Decomposition of a finite perimeter set in terms of indecomposable components

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Abstract

We present a notion of connectedness for finite perimeter sets, first introduced by Federer in the more general setting of currents. In particular we say that a finite perimeter set E is decomposable if we can write E as union of in two non-negligible, disjoint subsets E_0, E_1 such that the sum of the perimeters of E_0, E_1 is equal to the perimeter of E. Conversely, we say that E is indecomposable if it is not decomposable.

The main result we present states that every finite perimeter set can be written as a disjoint countable union of indecomposable subsets. Finally we will see how these indecomposable components are linked with the classical connected components of an open set.

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Definition 1. A finite perimeter set $E \subseteq \mathbb{R}^n$ is said to be *decomposable* if there exist two disjoint finite perimeter set E_0 and E_1 such that $E = E_0 \cup E_1$ and $\mathrm{Per}(E) = \mathrm{Per}(E_0) + \mathrm{Per}(E_1)$.

On the other hand, E is said to be *indecomposable* if it is not decomposable.

Remark 2. In general, given a finite perimeter set E and a partition $\{E_k\}_{k\in\mathbb{N}}$ of E into finite perimeter sets, it holds that $\mathrm{Per}(E) \le \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k)$. Indeed we have that

$$\mathrm{Per}(E) = \sup_{\varphi\in C_c^\infty(\mathbb{R}^n)\,,\ \|\varphi\|_\infty\le 1} \int_E \operatorname{div}\varphi(x)\,\mathrm{d}x = \sup_{\varphi\in C_c^\infty(\mathbb{R}^n)\,,\ \|\varphi\|_\infty\le 1} \sum_{k\in\mathbb{N}} \int_{E_k} \operatorname{div}\varphi(x)\,\mathrm{d}x \le$$

$$\le \sum_{k\in\mathbb{N}} \sup_{\varphi\in C_c^\infty(\mathbb{R}^n)\,,\ \|\varphi\|_\infty\le 1} \int_{E_k} \operatorname{div}\varphi(x)\,\mathrm{d}x = \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k)\,.$$

The aim of these notes is to prove the following theorem about decomposing a finite perimeter set into indecomposable parts.

Theorem 3. *Given a finite perimeter set $E \subseteq \mathbb{R}^n$ with finite Lebesgue measure, there exists a countable partition $\{E_k\}_{k\in\mathbb{N}}$ of E into indecomposable finite perimeter sets such that* $\mathrm{Per}(E) = \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k)$.

Remark 4. The theorem is still true without the assumption on the finiteness of the Lebesgue measure of E. In that case, the same proof works with some adjustments, but we are not interested in entering in those technical details.

From now on we denote with E a finite perimeter set in $\mathbb{R}^n$ with finite Lebesgue measure. Moreover, given a partition $\{E_k\}_{k\in\mathbb{N}}$ of E we will suppose that it is in descending order of volume, i.e. $|E_k| \ge |E_{k+1}|$ for every $k \in \mathbb{N}$.

Definition 5. We say that a countable (possibly finite) partition $\{E_k\}_{k\in\mathbb{N}}$ of $E \subseteq \mathbb{R}^n$ is *good* if E_k is a finite perimeter set for every $k \in \mathbb{N}$ and $\mathrm{Per}(E) = \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k)$.

We start by proving some technical lemmas, which will give us also an idea of what a good partition is.

Lemma 6. *Let $\{E_k\}_{k\in\mathbb{N}}$ be a partition of E, then $\{E_k\}_{k\le\mathbb{N}}$ is good if and only if $\{E_k\}_{k\le K} \cup \{E \setminus \cup_{k\le K} E_k\}$ is a good partition of E for every $K \in \mathbb{N}$.*

Proof. Let us assume that $\{E_k\}_{k\in\mathbb{N}}$ is a good partition of E and fix $K \in \mathbb{N}$. It is sufficient to prove the equality between the perimeters. Since one inequality is always true, we have only to notice that

$$\mathrm{Per}(E) = \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k) \ge \sum_{k\le K} \mathrm{Per}(E_k) + \mathrm{Per}(\cup_{k>K} E_k) = \sum_{k\le K} \mathrm{Per}(E_k) + \mathrm{Per}(E \setminus \cup_{k\le K} E_k)\,.$$

Since the other inequality is always true we have proven the result.

Viceversa let us assume that $\{E_k\}_{k\le K} \cup \{E \setminus \cup_{k\le K} E_k\}$ is a good partition of E for every $K \in \mathbb{N}$. Thus we have that

$$\mathrm{Per}(E) = \lim_{k\to\infty} \sum_{k\le K} \mathrm{Per}(E_k) + \mathrm{Per}(\cup_{k>K} E_k) \ge \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k)\,,$$

which is the nontrivial inequality which concludes thee proof.

□

Lemma 7. *Given E, F finite perimeter sets it holds that*

1. $\partial_*(E \cup F) \subseteq \partial_* E \cup \partial_* F$*;*
2. $\partial_*(E \cap F) \subseteq \partial_* E \cup \partial_* F$*.*

Proof. It easier to prove the opposite containments switching to the complements. Let $x \in (\partial_* E \cup \partial_* F)^c = (\partial_* E)^c \cap (\partial_* F)^c$, then the density of x with respect to E and F is 0 or 1. Let us divide in two cases:

- if $\Theta(E, x) = \Theta(F, x) = 0$, then easily $\Theta(E \cup F, x) = 0$;
- if $\Theta(E, x) = 1$ or $\Theta(F, x) = 1$, then $\Theta(E \cup F, x) = 1$.

In both cases we have proven that $x \in \partial_*(E \cup F)$.

The proof of the point 2 proceeds in the same way, being careful to divide in the cases in which $\Theta(E, x) = \Theta(F, x) = 1$ and $\Theta(E, x) = 0$ or $\Theta(F, x) = 0$.

□

Lemma 8. *If E, F are disjoint finite perimeter sets, the following are equivalent:*

1. $\mathrm{Per}(E \cup F) = \mathrm{Per}(E) + \mathrm{Per}(F)$*, i.e.* $\{E, F\}$ *is a good partition for* $E \cup F$*;*
2. $\partial_*(E \cup F) = \partial_* E \sqcup \partial_* F$ *up to* $\mathcal{H}^{n-1}$*-negligible sets;*
3. $\partial_*(E \cup F) = \partial_* E \cup \partial_* F$ *up to* $\mathcal{H}^{n-1}$*-negligible sets;*
4. $\partial_* E$ *and* $\partial_* F$ *are disjoint up to* $\mathcal{H}^{n-1}$*-negligible sets.*

Proof. Thank to Lemma 7, it is always true that $\partial_*(E \cup F) \subseteq \partial_* E \cup \partial_* F$. Let us now proof the single implications.

1 $\iff$ 2 We know that $\mathrm{Per}(E \cup F) = \mathcal{H}^{n-1}(\partial_*(E \cup F))$ and $\mathrm{Per}(E) + \mathrm{Per}(F) = \mathcal{H}^{n-1}(\partial_* E) + \mathcal{H}^{n-1}(\partial_* F)$ and that $\partial_*(E \cup F) \subseteq \partial_* E \cup \partial_* F$. Therefore, it is easy to see that $\mathrm{Per}(E \cup F) = \mathrm{Per}(E) + \mathrm{Per}(F)$ if and only if $\partial_*(E \cup F) = \partial_* E \sqcup \partial_* F$ up to $\mathcal{H}^{n-1}$-negligible sets.

2 $\implies$ 3 Trivial.

3 $\implies$ 4 Given a finite perimeter set E, we know that $\partial_* E$ coincides with the points x such that $\Theta_n(E, x) = \frac{1}{2}$ up to a $\mathcal{H}^{n-1}$-negligible set. Thus $\partial_* E \cap \partial_* F = \{x \in \mathbb{R}^n : \Theta_n(E, x) = \Theta_n(F, x) = \frac{1}{2}\}$ up to $\mathcal{H}^{n-1}$-negligible sets. However, if $\Theta_n(E, x) = \Theta_n(F, x) = \frac{1}{2}$, then $\Theta_n(E \cup F, x) = 1$ since E and F are disjoint and consequently $x \notin \partial_* E$.

Hence we have proven that, up to $\mathcal{H}^{n-1}$-negligible sets, $\partial_* E \cap \partial_* F \subseteq (\partial_*(E \cup F))^c$, which implies that $\partial_* E \cap \partial_* F$ is $\mathcal{H}^{n-1}$-negligible, together with hypothesis.

4 $\implies$ 2 It is sufficient to prove that the set of x such that $x \in \partial_* E \setminus \partial_*(E \cup F)$ is $\mathcal{H}^{n-1}$-negligible. For $\mathcal{H}^{n-1}$-almost every such x it holds that $\Theta(E, x) = \frac{1}{2}$ and $\Theta(E \cup F, x)$ and $\Theta(F, x)$ are 0 or 1 (since $\partial_* E$ and $\partial_* F$ are disjoint). However all the possible cases are easily impossible.

□

Proposition 9. *Let $F \subseteq E$ be a finite perimeter set and let $\{E_k\}_{k\in\mathbb{N}}$ be a good partition of E, then $\{E_k \cap F\}_{k\in\mathbb{N}}$ is a good partition of F.*

Proof. First of all let us prove the result in the case of a partition with two sets E_0 and E_1. Thanks to Lemma 7, we know that $\partial_* E \subseteq \partial_* E_0 \cup \partial_* E_1$. Consequently we obtain that

$$\begin{aligned}\mathrm{Per}(E_0) + \mathrm{Per}(E_1) = \mathrm{Per}(E) = \mathcal{H}^{n-1}(\partial_* E) \leq \\ \leq \mathcal{H}^{n-1}(\partial_* E_0) + \mathcal{H}^{n-1}(\partial_* E) = \mathrm{Per}(E_0) + \mathrm{Per}(E_1)\,.\end{aligned}$$

Thus all the inequalities must be equalities and $\partial_* E = \partial_* E_0 \cup \partial_* E_1$ up to a $\mathcal{H}^{n-1}$-negligible set.

Again applying Lemma 7, we have that $\partial_* F \subseteq \partial_*(F \cap E_0) \cup \partial_*(F \cap E_1)$. We want to prove that the converse is still true up to $\mathcal{H}^{n-1}$-negligible sets, which is equivalent to $(\partial_* F)^c \subseteq (\partial_*(F \cap E_0) \cup \partial_*(F \cap E_1))^c$ up to $\mathcal{H}^{n-1}$-negligible sets.

Let $x \notin \partial_* F$, then we prove what stated dividing into two cases:

- if $\Theta_n(F, x) = 0$, then obviously $\Theta_n(F \cap E_0, x) = 0$ and $\Theta_n(F \cap E_1, x) = 0$, which means that $x \notin \partial_*(F \cap E_0) \cup \partial_*(F \cap E_1)$;

- if $\Theta_n(F, x) = 1$, then $\Theta_n(E, x) = 1$ and consequently $x \notin \partial_* E$. Thus, up to a $\mathcal{H}^{n-1}$-negligible set, $x \notin \partial_* E_0 \cup \partial_* E_1$. This implies that $x \notin \partial_*(F \cap E_i) \subseteq \partial_* F \cup \partial_* E_i$ for $i = 0, 1$, exploiting again Lemma 7.

Hence we have obtained that $\partial_* F = \partial_*(F \cap E_0) \cup \partial_*(F \cap E_1)$, where the union is disjoint up to $\mathcal{H}^{n-1}$-negligible sets. This prove the lemma in the case of the two subsets, indeed

$$\begin{aligned}\mathrm{Per}(E) = \mathcal{H}^{n-1}(\partial_* F) = \mathcal{H}^{n-1}(\partial_*(F \cap E_0)) + \mathcal{H}^{n-1}(\partial_*(F \cap E_1)) = \\ = \mathrm{Per}(F \cap E_0) + \mathrm{Per}(F \cap E_1)\,.\end{aligned}$$

Now let us prove be induction on the dimension of the partition that the result is true for every finite partition $\{E_0, \dots, E_K\}$. Thanks to the inductive hypothesis we have that

$$\begin{aligned}\mathrm{Per}(F \cap E_0) + \ldots + \mathrm{Per}(F \cap E_{K-1}) + \mathrm{Per}(F \cap E_K) = \\ = \mathrm{Per}(F \cap E_0) + \ldots + \mathrm{Per}(F \cap (E_{K-1} \cup E_K)) = \mathrm{Per}(F)\,,\end{aligned}$$

which proves also the finite case.

Finally, let us generalize the result to a countable partition $\{E_k\}_{k\in\mathbb{N}}$. Thanks to the finite case and the Lemma 6, for every $K \in \mathbb{N}$ it holds that

$$\mathrm{Per}(F) = \sum_{k \leq K} \mathrm{Per}(F \cap E_k) + \mathrm{Per}(F \cap (\cup_{k>K} E_k))$$

and passing to the limit we obtain

$$\mathrm{Per}(F) \geq \sum_{k\in\mathbb{N}} \mathrm{Per}(F \cap E_k)\,.$$

Since the other inequality is always true we have concluded our proof.

□

Corollary 10. *Let $\{E_k\}_{k\in\mathbb{N}}$ and $\{F_h\}_{h\in\mathbb{N}}$ be good partitions of E, then $\{E_k \cap F_h\}_{k,h\in\mathbb{N}}$ is a good partition of E.*

Proof. Thanks to Proposition 9 applied to $\{E_k\}_{k\in\mathbb{N}}$ and F_h for every $h \in \mathbb{N}$, we have that

$$\sum_{h,k\in\mathbb{N}} \mathrm{Per}(E_k \cap E_h) = \sum_{h\in\mathbb{N}} \mathrm{Per}(F_h) = \mathrm{Per}(E)\,,$$

which proves what we need. $\square$

The following proposition is the key element of the proof of Theorem 3. The idea is that in every good partition there exists an element which volume "large enough".

Proposition 11. *Let $\{E_k\}_{k\in\mathbb{N}}$ be a good partition of $E \subseteq \mathbb{R}^n$, then there exists a constant $C > 0$ such that*

$$\left(\max_{k\in\mathbb{N}} |E_k|\right)^{\frac{1}{n}} \geq C\frac{|E|}{\mathrm{Per}(E)}\,.$$

Proof. Call $m = \max_{k\in\mathbb{N}}|E_k|$, thanks to the isoperimetric inequality we have that

$$\mathrm{Per}(E) = \sum_{k\in\mathbb{N}} \mathrm{Per}(E_k) \geq C\sum_{k\in\mathbb{N}}|E_k|^{\frac{n-1}{n}} = C\sum_{k\in\mathbb{N}}|E_k|\cdot|E_k|^{-\frac{1}{n}} \geq Cm^{-\frac{1}{n}}|E|\,,$$

which implies the sought inequality. $\square$

Remark 12. Since we are assuming that the E_k are in descending order of volume, we have that $\max_{k\in\mathbb{N}}|E_k| = |E_0|$.

Proposition 13. *Given E finite perimeter set, there exists $E_0 \subseteq E$ indecomposable finite perimeter set such that $\mathrm{Per}(E) = \mathrm{Per}(E_0) + \mathrm{Per}(E \setminus E_0)$ and $|E_0|^{\frac{1}{n}} \geq C\frac{|E|}{\mathrm{Per}(E)} > 0$.*

Proof. Let us define

$$m = \inf\left\{\max_{k\in\mathbb{N}}|E_k| \;:\; \{E_k\}_{k\in\mathbb{N}} \text{ is a good partition of } E\right\}.$$

Notice that, thanks to Proposition 11, it holds that $m^{\frac{1}{n}} \geq C\frac{|E|}{\mathrm{Per}(E)}$.

The idea is to prove that the infimum in the definition is a minimum, indeed this would easily implies that the E_0 of the partition which realizes the minimum is indecomposable.

Let $\{E_k^{(j)}\}_{k\in\mathbb{N}}$ be a sequence of good partitions such that $|E_0^{(j)}| \downarrow m$. Thanks to Corollary 10 we can assume that they are partitions more and more fine, i.e. for every $j \geq 1$ and $k \in \mathbb{N}$ there exists $h \in \mathbb{N}$ such that $E_k^{(j)} \subseteq E_h^{(j-1)}$. Indeed, given $\{E_k\}_{k\in\mathbb{N}}$ and $\{E'_k\}_{k\in\mathbb{N}}$ good partitions of E, the lemma tells that $\{E_k \cap E_h\}_{k,h\in\mathbb{N}}$ is a good partition as well.

Moreover, up to subsequence, we can assume that $E_k^{(j)} \supseteq E_k^{(j+1)}$ definitely in j for every $k \in \mathbb{N}$. To prove this fact, first of all let us notice that we have an inferior bound for the volume of the kth element of a partition, indeed thanks to Proposition 11 it holds that

$$|E_k^{(j)}|^{\frac{1}{n}} \geq C\frac{|E| - \sum_{h<k}|E_h^{(j)}|}{\mathrm{Per}(E \setminus \cup_{h<k}E_h^{(j)})} \geq C\frac{|E| - \sum_{h<k}|E_h^{(j)}|}{\mathrm{Per}(E)}\,. \tag{0.1}$$

Now, let us concentrate on the element $E_0^{(j)}$ of the partitions. Thanks to the estimate in the Equation (0.1), the element of the first partition $\{E_k^{(0)}\}_{k\in\mathbb{N}}$ which contains $E_0^{(j)}$ for a given $j \in \mathbb{N}$ must be one of the first K_0, since it must have volume greater or equal than $C\frac{|E|}{\mathrm{Per}(E)}$. Thus, for at least one between $E_0^{(0)}, \dots, E_{K_0}^{(0)}$, there exist infinitely many elements $E_0^{(j)}$ contained in it. We narrow to the partitions whose first element is one of these. Finally it is easy to see that definitely the first elements of these partitions must be contained one in the other (otherwise there would exist an element of volume at least $2m$ which contains their union, but definitely there isn't any element so large).

The argument applies similarly to the other elements of the partitions and thus we conclude by a diagonal argument.

At this point we can define

$$E_k = \liminf_{j\to\infty} E_k^{(j)} = \bigcup_{j=0}^{\infty}\bigcap_{i=j}^{\infty} E_k^{(i)} \,.$$

The same estimate of Equation (0.1) holds also for the E_k, namely

$$|E_k|^{\frac{1}{n}} \geq C\frac{|E| - \sum_{h<k}|E_h|}{\mathrm{Per}(E)} \,.$$

This tell us that for every $\varepsilon > 0$ we can choose $K \in \mathbb{N}$ such that $|E \setminus \cup_{h<k} E_h| < \varepsilon$. Otherwise definitely $|E_k|$ would be estimate from below and E would not have finite volume.

At this point, fix such $K \in \mathbb{N}$ such that $|\cup_{k>K} E_h| < m$. First of all we prove that

$$\{E_0, \dots, E_K, E \setminus (\cup_{k\leq K} E_k)\}$$

is still a good partition of E. To this aim, it suffices to show that $\sum_{k=0}^K \mathrm{Per}(E_k) + \mathrm{Per}(E \setminus (\cup_{k\leq K} E_k)) \leq \mathrm{Per}(E)$, but this is easily verified since

$$\begin{aligned}\sum_{k=0}^{K} \mathrm{Per}(E_k) + &\mathrm{Per}(E \setminus (\cup_{k\leq K} E_k)) \leq \\ &\leq \sum_{k=0}^{K} \liminf_{j\to\infty} \mathrm{Per}(E_k^{(j)}) + \liminf_{j\to\infty} \mathrm{Per}(E \setminus (\cup_{k\leq K} E_k^{(j)})) \leq \\ &\leq \liminf_{j\to\infty} \sum_{k=0}^{K} \mathrm{Per}(E_k^{(j)}) + \mathrm{Per}(E \setminus (\cup_{k\leq K} E_k^{(j)})) = \mathrm{Per}(E)\,.\end{aligned}$$

Besides, we know that $|E_0^{(j)}| \downarrow m$ and thus $|E_0| = m$. This conclude our proof, indeed all the elements of this decomposition have volume less or equal than m, thus at least one of them with volume m (among which there is at least E_0) must be indecomposable. $\square$

Proof of Theorem 3. Let us define

$$m = \inf\{|F| \;:\; \{E_k\}_{k\leq K} \cup \{F\} \in \mathcal{F}\} \,,$$

where $\mathcal{F}$ is the set of good partitions $\{E_k\}_{k\leq K} \cup \{F\}$ of E such that E_k is indecomposable for every $k \leq K$.

First of all let us prove that m is 0. Suppose by contradiction that it is not and consider a good partition $\{E_k\}_{k\le K}\cup\{F\}\in\mathcal{F}$ such that $|F|-m$ is sufficiently small. Thanks to Proposition 13, there exists an indecomposable finite perimeter $E_{K+1}\subseteq F$ such that $|E_{K+1}|^{\frac{1}{n}}\ge\frac{Cm}{\mathrm{Per}(E)}$ and $\{E_{K+1},F\setminus E_{K+1}\}$ is a good partition of F. Thus, choosing properly $|F|-m$, $\{E_k\}_{k\le K+1}\cup\{F\setminus E_{K+1}\}$ still belongs to $\mathcal{F}$, but $|F\setminus E_{K+1}|<m$, which give a contradiction.

Now let us consider a sequence $P_j=\{E_k^{(j)}\}_{k\le K}\cup\{F^{(j)}\}\in\mathcal{F}$ such that $|F^{(j)}|\downarrow 0$. By Corollary 10, since the $E_k^{(j)}$ are indecomposable, we can assume that $P_j=\{E_k\}_{k\le j}\cup\{F^{(j)}\}$ with the E_k indecomposable sets in common to all the partitions P_j. Therefore, thanks to Lemma 6, it turns out that $\{E_k\}_{k\in\mathbb{N}}$ is a good partition of E itself up to a $\mathcal{H}^{n-1}$-negligible set, but this conclude our proof. □

In conclusion, we want to give a glimpse into the connection between this notion of components and the classical notion of connected components of an open set.

Proposition 14. *Let $A\subseteq\mathbb{R}^n$ be an open connected set with finite perimeter and finite Lebesgue measure. Then A is indecomposable.*

Proof. Assume by contradiction that there exist a nontrivial good partition $\{E_0,E_1\}$ of A. Thanks to Lemma 8, it holds that $\partial_*A=\partial_*E_0\sqcup\partial_*E_1$ and in particulare $\partial_*E_0\subseteq\partial_*A\subseteq\partial A$. Consequently we have that

$$\mathcal{H}^{n-1}(\partial_*E_0\cap A)\le\mathcal{H}^{n-1}(\partial A\cap A)=0\,,$$

since $\partial A\cap A=\emptyset$. This tells that $\mathrm{D}\chi_{E_0}=0$ on A. Hence χ_{E_0} is locally equivalent to a constant in A and, by the connectedness of A, it is globally equivalent to a constant. Therefore $E_0=\emptyset$ or $E_0=A$ up to $\mathcal{H}^{n-1}$-negligible sets, which leads to a contradiction. □

Lemma 15. *Let $A\in\mathbb{R}^n$ be an open set with finite perimeter and finite Lebesgue measure such that $\partial A=\partial_*A$ up to $\mathcal{H}^{n-1}$-negligible sets. Then the decomposition of A in indecomposable components coincides with its decomposition in connected components.*

However, it is not true in general that the connected components of an open set coincides with its indecomposable components. Let us see why throught the following example.

Example. Let $K\in\mathbb{R}^2$ be a compact subset of $(\,0,\,1\,)\times\{0\}$ with empty interior and $\mathcal{H}^1(K)>0$. Let us write $(\,0,\,1\,)\times\{0\}\setminus K=\cup_{i\in I}(\,a_i,\,b_i\,)$ and define $B_i=B(\frac{a_i+b_i}{2},\frac{b_i-a_i}{2})$ the ball with diameter $(\,a_i,\,b_i\,)$. Then consider $A\coloneqq(\,0,\,1\,)\times(\,-1,\,1\,)\setminus\overline{\cup_{i\in I}B_i}$.

Since K has empty interior, the open set A has easily two connected components $E_0=A\cap(\,0,\,1\,)\times(\,0,\,1\,)$ and $E_1=A\cap(\,0,\,1\,)\times(\,-1,\,0\,)$. Thus, thanks to Proposition 14, the only two possibilities are that A is indecomposable or that $\{E_0,E_1\}$ is the good partition in indecomposable sets. However $K\subseteq\partial_*E_0\cap\partial_*E_1$ and consequently, by Lemma 8 and the fact that $\mathcal{H}^1(K)>0$, $\{E_0,E_1\}$ is not a good partition for A, which therefore is indecomposable.