# Calcolo diff. e p.ti critici

## Es. 4

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\mathrm{Dom}(f)=\mathbb{R}^2$$

$$\frac{\partial f}{\partial x}(x,y)=\frac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2}=$$

$$=\frac{y^3-x^2y}{(x^2+y^2)^2} \quad \text{per } (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{x^3-xy^2}{(x^2+y^2)^2} \quad \text{per } (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=0$$

$$\frac{\partial f}{\partial y}(0,0)=0 \quad \text{analogamente}$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^3 - x^2 y}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3 - x y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-2xy(x^2+y^2)^2 - (y^3 - x^2 y)\,4(x^2+y^2)x}{(x^2+y^2)^4}$$

$$\text{per } (x,y) \neq (0,0)$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(t,0) - \frac{\partial f}{\partial x}(0,0)}{t} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \frac{\partial f}{\partial y}(0,0)}{t} =$$

$$= \lim_{t \to 0} \frac{t^3/t^4 - 0}{t} = +\infty$$

ESERCIZIO. Mostrare che f non è diff. in (0,0)

## Es. 8

$$f(x,y,z) = \sqrt{x^2+y^2+z^2} \qquad Dom(f) = \mathbb{R}^3$$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}} \quad \text{per } (x,y,z) \neq \underline{0}$$

$$\frac{\partial f}{\partial x}(\underline{0}) = \lim_{t\to 0} \frac{\sqrt{t^2}-0}{t} = \lim_{t\to 0} \frac{|t|}{t} \quad \nexists$$

Analogamente per le altre der. parziali

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \left(\sqrt{x^2+y^2+z^2} - x\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\frac{1}{x^2+y^2+z^2} =$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{per } (x,y,z) \neq \underline{0}$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y,z) = \left(-x\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\frac{1}{x^2+y^2+z^2} =$$

$$= -\frac{xy}{(x^2+y^2+z^2)^{3/2}} \quad \text{per } (x,y,z) \neq \underline{0}$$

$$\frac{\partial^2 f}{\partial z\,\partial x}(x,y,z) = -\frac{xz}{(x^2+y^2+z^2)^{3/2}} \quad \text{per } (x,y,z) \neq \underline{0}$$

## Es. 2

$f(x,y) = x^2 + y^2 + y \qquad Dom(f) = \mathbb{R}^2$

$\nabla f(x,y) = (2x, 2y+1)$

$H_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ sempre def. $> 0$

$\nabla f(x,y) = \underline{0} \iff \begin{cases} 2x = 0 \\ 2y+1 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = -\frac{1}{2} \end{cases}$

p.t. critico $(0, -\frac{1}{2})$ ed è minimo loc.

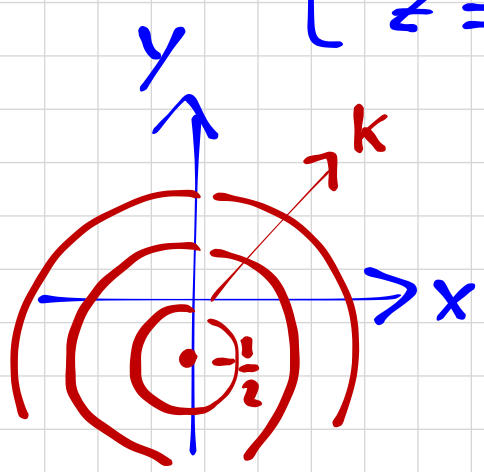
$$f(x,y) - f\left(0, -\tfrac{1}{2}\right) = x^2 + y^2 + y + \tfrac{1}{4} =$$

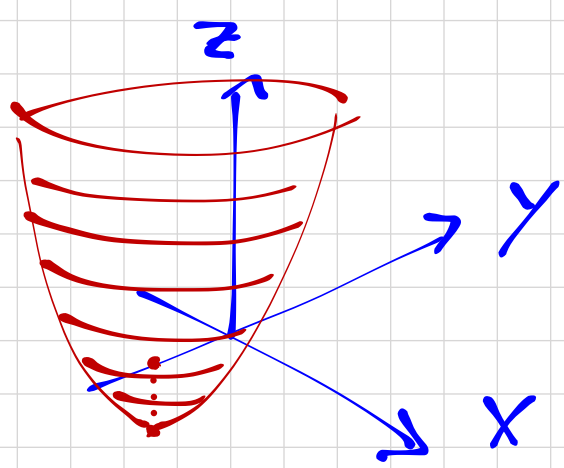
$$= x^2 + \left(y + \tfrac{1}{2}\right)^2 \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$$

$\Rightarrow$ $(0, -\frac{1}{2})$ è p.t. di minimo ass.

$\{z = k\} \cap Graf(f)$ $\begin{cases} z = x^2 + y^2 + y \\ z = k \end{cases} \iff$

$\iff \begin{cases} x^2 + y^2 + y = k \\ z = k \end{cases}$



## Es. 5

$$f(x,y) = \sqrt{x^2+y^2}\, e^{x+y} \qquad Dom(f) = \mathbb{R}^2$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \cdot e^{x+y} + \sqrt{x^2+y^2}\, e^{x+y} =$$

$$= e^{x+y}\, \frac{x^2+y^2+x}{\sqrt{x^2+y^2}} \qquad \text{per } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y}(x,y) = e^{x+y}\, \frac{y^2+y+x^2}{\sqrt{x^2+y^2}} \qquad \text{per } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{\sqrt{t^2}\, e^t}{t} = \lim_{t\to 0} \frac{|t|}{t} e^t \quad \nexists$$

$\frac{\partial f}{\partial y}(0,0)$ $\nexists$ $\Rightarrow$ in $(0,0)$ non esistono le der. parziali

(i) $\nabla f(x,y) = \underline{0} \iff \begin{cases} x^2+y^2+x = 0 \\ y^2+y+x^2 = 0 \end{cases} \iff$

$$\iff \begin{cases} x = y \\ 2x^2+x = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = -\frac{1}{2} \\ y = -\frac{1}{2} \end{cases}$$

non acc.

$f$ ti critico $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x+y}\left(\frac{x^2+y^2+x}{\sqrt{x^2+y^2}} + \frac{(2x+1)\sqrt{x^2+y^2} - \frac{(x^2+y^2+x)x}{\sqrt{x^2+y^2}}}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = e^{x+y}\left(\frac{x^2+y^2+y}{\sqrt{x^2+y^2}} + \frac{(2y+1)\sqrt{x^2+y^2} - \frac{(x^2+y^2+y)x}{\sqrt{x^2+y^2}}}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = e^{x+y}\left(\frac{x^2+y^2+x}{\sqrt{x^2+y^2}} + \frac{2y\sqrt{x^2+y^2} - \frac{(x^2+y^2+x)y}{\sqrt{x^2+y^2}}}{x^2+y^2}\right)$$

$$H_f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{e} \\ -\frac{\sqrt{2}}{e} & 0 \end{pmatrix}$$

$$\det H_f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -\frac{2}{e^2}$$

$$\operatorname{Tr} H_f\left(-\frac{1}{2}, -\frac{1}{2}\right) = 0$$

$\Rightarrow$ p.l. caratt.

$$\lambda^2 - \frac{2}{e^2} = 0$$

$\Rightarrow$ due autoval, reali discordi

$\Rightarrow \left(-\frac{1}{2}, -\frac{1}{2}\right)$ p.t. di sella

(ii) Vediamo in $(0,0)$ (p.to di non diff.)

$$f(0,0) = 0$$

$$f(x,y) - f(0,0) = \underbrace{\sqrt{x^2+y^2}}_{\geq 0}\,\underbrace{e^{x+y}}_{>0} \geq 0$$

$\Rightarrow$ $(0,0)$ p.to di min. assoluto

## Es. 13

$$f(x,y) = \sin(x+y) - \sin(x-y)$$

$$f : X \to \mathbb{R}, \quad X = \{0 < x < 2\pi \wedge 0 < y < 2\pi\}$$

$$\nabla f(x,y) = \begin{pmatrix} \cos(x+y) - \cos(x-y) \\ \cos(x+y) + \cos(x-y) \end{pmatrix}$$

$$\nabla f(x,y) = \underline{0} \iff \begin{cases} \cos(x+y) - \cos(x-y) = 0 \\ \cos(x+y) + \cos(x-y) = 0 \end{cases}$$

$$\iff \begin{cases} \cos(x+y) = 0 \\ \cos(x-y) = 0 \end{cases} \iff$$

$$\iff \begin{cases} \cos x \cos y - \sin x \sin y = 0 \\ \cos x \cos y + \sin x \sin y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \cos x \cos y = 0 \\ \sin x \sin y = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos x = 0 \\ \sin x = 0 \end{cases} \vee \begin{cases} \cos x = 0 \\ \sin y = 0 \end{cases} \vee$$

$$\vee \begin{cases} \cos y = 0 \\ \sin x = 0 \end{cases} \vee \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \pi/2 \\ y = 0 \end{cases} \notin X \quad \vee \begin{cases} x = 3/2\,\pi \\ y = 0 \end{cases} \notin X \quad \vee$$

$$\vee \begin{cases} x = \pi/2 \\ y = \pi \end{cases} \vee \begin{cases} x = 3/2\,\pi \\ y = \pi \end{cases} \vee$$

$$\vee \begin{cases} y = \pi/2 \\ x = 0 \end{cases} \notin X \quad \vee \begin{cases} y = 3/2\,\pi \\ x = 0 \end{cases} \notin X$$

$$\vee \begin{cases} y = \pi/2 \\ x = \pi \end{cases} \vee \begin{cases} y = 3/2\,\pi \\ x = \pi \end{cases}$$

## ESERCIZIO

Finire la classific. dei p.ti critici