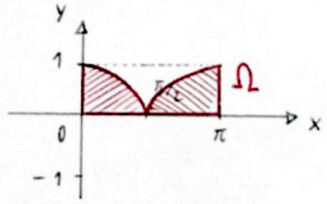
(8) $\iint_\Omega xy\sin x \, dx\,dy =$ $\Omega = \{(x,y) \mid 0 \leq x \leq \pi, \ 0 \leq y \leq |\cos x|\}$

$$= \int_0^\pi \left(\int_0^{|\cos x|} xy\sin x \ dy \right) dx = \int_0^\pi x\sin x \cdot \frac{\cos^2 x}{2} \, dx =$$



$$= \int_0^\pi \frac{1}{2} x (\sin x - \sin^3 x) dx = \pi/6$$

(9) $\iint_\Omega y \, dx\,dy$ $\Omega = \{(x,y) \mid -\sqrt{x^3(1-x)^3} \leq y \leq \sqrt{x^3(1-x)^3}\}$

$g(x) := \sqrt{x^3(1-x)^3}$
- campo di esistenza $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
- $g(x) \geq 0 \quad \forall x \in [0,1]$

$$\int_0^1 \int_{-g(x)}^{g(x)} y \, dy\,dx = 0$$ per simmetria (o anche per calcolo diretto)

(15) $\iint_\Omega (x^2+y^2)^\alpha \, dx\,dy$ $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}, \ \alpha \in \mathbb{R}$

$$\iint_\Omega (x^2+y^2)^\alpha \, dx\,dy = \int_0^{2\pi}\int_0^1 \rho^{2\alpha} \cdot \rho \, d\rho \, d\vartheta = 2\pi \int_0^1 \rho^{2\alpha+1} \, d\rho =$$

$$= 2\pi \int_0^1 \frac{1}{\rho^{-2\alpha-1}} \, d\rho$$ è un integrale improprio $\mathcal{J}$ in $\rho = 0$, che converge se e solo se $-2\alpha - 1 < 1 \iff \alpha > -1$

- $\alpha > -1 \Rightarrow \mathcal{J} = \cancel{2}\pi \left. \frac{\rho^{2(\alpha+1)}}{\cancel{2}(\alpha+1)} \right|_0^1 = \frac{\pi}{1+\alpha}$
- $\alpha \leq -1 \Rightarrow \mathcal{J} = +\infty$.

(16) $\iint_\Omega x^2 y^2 \, dx\,dy$ $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4\}$

$2\vartheta = \varphi$
$2\,d\vartheta = d\varphi$

$$\iint_\Omega x^2y^2 \, dx\,dy = \int_0^{2\pi}\int_1^2 \rho^2\cos^2\vartheta \, \rho^2\sin^2\vartheta \cdot \rho \, d\rho \, d\vartheta =$$

$$= \int_1^2 \rho^5 \, d\rho \cdot \int_0^{2\pi} \sin^2\vartheta \cos^2\vartheta \, d\vartheta =$$

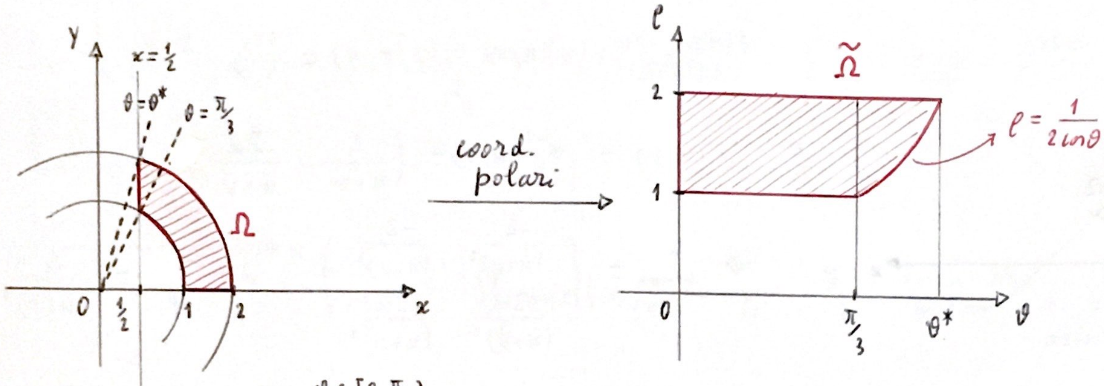
$$\begin{cases} \int \sin^2\vartheta\cos^2\vartheta \, d\vartheta = \frac{1}{4}\int \sin^2 2\vartheta \, d\vartheta = \\ = \frac{1}{4}\cdot\frac{1}{2}\int \sin^2\varphi \, d\varphi = \\ = \frac{1}{8} \, \frac{\varphi - \sin\varphi\cos\varphi}{2} = \\ = \frac{2\vartheta - \sin 2\vartheta\cos 2\vartheta}{16} \end{cases}$$

$$= \left. \frac{1}{6}\rho^6 \right|_1^2 \cdot \left. \frac{2\vartheta - \sin 2\vartheta \cos 2\vartheta}{16} \right|_0^{2\pi} =$$

$$= \frac{\cancel{64}^{\cancel{16}\,8}}{\cancel{6}_3} \cdot \frac{1}{\cancel{4}}\pi - \frac{1}{6} \cdot \frac{\pi}{4} = \frac{21}{8}\pi.$$

(6)

$$\iint_{\Omega} \frac{x+y}{x^2+y^2}\,dx\,dy, \quad \Omega = \{(x,y) \in \mathbb{R}^2 \mid x \geq \tfrac{1}{2},\ y \geq 0,\ 1 \leq x^2+y^2 \leq 4\}$$



$$\vartheta \in [0, \tfrac{\pi}{2})$$

$$x \geq \frac{1}{2} \iff \rho\cos\theta \geq \frac{1}{2} \iff \rho \geq \frac{1}{2\cos\theta}$$

$$\frac{1}{2\cos\theta} = 1 \iff \theta = \pi/3$$

$$\frac{1}{2\cos\theta} = 2 \iff \vartheta = \vartheta^* \ \text{con} \ \cos\vartheta = \tfrac{1}{4}$$

$$\Rightarrow \tilde{\Omega} = \{(\rho,\theta) \mid 1 \leq \rho \leq 2,\ 0 \leq \theta \leq \pi/3\} \cup \{(\rho,\vartheta) \mid \frac{1}{2\cos\theta} \leq \rho \leq 2,\ \pi/3 \leq \vartheta \leq \vartheta^*\}$$

$$\iint_{\Omega} \frac{x+y}{x^2+y^2}\,dx\,dy = \iint_{\tilde{\Omega}} \frac{\cancel{\rho}(\cos\theta+\sin\theta)}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho\,d\theta =$$

$$= \int_0^{\pi/3}\int_1^2 (\sin\theta+\cos\theta)\,d\rho\,d\theta + \int_{\pi/3}^{\theta^*}\int_{\frac{1}{2\cos\theta}}^{2} (\sin\theta+\cos\theta)\,d\rho\,d\theta =$$

$$= (-\cos\theta+\sin\theta)\Big|_0^{\pi/3} + \int_{\pi/3}^{\theta^*} \left(2-\frac{1}{2\cos\theta}\right)(\cos\theta+\sin\theta)\,d\theta =$$

$$= -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} + 1 + 2(\sin\vartheta - \cos\vartheta)\Big|_{\pi/3}^{\theta^*} - \frac{1}{2}\int_{\pi/3}^{\theta^*}(1+\operatorname{tg}\theta)\,d\theta =$$

$$\sin\vartheta^* = \frac{\sqrt{15}}{4} \qquad \operatorname{tg}\vartheta^* = \sqrt{15}$$

$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} + 2\left(\frac{\sqrt{15}}{4} - \frac{1}{4} - \frac{\sqrt{3}}{2} + \frac{1}{2}\right) - \frac{1}{2}\left[\vartheta - \log(\cos\theta)\right]\Big|_{\pi/3}^{\theta^*} =$$

$$= \cancel{\tfrac{1}{2}} + \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{15}}{2} - \cancel{\tfrac{1}{2}} - \sqrt{3} + 1 - \frac{1}{2}\left(\vartheta^* - \log\frac{1}{4} - \frac{\pi}{3} + \log\frac{1}{2}\right) =$$

$$= -\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{15}}{2} + 1 - \tfrac{1}{2}\operatorname{arctg}\sqrt{15} + \tfrac{1}{2}\log\tfrac{1}{4} + \tfrac{\pi}{6} - \tfrac{1}{2}\log\tfrac{1}{2} =$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \operatorname{arctg}\sqrt{15} + \sqrt{15} - \sqrt{3} + 2 - \log 2\right)$$