := |\Re(\mathrm{Hol}(\gamma))| \cdot |\Im(\mathrm{Hol}(\gamma))| = |\Re(\mathrm{Hol}_t(\gamma))| \cdot |\Im(\mathrm{Hol}_t(\gamma))| \text{ for all } t; \quad \cot_t(\gamma) := \frac{|\Re(\mathrm{Hol}_t(\gamma))|}{|\Im(\mathrm{Hol}_t(\gamma))|}.$$

The first fact to note is that

For all t, the equality $A(\gamma) = |\mathrm{Hol}_t(\gamma)|^2 \frac{\cot_t(\gamma)}{1+\cot_t(\gamma)^2}$ holds.

We deduce from that:

Let $t > 0$ be an instant such that $|\mathrm{Hol}_t(\gamma)| < 1$. Then $t > \log|\mathrm{Hol}_0(\gamma)|$.

Indeed, from the equality above and the fact that $\cot_t(\gamma) = e^{2t}\cot_0(\gamma)$, we have

$$|\mathrm{Hol}_0(\gamma)|^2 \frac{1}{1+\cot_0(\gamma)^2} = \frac{A(\gamma)}{\cot_0(\gamma)} = |\mathrm{Hol}_t(\gamma)|^2 \frac{e^{2t}}{1+\cot_t(\gamma)^2} < \frac{e^{2t}}{1+\cot_0(\gamma)^2}.$$

Now, we define $\tau = \tau(X,\gamma)$ to be the instant when $|\mathrm{Hol}_t(\gamma)|$ reaches its minimum for $t \in \mathbb{R}$.

The instant τ is uniquely determined and equals $-(1/2)\log\cot_0(\gamma)$. Moreover $A(\gamma) = |\mathrm{Hol}_\tau(\gamma)|^2/2 \leq |\mathrm{Hol}_t(\gamma)|^2/2$ for all t.

The instant τ is indeed the only one that satisfies $1 = \cot_\tau(\gamma) = e^{2\tau}\cot_0(\gamma)$. The second part is simply a consequence of the formula for $A(\gamma)$ mentioned above.

Fix an $\varepsilon > 0$. Then for almost every $X \in \mathcal{H}(S,\Sigma,h)$, and for all but finitely many saddle connections γ on X, it holds that $\tau(X,\gamma) \leq (1+\varepsilon)\log|\mathrm{Hol}(\gamma)|$.

This statement is a consequence of the convergent part of Theorem 5.1.2 applied to $\phi(t) = \min\{1, t^{-(1+2\varepsilon)}\}$. Indeed it implies that all but finitely many saddle connections γ satisfy $|\Re(\mathrm{Hol}(\gamma))| \geq |\mathrm{Hol}(\gamma)|^{-(1+2\varepsilon)}$, so

$$\tau = -\frac{1}{2}\log\frac{|\Re(\mathrm{Hol}(\gamma))|}{|\Im(\mathrm{Hol}(\gamma))|} \leq -\frac{1}{2}\log|\mathrm{Hol}(\gamma)|^{-2(1+\varepsilon)}.$$

§ 5.4.E Proof of the convergent case

Step 1 – Statement of the contradiction: The statement of the theorem is equivalent to the apparently weaker claim that, for almost every X, and t sufficiently big, one has

$$\frac{\mathrm{Sys}(g^t X)}{\sqrt{\psi(t)}} \geq 1. \tag{5.4}$$

Indeed suppose that, for every ψ as in the statement, this is true for a subset $E(\psi) = \mathcal{H}^{(1)}(S,\Sigma,h)$ of full measure (depending on ψ). Then the elements of $\bigcap_{N\in\mathbb{N}} E(N^2\psi)$, which also has full measure, satisfy the original statement.

Now, take X such that inequality 5.4 is false: so an increasing sequence $t_n \to +\infty$ exists such that $\mathrm{Sys}(g^{t_n}X) < \sqrt{\psi(t_n)}$. For each n, let γ_n be a saddle connection on X such that $|\mathrm{Hol}_{t_n}(\gamma_n)| = \mathrm{Sys}(g^{t_n}X)$. In particular the sequence $(\mathrm{Hol}_{t_n}(\gamma_n))_{n\in\mathbb{N}}$ is bounded.

Step 2 – Properties of the contradicting sequence: We now need some observations about the saddle connections γ_n. First of all they have to be infinitely many distinct ones, because for a single connection γ it always holds that $\lim_{t\to+\infty}|\mathrm{Hol}_t(\gamma)| = +\infty$. Up to extracting subsequences, we will also have $|\mathrm{Hol}_0(\gamma_n)| \to +\infty$, because, according to the Masur's result we quoted in the proof of the convergent case of Theorem 5.1.2, only finitely many saddle connections have bounded holonomy; so, we suppose $|\mathrm{Hol}_0(\gamma_n)| > 1$ for all n.

We now see that, for each subsequence (γ_k) such that $\lim_{k\to+\infty}\cot_0(\gamma_k)$ is defined, this limit is either 0 or ∞. Indeed a subsequence with $\cot_0(\gamma_k) \to c \in \mathbb{R}_+$ would yield $A(\gamma_k) \to +\infty$, according to the formula given in paragraph 5.4.D; but we know that $A(\gamma_k) \leq |\mathrm{Hol}_{t_k}(\gamma_k)|^2/2$, therefore the sequence of the areas is bounded.

But also $\cot_0(\gamma_k) \to +\infty$ is impossible: in paragraph 5.4.D we saw that $|\mathrm{Hol}_t(\gamma_k)|$ reaches its minimum for $t = \tau_n = -(1/2)\log\cot_0(\gamma_n)$: so, the sequence τ_n would be negative from a certain point on, and therefore $|\mathrm{Hol}_{t_k}(\gamma_k)| > |\mathrm{Hol}_0(\gamma_k)| \to +\infty$, contradicting our choice.

Step 3 – Application of Theorem 5.1.2: Since $\cot_0(\gamma_n) \to 0$, for any $\varepsilon > 0$, and n big enough, we have

$$\frac{|\Im(\mathrm{Hol}_0(\gamma_n))|}{|\mathrm{Hol}_0(\gamma_n)|} > \frac{1}{1+\varepsilon} \qquad \text{and therefore}$$

$$|\Re(\mathrm{Hol}_0(\gamma_n))| = \frac{A(\gamma_n)}{|\Im(\mathrm{Hol}_0(\gamma_n))|} < (1+\varepsilon)\frac{A(\gamma_n)}{|\mathrm{Hol}_0(\gamma_n)|} < \frac{1+\varepsilon}{2}\frac{|\mathrm{Hol}_{t_n}(\gamma_n)|^2}{|\mathrm{Hol}_0(\gamma_n)|} < \frac{1+\varepsilon}{2}\frac{\psi(t_n)}{|\mathrm{Hol}_0(\gamma_n)|}.$$

For n big enough being $|\mathrm{Hol}_{t_n}(\gamma_n)| < 1$, we must have $t_n > \log|\mathrm{Hol}_0(\gamma_n)|$, according to what seen in paragraph 5.4.D. Hence

$$|\Re(\mathrm{Hol}_0(\gamma_n))| < \frac{1+\varepsilon}{2}\frac{\psi(\log|\mathrm{Hol}_0(\gamma_n)|)}{|\mathrm{Hol}_0(\gamma_n))|}.$$

Remark 5.4.8. For any decreasing function $\psi : [0,+\infty) \to (0,+\infty)$, if one defines $\phi : [1,+\infty) \to [0,+\infty)$ by setting $\phi(x) = \psi(\log x)/x$, then $x\phi(x)$ will be decreasing.

Conversely, starting from a function ϕ such that $x\phi(x)$ is decreasing, if one defines $\psi(t) = e^t\phi(e^t)$, this will be a decreasing function. For both constructions it will hold that $\int_0^{+\infty}\psi(t)\mathrm{d}t = \int_1^{+\infty}\phi(x)\mathrm{d}x$. ◇

In conclusion, the set $\{\mathrm{Hol}_0(\gamma_n)|n \in \mathbb{N}\} \subseteq \mathrm{Hol}(X)$ is an infinite set of solutions to $|\Re(z)| < \tilde{\phi}(|z|)$, where $\tilde{\phi}(x) = (1+\varepsilon)\psi(\log x)/(2x)$ is a multiple of the ϕ in the Remark above. According to Theorem 5.1.2 (and to Remark 5.4.7), this can happen only for the X belonging to a subset of $\mathcal{H}^{(1)}(S,\Sigma,h)$ with measure zero. □

§ 5.4.F Proof of the divergent case

Step 1 – Explicit construction of a sequence: In the same way as before, it is sufficient to prove a weaker claim, namely that for almost every $\hat{X}$ one has

$$\liminf_{t\to+\infty}\frac{\mathrm{Sys}^{p_i,l}_{p_j,m}(g^t\hat{X})}{\sqrt{\psi(t)}} \leq 1.$$

Indeed if this is true for every $\hat{X}$ belonging to a subset $E'(\psi)$ with full measure, one has the original statement for every $\hat{X}$ in the set $\bigcap_{N\in\mathbb{N}} E'(\psi/N^2)$. We fix an $\varepsilon > 0$, and define

$$\tilde{\phi}(t) := \frac{\psi\left((1+\varepsilon)\log t\right)}{2t};$$

up to multiplicative constants, this is the same as the function ϕ defined in Remark 5.4.8: thus it satisfies the hypotheses on the function appearing in the divergent case of Theorem 5.1.2.

Hence — recalling also Remarks 5.2.2 and 5.4.7 — for every $\hat{X}$ belonging to an appropriate subset of $\hat{\mathcal{H}}^{(1)}(S,\Sigma,h)$ with full measure, the bundle $\mathcal{B}^{p_i,l}_{p_j,m}$ will contain infinitely many saddle connections $\{\gamma_n|n \in \mathbb{N}\}$ such that $0 < |\Re(\mathrm{Hol}(\gamma_n))| < \tilde{\phi}(|\mathrm{Hol}(\gamma_n)|)$. We consider $\hat{X}$ as above.

For each n, let $\tau_n = \tau(X,\gamma_n)$ be the instant when the length of γ_n reaches its minimum, as defined in paragraph 5.4.D. We will show that the claim is verified by the sequence (τ_n).

Step 2 – Estimates: First of all, we need that the sequence τ_n goes to infinity: but, since $\{|\mathrm{Hol}(\gamma_n)|\,|\,n \in \mathbb{N}\}$ is an infinite set, it is necessarily unbounded (again because of Masur's theorem used to prove Theorem 5.1.2). So, we can suppose $|\mathrm{Hol}(\gamma_n)| \to +\infty$; whereas $|\Re(\mathrm{Hol}(\gamma_n))| < \tilde{\phi}(|\mathrm{Hol}(\gamma_n)|)$ is bounded independently of n, so $\tau_n = -(1/2)\log\cot_0(\gamma_n) \to +\infty$.

We now perform some estimates: the bound on $|\Re(\mathrm{Hol}(\gamma_n))|$ implies

$$\mathrm{Sys}^{p_i,l}_{p_j,m}(g^{\tau_n}\hat{X})^2 \leq |\mathrm{Hol}_{\tau_n}(\gamma_n)|^2 = 2A(\gamma_n) < 2|\Im(\mathrm{Hol}(\gamma_n))|\,\tilde{\phi}\,(|\mathrm{Hol}(\gamma_n)|) < 2|\mathrm{Hol}(\gamma_n)|\,\tilde{\phi}\,(|\mathrm{Hol}(\gamma_n)|)$$

and we can suppose also that $\hat{X}$ (or, to better say, its non-framed version) belongs to the set of full measure such that the last statement in paragraph 5.4.D holds: for n sufficiently big we have $\tau_n \leq (1+\varepsilon)\log|\mathrm{Hol}(\gamma_n)|$, that is $|\mathrm{Hol}(\gamma_n)| \geq \exp(\tau_n/(1+\varepsilon))$. As $t\tilde{\phi}(t)$ is a decreasing function, we continue the chain of inequalities

$$2|\mathrm{Hol}(\gamma_n)|\tilde{\phi}\left(|\mathrm{Hol}(\gamma_n)|\right) \leq 2\exp\left(\frac{\tau_n}{1+\varepsilon}\right)\tilde{\phi}\left(\exp\left(\frac{\tau_n}{1+\varepsilon}\right)\right) = \psi(\tau_n).$$

So, for n big enough we have $\mathrm{Sys}^{p_i,l}_{p_j,m}(g^{\tau_n}\hat{X}) < \sqrt{\psi(\tau_n)}$, as desired. □

Acknowledgements

I would like to thank everyone who helped me to deepen the theory of Teichmüller dynamics, and to draft this work.

The first person I have to mention is, of course, my supervisor, **Stefano Marmi**. Throughout the last two years he guided my studies of this subject, providing me precious advice on how to orient myself among the immense literature written in the last decades; he allowed me to understand the ramifications and the current research themes within this subject, as well as the very aim of dynamical systems. Thanks to him, moreover, I had the opportunity of establishing a direct contact with some of the main contributors to this theory.

An irreplaceable source of ideas for this thesis have been, in particular, my several discussions with the author of the papers I read and revised, **Luca Marchese**. In those occasions I recognised that no written work may substitute a direct talk to understand the ideas and the motivations concealed behind the statement, or the proof of a theorem. I sincerely thank him for the further references he gave me, and for how confidential he has been with me.

I received other, very relevant, cues from the course kept by **Jean-Christophe Yoccoz** and **Carlos Matheus** at the ICTP, Trieste in May 2012. Attendance of it has been very helpful for me to appreciate some fundamental details.

Last but not least, I am grateful to my counter-supervisor, **Carlo Carminati**, for his quick but thorough research for errors in this work.

Bibliography

[AGY06] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.*, (104):143–211, 2006.

[Ath08] J. S. Athreya. Logarithm laws and shrinking target properties. *ArXiv e-prints*, August 2008.

[AV07] Artur Avila and Marcelo Viana. Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture. *Acta Math.*, 198(1):1–56, 2007.

[Boi10] C. Boissy. Ends of strata of the moduli space of quadratic differentials. *ArXiv e-prints*, April 2010.

[Cha11] Jon Chaika. Shrinking Targets For IETS: Extending a Theorem of Kurzweil. *Geometric And Functional Analysis*, 21:1020–1042, 2011. 10.1007/s00039-011-0130-y.

[CM08] Carlo Carminati and Stefano Marmi. An introduction to dynamical systems. `http://homepage.sns.it/marmi/lezioni/dynsys_ergodic.pdf`, November 2008. Preliminary version.

[EMZ02] A. Eskin, H. Masur, and A. Zorich. Moduli Spaces of Abelian Differentials: The Principal Boundary, Counting Problems and the Siegel–Veech Constants. *ArXiv Mathematics e-prints*, February 2002.

[FM11] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton Mathematical Series. Princeton University Press, 2011.

[Khi97] A. Ya. Khinchin. *Continued fractions*. Dover Publications Inc., Mineola, NY, russian edition, 1997. With a preface by B. V. Gnedenko, Reprint of the 1964 translation.

[KMS86] Steven Kerckhoff, Howard Masur, and John Smillie. Ergodicity of billiard flows and quadratic differentials. *The Annals of Mathematics*, 124(2):pp. 293–311, 1986.

[Mar] Luca Marchese. Shrinking targets for interval exchange transformations and translation surfaces via Khinchin-type theorems. Received directly from the author.

[Mar10] Luca Marchese. Khinchin type condition for translation surfaces and asymptotic laws for the Teichmuller flow. *ArXiv e-prints*, March 2010.

[Mar11] Luca Marchese. The Khinchin theorem for interval-exchange transformations. *J. Mod. Dyn.*, 5(1):123–183, 2011.

[Mar12] Bruno Martelli. Lezioni di geometria iperbolica. `http://www.dm.unipi.it/~martelli/didattica/matematica/2012/Geometria%20iperbolica.pdf`, January 2012.

[Mas82] Howard Masur. Interval exchange transformations and measured foliations. *The Annals of Mathematics*, 115(1):pp. 169–200, 1982.

[Mas88] Howard Masur. Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential. In *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, volume 10 of *Math. Sci. Res. Inst. Publ.*, pages 215–228. Springer, New York, 1988.

[Mas93] Howard Masur. Logarithmic law for geodesics in moduli space. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 229–245. Amer. Math. Soc., Providence, RI, 1993.

[MY12] Carlos Matheus and Jean-Christophe Yoccoz. Birkhoff sums of i.e.t.'s: the Kontsevich-Zorich cocycle. `http://cdsagenda5.ictp.trieste.it/full_display.php?ida=a11165` and `http://matheuscmss.wordpress.com/2012/06/01/school-and-conference-in-dynamical-systems-2012-ictp-trieste-italy/`, May 2012. Slides of a mini-course at the School and Conference in Dynamical Systems, ICTP, Trieste.

[Vee82] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *The Annals of Mathematics*, 115(2):pp. 201–242, 1982.

[Vee86] William A. Veech. The Teichmüller Geodesic Flow. *The Annals of Mathematics*, 124(3):pp. 441–530, 1986.

[Yoc07] Jean-Christophe Yoccoz. Interval exchange maps and translation surfaces. `http://www.college-de-france.fr/media/jean-christophe-yoccoz/UPL15305_PisaLecturesJCY2007.pdf`, 2007. Notes of a course at the Scuola Normale Superiore, Pisa.

[Zor96] Anton Zorich. Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents. *Ann. Inst. Fourier (Grenoble)*, 46(2):325–370, 1996.

2007–2012

Pare proprio che la mia vita stia per cambiare capitolo. Certo non è la prima volta che mi capita, ma credo che le volte precedenti non mi sia mai dispiaciuto quanto ora. I miei anni universitari sono stati ricchissimi di nuove esperienze, e mi hanno cambiato nel profondo.

È pratica comune degli studenti universitari aggiungere alla parola “Pisa” un certo suffisso ben preciso imparato dai Livornesi, ma la verità è che mi sono affezionato a questa cittadina. Non certo perché la ritenga perfetta, ma perché associo a Pisa tante occasioni piacevoli, e il raggiungimento di tanti obiettivi che per me erano importanti. Insomma, in qualche senso a Pisa ci sono cresciuto. Qui ho avuto la mia prima vera esperienza di autonomia; e il suo bilancio mi rende davvero orgoglioso di me.

Certo, la vita pisana mi ha fatto abbastanza sudare. Non so quante volte mi sono sentito un prigioniero dello studio, ed ho temuto che a causa dei vincoli di tempo che la Normale mi imponeva mi sarei perso gli anni migliori della mia vita; anche perché attorno a me ho visto persone la cui esistenza è totalmente riempita dalla materia che studiano (magari insieme a qualche videogame o qualche altro tipo di nerdata). Chi mi conosce sa che ho il vizio di piangermi addosso (magari una volta più di adesso), però ora mi rendo conto che dopo lo sconforto mi sono sempre rimboccato le maniche. Tra poco probabilmente me ne andrò via da questa città, a malincuore ma soddisfatto.

Sarò soddisfatto non solo di essere cresciuto (anche se ho capito che in realtà non sarò mai maturo abbastanza), ma anche perché la multiforme vita universitaria mi ha consentito di diventare meno provinciale e più aperto. Ci sono tante cose che non avrei imparato, tante realtà che non avrei mai conosciuto restando a casa. Basti pensare alle cose più immediate: la musica, i fumetti, il cibo. Oppure le tante gite che ho fatto (Firenze, Lucca, Castiglioncello, Siena), e i viaggi veri e propri che ho fatto con i pisani (Ginevra, Parigi, Oslo). Per tutto questo, quando me ne sarò reso veramente conto, mi farà male pensare che nella mia routine non ci saranno più il Dip, la Normale e la sua mensa, il Carducci, la mia stanza al ponte Solferino, le Vettovaglie, Piazza della Pera, i lungarni.

Ma adesso veniamo al dunque, al vero motivo per cui tu lettore sei arrivato a questa pagina: i ringraziamenti a tutte le persone che sono state significative per me in questi anni, siano essi “pisani” o no. Una premessa necessaria: mi sono accorto di non essere in grado di gestire un grande numero di relazioni sociali. Ho trascorso cinque anni praticamente senza stare fermo un attimo. Così facendo non solo ho trascurato un sacco di amici lontani (nonostante Facebook) ma, a causa dello stress, anche molte delle mie conoscenze pisane. Mi dispiace molto di essermi allontanato da svariate persone che stimavo e con cui forse avrei potuto coltivare rapporti migliori, ma *c’est la vie*.

Ogni serie di ringraziamenti che si rispetti non può che cominciare da papà Francesco e da mamma Michelina. Non serve certo scriverlo qui per farglielo sapere, ma nessuno vuole loro bene come i loro figli. Un’antica tradizione vuole che il genitore stia in continuazione a chiedere al figlio lontano se ha mangiato e che tempo fa, e che questo faccia imbestialire il figlio che si sente trattato come un bambino. Però questo non significa nulla: fa parte del gioco, e crescere significa anche capire i tuoi genitori. La mia speranza è di essere un buon figlio, soprattutto da qualche mese a questa parte. E spero i miei sappiano che, anche se sto cercando di costruirmi la mia vita da adulto, e le cose da fare sembrano sempre troppe, non mi tirerò mai indietro per loro. E di nessun altro potrò mai fidarmi come di loro.

Dai miei genitori passiamo, ovviamente, a mia sorella Maria. È bellissimo poco a poco scoprire che la piccola della famiglia sta crescendo, e si sta costruendo una personalità piuttosto affascinante. Per me è un'esperienza nuova condividere un mucchio di interessi con mia sorella, scherzarci in libertà, considerarla una persona responsabile. Mi avevano detto che sarebbe arrivato un momento in cui sarei diventato geloso di lei: e temo proprio che sia arrivato.

Ogni serie di ringraziamenti che si rispetti non può che continuare con la mia dolce (ma davvero dolce) metà Elisa, una delle cose più belle che mi siano mai capitate. Spesso mi chiedo se, con i miei modi di fare così lunatici, con la totale cancellazione del concetto di relax dalla mia vita, mi merito davvero una fortuna così grande. E invece lei prima mi chiama «odioso», poi cerca le mie carezze. Chi al suo posto avrebbe potuto darmi il sorriso in questi mesi così complicati, semplicemente accompagnando con il suo affetto e la sua bellezza la mia routine quotidiana? La nostra storia non è cominciata tantissimo tempo fa, ma si fa presto ad abituarsi alle cose che fanno stare bene. Probabilmente lei è fra i motivi per cui tanta gente, negli ultimi tempi, mi ha visto in giro molto di meno; ma adorare la propria ragazza mi sembra una giustificazione validissima.

Da Elisa al principale responsabile della nostra storia il passo è breve. Il grande Michele (Maicolcercil) merita prima di tutto un enorme grazie per i consigli che mi ha dato, così come per essere stato un confidente schietto, pronto ad ascoltarmi quando mi è sembrato di trovarmi in un vicolo cieco. E lo ringrazio anche per avermi dato modo di ricambiare, in tempi più recenti, la fiducia che gli ho dato; o semplicemente per i pranzi a mensa, i giri di sera ritagliati nei buchi di tempo, l'escursione alle Cinque Terre. Se la parola "amico" ha ancora un valore, lui merita questo appellativo.

Ma lo stesso posso dire del buon vecchio Gerardo (Oompa). Non esagero dicendo che mi ha visto crescere, anche se per forza di cose non vedendoci molto spesso. A lui va un grande ringraziamento soprattutto per avermi sempre tenuto in considerazione, ed essere stato spesso una "voce della saggezza" molto più esperta di me. Tra i momenti passati con lui, meritano una citazione il gran premio di Monza e due giri a Firenze. Ma non è l'unica persona importante rimastami dal gruppo delle OliMat del liceo, che ogni tanto becco qui a Pisa o nella vecchia Benevento: indispensabile nominare il sempre indaffarato Teo (con grandi complimenti per i risultati di marcia, che con lui vanno sempre di moda); quel sotterraneo nerd che è Valerio; Leucio e Cristian.

Grazie ai miei amici caudini, per la grande accoglienza che mi riservano ogni volta che scendo a casa. Anche con loro i momenti passati insieme non sono frequenti, anzi tendono a diminuire con il passare del tempo, ma sono sempre un'occasione immancabile per me: la prova vivente che le cose semplici, siano esse una pizza, una gita o un giro in bici, sono le migliori. Facendo i nomi: Gabriele ed Angelica (con una figlia appena arrivata o che sta per arrivare nel momento in cui discuto; e di cui ancora mi vanto di essere stato testimone di nozze), Gianni (non scorderò mai il mio giro da lui in Trentino), Titta, Teodora ed Antonio, Anna, Ciro e Carmelina (che domani si sposeranno in comune). Nonché Edoardo e Gina, Mary, Elisabetta e Cesare.

Passiamo quindi ai matematici: il primo pensiero va sicuramente al vecchio gruppo dei "normalisti scialli", che è scomparso pian piano: Nico (grazie al quale ancora oggi, ogni tanto, parlo mezzo romano), Lin, il Marche, Riccardo/Zoidberg, tutti emigrati, e Matthew sono tra le persone con cui ho potuto condividere di più nei primi anni di università. Soprattutto Nico e Lin per me rimangono esempi ineguagliati di come si può essere contemporaneamente dei grandi matematici e dei cazzari ancora più grandi: spero mi abbiano insegnato qualcosa. A tutti loro devo aggiungere la new entry Andre (universalmente noto come il Petracci), che è stato per me un importante punto di riferimento in tempi più recenti; e la ragazza del Marche, Giulia.

C'è l'immenso mondo di confine tra matematica e fisica: l'appartamento a Porta San Zeno di Leo, Cek, Mauro e i Garfagnini, la cui ospitalità infinita mi ha sempre colpito; e tutti i suoi frequentatori Fra Avanzi, Pietro, Tommy, Marco, il Massei, il Borgo, Marco Mariti, Domenico, Fra Colangelo, il Marini; e le ragazze Costy, Veronica, Eleonora, Sara, Annina, Elisa, Giulia (e tanti altri...). Mi dispiace di essere subentrato abbastanza stabilmente in questo microuniverso solo in un secondo momento, e di non essere mai riuscito a cogliere fino in fondo lo sterminato bagaglio di citazioni demenziali che ogni tanto qualcuno sfodera, ma ricordo comunque con grande piacere le epoche delle manifestazioni; le cene, vegane e non... e il giorno in cui Massei, Pietro e Marraccini mi diedero una mano a traslocare!

C'è il gruppone storico di matematica, anche questo molto rimaneggiato dai trasferimenti, al quale ricollego una parte consistente dei miei ricordi dell'università, delle mie serate in compagnia, e dei video di Maccio Capatonda: Luca (ottimo compagno di studio e per sparare cazzate, a cui vanno ancora una volta infinite grazie per avermi spronato tanto in palestra da farmi perdere 20 kg) e la sua ragazza Claudia; Umberto (l'inimitabile pivello spaccone, che ho rivisto con grande piacere quando sono cascato a Trieste) e il suo coinquilino Biagio; il Paglia (un buon confidente; è stata bella la zingarata al Woodstock 5 stelle a Cesena con lui, i suoi amici e il già citato Luca); Alex il Neri e Aurora; Felice; Simone Lagrange; il Tambe e Clio; Sapio; il Mariotti.

Come non ricordare poi Eugenio (già noto come il Panda), anche se lo vedo una volta ogni morte di papa; l'aula studenti, che ho ingiustamente abbandonato, a cominciare dai miei mentori Paoloaceto e Springfield, e continuando con Agnese, Bob, Daniela, Matteo, Gloria, il Capu, il Monta, Giulia, Michele, Carla, Cecilia, Sapiens, Marcella, Aurora, Mufasa, il Sergente; gli altri normalisti Maria, Eleonora, Roberto, Denis, e tanti altri: il Codenotti, il Grane, Tiz, Alberto, Danny, Daniele, Puppupulu, Gaia, Isotta, Lilla, Giulia, Silvia, Cecilia, Hjalmar, Scala, Veronica, Alessandra, Leo Robol, Milena, Luigi, Sabino eccetera eccetera. Menzione d'onore per Alessandro Sisto: devo a lui il consiglio di contattare il prof. Lackenby, che sarà il mio relatore ad Oxford. E parlando di relatori, a due anni di distanza vorrei rinnovare la mia riconoscenza a Roberto Frigerio.

Se sei stanco di leggere mi dispiace, ma non ho ancora finito, perché esauriti i matematici mi rimane sempre da ringraziare l'altro mondo, non meno importante, dei normalisti. Qui il primo grazie va probabilmente a Casa Rainaldo ("Villa di Lato"): il sorridente ed enigmatico David, Chioma "ngulo!", l'inalberato Feller, Farace che ne sa a pacchi; e le rispettive "scucchie". una massoneria più che una casa, in cui chi è entrato non è mai più stato quello di prima, e che da un anno ci regala aneddoti esilaranti ai limiti del reale.

Non ho citato Mattiacarlo per poterlo inserire in un altro gruppo, insieme a Greta, a Lucatosti, a Süzzi. A tutti questi volevo dire che, anche se sono ancora convinti che il viaggio a Oslo per me sia stato uno shock irreparabile, in realtà sarà un ricordo che conserverò volentieri.

Vado avanti con l'evanescente Ettore, Matteo Vezza, Sara: conoscenze che ho approfondito grazie a Michele, e sono lieto di averlo fatto. Gli altri più o meno abitudinari del caffè dopo pranzo: il tranquillissimo Bolzo, l'eigen Ilario, il Cera (che ringrazio anche per quella giornata in gruppo sulle montagne liguri), Enri, Simo, Claudia, Nico Grilli, Laura, Brian De Palma, Yak; e poi gli altri del mio anno: Mister Miazzi, l'Alf con le sue infinite fisse, Fede, il Soba, il Reca, il Barat.

E poi il popolo dell'Ottantasette: Kerrison, Deriusrascian, Marco Peruzzi, ACM, Ugo, Gabbo, Lorenzo, Sophie, Simo, Mara, Cavazzani. Cercare di mettere ordine fra tutte le persone che ho citato non ha veramente senso: con loro ho passato giornate di studio alleggerite dalla compagnia, altre piccole scene di quotidianità, grigliate sul terrazzo del Carducci, pizzate di proporzioni enormi, secchiate più o meno della stessa enormità, feste al Faedo.

Infine, meritano un grazie gli amici di Elisa: Elisabetta, Federica, Benedetta (x2), Ciccio, Dario, Angelini, Nino, Benedetto; quelli dei biolochimici, in particolare Anastasia; gli schermidori: Cino e Martina; i santannini: Lenzi, Wolf, Lorenzo, la Giunti, Benetton, Laura e chi più ne ha più ne metta.

In questa infinita carrellata di persone, vista l'ora in cui scrivo, avrò sicuramente dimenticato qualcuno di importante; quando me ne accorgerò, mi chiederò come ho potuto.

Ancora grazie a tutti.

Antonio (Decan, Anthony Capuano, 'u prufssor, Antoniobiennio, biondo)