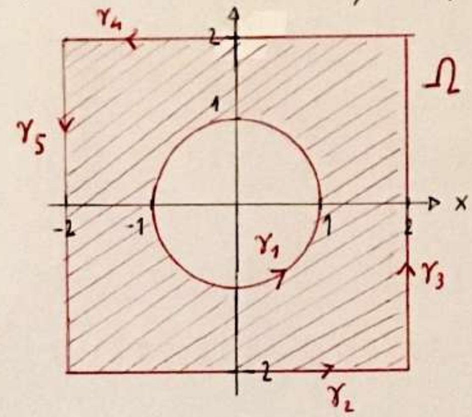
(9) $f(x,y) = x^2 - xy + 1$ $\qquad \Omega = \{(x,y) \mid -2 \le x \le 2,\ -2 \le y \le 2,\ x^2 + y^2 \ge 1\}$



① parte interna $\mathring{\Omega}$

$$\nabla f(x,y) = \underline{0} \iff \begin{cases} 2x - y = 0 \\ -x = 0 \end{cases}$$

$$(x,y) = (0,0) \notin \mathring{\Omega}$$

② frontiera $\partial\Omega = \bigcup_{i=1}^{5} \mathrm{Imm}(\gamma_i)$

- $\mathrm{Imm}\,\gamma_1 = \{(x,y) \mid x^2 + y^2 - 1 = 0\}$ $\qquad g(x,y) = x^2 + y^2 - 1$ e $\nabla g(x,y) \neq \underline{0}$ $\forall (x,y) \in \mathrm{Imm}\,\gamma_1$

$$\begin{cases} 2x - y = \lambda \cdot 2x \\ -x = \lambda \cdot 2y \\ x^2 + y^2 - 1 = 0 \end{cases} \qquad \begin{cases} y = 2x(1-\lambda) \\ x = -\lambda \cdot 2y \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$\begin{cases} y = -4\lambda y(1-\lambda) \\ x = -2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases} \iff \cancel{\begin{cases} y = 0 \\ x = 0 \\ 0 = 1 \end{cases}} \ \vee \ \begin{cases} 1 = -4\lambda(1-\lambda) \\ x = -2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases} \quad 4\lambda^2 - 4\lambda - 1 = 0$$

$$\lambda = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{2}$$

$$\begin{cases} \lambda = \frac{1}{2} + \frac{1}{2}\sqrt{2} \\ x = (-1-\sqrt{2})y \\ (1 + 2 + 2\sqrt{2})y^2 + y^2 - 1 = 0 \end{cases} \ \vee \ \begin{cases} \lambda = \frac{1}{2} - \frac{1}{2}\sqrt{2} \\ x = (-1+\sqrt{2})y \\ (1 + 2 - 2\sqrt{2})y^2 + y^2 - 1 = 0 \end{cases}$$

$$y^2 = \frac{1}{2(2+\sqrt{2})} \qquad\qquad y^2 = \frac{1}{2(2-\sqrt{2})}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$y = \pm\sqrt{\frac{1}{2(2+\sqrt{2})}} = \pm\alpha \qquad\qquad y = \pm\sqrt{\frac{1}{2(2-\sqrt{2})}} = \pm\beta$$

$\underline{P}_1 = \left((-1-\sqrt{2})\alpha,\ \alpha\right)$ $\qquad\qquad$ $\underline{P}_3 = \left((-1+\sqrt{2})\beta,\ \beta\right)$

$\underline{P}_2 = \left((1+\sqrt{2})\alpha,\ -\alpha\right)$ $\qquad\qquad$ $\underline{P}_4 = \left((1-\sqrt{2})\beta,\ -\beta\right)$

$$f(\underline{P}_1) = (3+2\sqrt{2})\alpha^2 - (-1-\sqrt{2})\alpha^2 + 1 = (4+3\sqrt{2}) \cdot \frac{1}{2(2+\sqrt{2})} + 1 \simeq 2.2071$$

$$f(\underline{P}_2) = f(\underline{P}_1)$$

$$f(\underline{P}_3) = (3-2\sqrt{2})\beta^2 - (-1+\sqrt{2})\beta^2 + 1 = (4-3\sqrt{2})\frac{1}{2(2-\sqrt{2})} + 1 \simeq 0.79289$$

$$f(\underline{P}_4) = f(\underline{P}_3)$$

$\gamma_2(t) = (t,-2),\ t \in [-2,2]$ $\qquad g_2(t) = t^2+2t+1 \qquad g_2'(t) = 2t+2 = 0 \iff t=-1$

$\gamma_3(t) = (2,t),\ t \in [-2,2]$ $\qquad g_3(t) = 4-2t+1 \qquad g_3'(t) = -2 \neq 0 \quad \forall t$

$\gamma_4(t) = (-t,2),\ t \in [-2,2]$ $\qquad g_4(t) = t^2+2t+1 \qquad g_4'(t) = 0 \iff t=-1$

$\gamma_5(t) = (-2,-t),\ t \in [-2,2]$ $\qquad g_5(t) = 4-2t+1 \qquad g_5'(t) = -2 \neq 0 \quad \forall t$

$\underline{q}_2 = (-1,-2) \qquad f(\underline{q}_2) = 1-2+1 = 0$

$\underline{q}_4 = (1,2) \qquad f(\underline{q}_4) = 1-2+1 = 0$

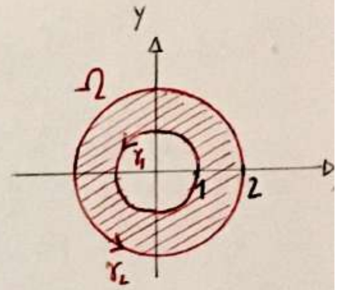
VERTICI

$f(2,2) = 1 \qquad f(2,-2) = 9$

$f(-2,2) = 9$

$f(-2,-2) = 1$

$$\Rightarrow \max_{\Omega} f = 9 \quad , \quad \min_{\Omega} f = 0$$



(10) $f(x,y) = x^2 - y^3 \qquad \Omega = \{ (x,y) \mid 1 \leq x^2+y^2 \leq 4 \}$

① parte interna $\mathring{\Omega}$

$$\nabla f = \underline{0} \iff \begin{cases} 2x = 0 \\ -3y^2 = 0 \end{cases} \qquad (x,y) = (0,0) \notin \mathring{\Omega}$$

② $\partial\Omega = \mathrm{Imm}\,\gamma_1 \cup \mathrm{Imm}\,\gamma_2 = \Gamma_1 \cup \Gamma_2$

$\Gamma_1 = \{(x,y) \mid x^2+y^2 = 1\} \quad \nabla g_1 \neq \underline{0}$ su Γ_1

$\Gamma_2 = \{(x,y) \mid x^2+y^2 = 4\} \quad \nabla g_2 \neq \underline{0}$ su Γ_2

$$\begin{cases} 2x = \lambda \cdot 2x \\ -3y^2 = \lambda \cdot 2y \\ x^2+y^2-1 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = \pm 1 \\ 3y^2 = -2\lambda y \end{cases} \vee \begin{cases} \lambda = 1 \\ " \\ " \end{cases}$$

$$\begin{cases} x=0 \\ y=1 \\ \lambda = -\frac{3}{2} \end{cases} \vee \begin{cases} x=0 \\ y=-1 \\ \lambda = \frac{3}{2} \end{cases} \vee \begin{cases} \lambda = 1 \\ y = 0 \\ x = \pm 1 \end{cases} \vee \begin{cases} \lambda = 1 \\ y = -\frac{2}{3} \\ x = \pm\frac{1}{3}\sqrt{5} \end{cases}$$

$\underline{P}_1 = (0,1) \qquad f(\underline{P}_1) = -1$

$\underline{P}_2 = (0,-1) \qquad f(\underline{P}_2) = 1$

$\underline{P}_3 = (1,0) \qquad f(\underline{P}_3) = 1$

$\underline{P}_4 = (-1,0) \qquad f(\underline{P}_4) = 1$

$\underline{P}_5 = \left(\frac{1}{3}\sqrt{5}, -\frac{2}{3}\right) \qquad f(\underline{P}_5) = 7/27$

$\underline{P}_6 = \left(-\frac{1}{3}\sqrt{5}, -\frac{2}{3}\right) \qquad f(\underline{P}_6) = 7/27$

$$\begin{cases} 2x = \lambda 2x \\ -3y^2 = \lambda 2y \\ x^2+y^2-4 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = \pm 1 \\ -3y^2 = \lambda 2y \end{cases} \vee \begin{cases} \lambda = 1 \\ -3y^2 = 2y \\ x^2+y^2 = 4 \end{cases}$$

$$\begin{cases} x=0 \\ y=\pm 2 \\ \lambda = \mp \end{cases} \vee \begin{cases} \lambda = 1 \\ y = 0 \\ x = \pm 2 \end{cases} \vee \begin{cases} \lambda = 1 \\ y = -\frac{2}{3} \\ x = \pm\sqrt{2}\cdot\frac{4}{3} \end{cases}$$

$\underline{q}_1 = (0,2) \qquad f(\underline{q}_1) = -8$

$\underline{q}_2 = (0,-2) \qquad f(\underline{q}_2) = 8$

$\underline{q}_3 = (2,0) \qquad f(\underline{q}_3) = 4$

$\underline{q}_4 = (-2,0) \qquad f(\underline{q}_4) = 4$

$\underline{q}_5 = \left(\frac{4}{3}\sqrt{2}, -\frac{2}{3}\right) \qquad f(\underline{q}_5) = 88/27$

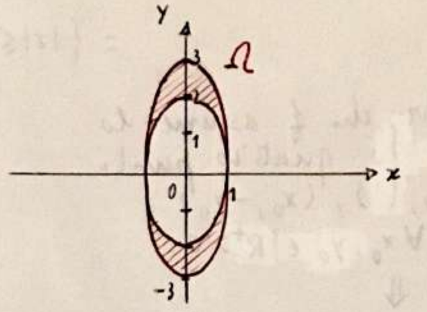
$\underline{q}_6 = \left(-\frac{4}{3}\sqrt{2}, -\frac{2}{3}\right) \qquad f(\underline{q}_6) = 88/27$

$$\Longrightarrow \max_{\Omega} f = 8 \quad , \quad \min_{\Omega} f = -8$$

(11) $f(x,y) = \log(x^2+y^2)$, $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2/4 \geq 1 \ , \ x^2 + y^2/9 \leq 1\}$

① in $\mathring{\Omega}$

$$\nabla f(x,y) = \underline{0} \iff \begin{cases} \frac{2x}{x^2+y^2} = 0 \\ \frac{2y}{x^2+y^2} = 0 \end{cases} \qquad (0,0) \notin \mathring{\Omega}$$



② $\partial\Omega = \operatorname{Imm}\gamma_1 \cup \operatorname{Imm}\gamma_2$

$\gamma_1(t) = (\cos t, 2\sin t)$
$t \in [0, 2\pi)$

$g_1(t) = (f \circ \gamma_1)(t) = \log(\cos^2 t + 4\sin^2 t) =$
$= \log(1 + 3\sin^2 t)$

$g_1'(t) = \dfrac{6 \sin t \cos t}{1 + 3\sin^2 t} = 0 \iff t = 0 \lor t = \frac{\pi}{2} \lor t = \pi \lor t = \frac{3}{2}\pi$

$\underline{P}_1 = (1,0)$, $\underline{P}_2 = (0,2)$, $\underline{P}_3 = (-1,0)$, $\underline{P}_4 = (0,-2)$

$\gamma_2(t) = (\cos t, 3\sin t)$
$t \in [0, 2\pi)$

$g_2(t) = (f \circ \gamma_2)(t) = \log(1 + 8\sin^2 t)$

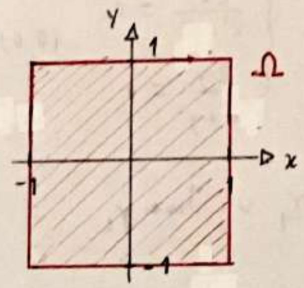
$g_2'(t) = \dfrac{16 \sin t \cos t}{1 + 8\sin^2 t} = 0 \iff t = 0 \lor t = \frac{\pi}{2} \lor t = \pi \lor t = \frac{3}{2}\pi$

$\underline{Q}_1 = (0,3)$, $\underline{Q}_2 = (0,-3)$

$$\left.\begin{array}{ll} f(\underline{P}_1) = 0 & f(\underline{Q}_1) = \log 9 \\ f(\underline{P}_2) = \log 4 & \\ f(\underline{P}_3) = 0 & f(\underline{Q}_2) = \log 9 \\ f(\underline{P}_4) = \log 4 & \end{array}\right\} \Rightarrow \begin{array}{l} \max_{\Omega} f = \log 9 = 2\log 3 \\ \min_{\Omega} f = 0 \end{array}$$

(12) $f(x,y) = \sin|xy|$, $\Omega = \{(x,y) \in \mathbb{R}^2 \mid \max\{|x|,|y|\} \leq 1\} =$

$= \{|x| \leq |y| \wedge |y| \leq 1\} \cup \{|x| \geq |y| \wedge |x| \leq 1\}$



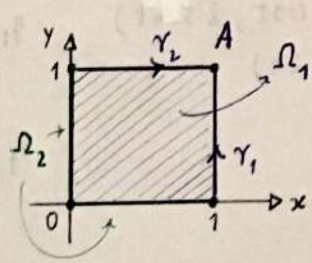
Si deve osservare che f assume lo stesso valore sui quattro punti (x_0, y_0), $(-x_0, y_0)$, $(x_0, -y_0)$ e $(-x_0, -y_0)$ $\forall x_0 > 0$ $\forall y_0 > 0$

$\Downarrow$

È sufficiente studiare f su

$\Omega \cap \{x > 0, y > 0\} =: \Omega_1$

$\{x = 0, 0 \leq y \leq 1\} \cup \{y = 0, 0 \leq x \leq 1\} =: \Omega_2$



① $f\big|_{\Omega_2} = 0$

② in $\mathring{\Omega}_1$ $\quad f(x,y) = \sin xy$ in $\mathring{\Omega}_1$ è >0

$\nabla f(x,y) = \underline{0} \iff \begin{cases} \cancel{y} \cos xy = 0 \\ \cancel{x} \cos xy = 0 \end{cases} \iff \cos xy = 0$ e questo non avviene mai perché in $\mathring{\Omega}_1$

$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \implies 0 < xy < 1 < \frac{\pi}{2}$

③ $\partial \Omega_1 = \operatorname{Imm} \gamma_1 \cup \operatorname{Imm} \gamma_2$

$\gamma_1(t) = (1, t)$ $\quad g_1(t) = (f \circ \gamma_1)(t) = \sin t$
$t \in [0,1]$ $\quad g_1'(t) = 0 \iff \cos t = 0$ e questo non avviene in $[0,1]$

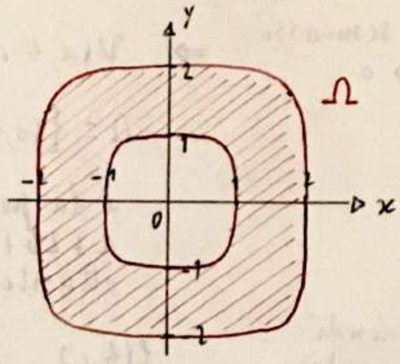
$\gamma_2(t) = (t, 1)$ $\quad$ analogo
$t \in [0,1]$

$f\big|_{\Omega_2} = 0 \quad f(\underline{A}) = \sin 1 \implies \max_{\Omega} f = \sin 1$

$\min_{\Omega} f = 0$

(13) $f(x,y) = x^2 - y^2$ $\qquad \Omega = \{ (x,y) \in \mathbb{R}^2 \mid 1 \leq x^4 + y^4 \leq 16 \}$

① in $\mathring{\Omega}$



$$\nabla f(x,y) = \underline{0} \iff \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (0,0) \notin \Omega$$

② $\partial\Omega = \Gamma_1 \cup \Gamma_2$

$\Gamma_1 := \{ x^4 + y^4 - 1 = 0 \}$

$\Gamma_2 := \{ x^4 + y^4 - 16 = 0 \}$

$$\begin{aligned} g_1(x,y) &= x^4 + y^4 - 1 \\ g_2(x,y) &= x^4 + y^4 - 16 \end{aligned} \Rightarrow \underset{\text{su } \Gamma_1}{\nabla g_1(x,y) \neq 0}, \ \underset{\text{su } \Gamma_2}{\nabla g_2(x,y) \neq 0}$$

$$\Gamma_1 : \begin{cases} 2x = \lambda \cdot 4x^3 \\ 2y = \lambda \cdot 4y^3 \\ x^4 + y^4 = 1 \end{cases} \iff \begin{cases} x(1 - \lambda \cdot 2x^2) = 0 \\ y(1 - \lambda \cdot 2y^2) = 0 \\ x^4 + y^4 = 1 \end{cases} \iff \begin{cases} x = 0 \\ y^2 = \frac{1}{2\lambda} \\ y = \pm 1 \end{cases} \vee \begin{cases} x^2 = \frac{1}{2\lambda} \\ y = 0 \\ x = \pm 1 \end{cases} \vee \begin{cases} x^2 = \frac{1}{2\lambda} \\ y^2 = \frac{1}{2\lambda} \\ \lambda^2 = \frac{1}{2} \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = \pm 1 \\ \lambda = \frac{1}{2} \end{cases} \vee \begin{cases} x = \pm 1 \\ y = 0 \\ \lambda = \frac{1}{2} \end{cases} \vee \begin{cases} x = \pm \frac{1}{\sqrt[4]{2}} \\ y = \pm \frac{1}{\sqrt[4]{2}} \\ \lambda = \frac{1}{\sqrt{2}} \end{cases}$$

$$\underline{P}_{1,2} = (0, \pm 1), \quad \underline{P}_{3,4} = (\pm 1, 0), \quad \underline{P}_{5,6,7,8} = \left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}} \right)$$

Γ_2 : in modo del tutto analogo $\quad \underline{Q}_{1,2} = (0, \pm 2), \ \underline{Q}_{3,4} = (\pm 2, 0), \ \underline{Q}_{5,6,7,8} = (\pm \sqrt[4]{8}, \pm \sqrt[4]{8})$

$$\left.\begin{aligned} f(\underline{P}_{1,2}) &= -1 & f(\underline{Q}_{1,2}) &= -4 \\ f(\underline{P}_{3,4}) &= 1 & f(\underline{Q}_{3,4}) &= 4 \\ f(\underline{P}_{5,6,7,8}) &= 0 & f(\underline{Q}_{5,6,7,8}) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \max_{\Omega} f &= 4 \\ \min_{\Omega} f &= -4 \end{aligned}$$