The Malgrange-Ehrenpreis theorem

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Abstract

We present two proofs of the Malgrange-Ehrenpreis theorem about the existance of fundamental solutions of linear partial differential equations. These notes have been written for the seminary of the exam "Analisi Superiore", taught by professor Luigi De Pascale at University of Pisa on the academic year 2015/2016.

Two proofs of the Malgrange Ehrenpreis theorem

The Malgrange-Ehrenpreis theorem states that every linear partial differential operator with constant coefficients admits a fundamental solution.

Theorem 1 (Malgrange-Ehrenpreis)**.** *For every nonzero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$, there exists $G \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(\mathrm{D})G = \delta$.*

We will present two different solutions of the theorem. The former is the original proof by Malgrange and Ehrenpreis and uses the Hahn-Banach theorem to construct G starting from defining it on a subset of $\mathcal{D}(\mathbb{R}^n)$ in the most natural way. The latter, due to Hörmander, gives a more explicit construction through the Fourier transform.

Proof. Let $F : P(-\mathrm{D})\mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ be the functional defined by $P(-\mathrm{D})\varphi \mapsto \varphi(0)$, where $P(-\mathrm{D})\mathcal{D}(\mathbb{R}^n) = \{P(-\mathrm{D})\varphi \ : \ \varphi \in \mathcal{D}(\mathbb{R}^n)\}$ is the image of $\mathcal{D}(\mathbb{R}^n)$ through $P(-D)$. First notice that the function F is well-defined. Indeed, if $P(-\mathrm{D})\varphi = 0$ than $P(-i\xi)\hat{\varphi} = \mathcal{F}(P(-\mathrm{D})\varphi) = 0$ and consequently $\hat{\varphi} = 0$, which gives $\varphi = 0$.

Note that if $G \in \mathcal{D}'(\mathbb{R}^n)$ is an extension of F it will be a fundamental solution, since for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we will have

$$\langle P(\mathrm{D})G, \varphi \rangle = \langle G, P(-\mathrm{D})\varphi \rangle = \langle F, P(-\mathrm{D})\varphi \rangle = \varphi(0)\,.$$

Therefore, let's show that F is linear and continuos with respect to the topology induced by $\mathcal{D}(\mathbb{R}^n)$, so that the Hahn-Banach theorem can give us the sought extension.

Lemma 2. *Let $F : \mathbb{C}^n \to \mathbb{C}$ be an entire function, $P : \mathbb{C}^n \to \mathbb{C}$ a polynomial with degree m and $\rho : \mathbb{C}^n \to \mathbb{C}$ a measurable positive function with compact support depending only on $|z_1|, \dots, |z_n|$. Then, for every $\alpha \in \mathbb{N}^n$, there exists $C_{m,|\alpha|}$ such that*

$$|F(0)||P^{(\alpha)}(0)| \cdot \int |z^\alpha| \rho(z)\,\mathrm{d}z \le C_{m,|\alpha|} \int |F(z)P(z)|\rho(z)\,\mathrm{d}z\,.$$

Proof. Let's prove the lemma in the one-dimensional case, then the general one will follow easily.

If $q(z) = a_k z^k + \ldots + a_1 z + a_0$ is a polynomial and $g(z)$ is an entire function, thanks to the Cauchy's integral formula applied to $z^k \bar{q}(1/z) g(z)$, for every real number $r > 0$ we have

$$r^k |a_k g(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta}) q(re^{i\theta})| \, \mathrm{d}\theta \, .$$

Now, if $P(z) = \prod_{j=1}^m (z + z_j)$, applying the last formula with $q(z) = \prod_{j=1}^k (z + z_j)$ and $g(z) = F(z) \prod_{j=k+1}^m (z + z_j)$, we have

$$r^k |F(0) \prod_{j=k+1}^m z_j| \leq \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta}) P(re^{i\theta})| \, \mathrm{d}\theta \, .$$

Since this inequality holds for any product of $m - k$ of the numbers z_j and $P^{(k)}(0)$ is sum of such terms, we obtain

$$r^k |F(0) P^{(k)}(0)| \leq C_{m,k} \int_0^{2\pi} |F(e^{i\theta}) P(e^{i\theta})| \, \mathrm{d}\theta \, .$$

Therefore, multiplying by $r\rho(r)$ and integrating with respect to r, we obtain the thesis

$$|F(0) P^{(k)}(0)| \int |z^k| \rho(z) \, \mathrm{d}z \leq C_{m,k} \int |F(z) P(z)| \rho(z) \, \mathrm{d}z \, .$$

□

Setting $\psi \coloneqq P(\mathrm{D})\varphi$, we have $\hat{\psi}(\xi) = P(i\xi)\hat{\varphi}(\xi)$. Our aim is to estimate $\varphi(0)$ with $|\psi|_{C^k}$ for some $k \in \mathbb{N}$.

Fixed $\xi \in \mathbb{R}^n$, we apply the Lemma 2 to the entire function $F(z) = \hat{\varphi}(\xi + z)$, the polynomial $P(i(\xi + z))$ and the characteristic function of the set $\{|z| < 1\}$ as $\rho(z)$. Choosing α such that $P^{(\alpha)}$ is costant, we obtain

$$|\hat{\varphi}(\xi)| \leq C_{n,P} \int |\hat{\varphi}(\xi + z) P(i(\xi + z))| \rho(z) \, \mathrm{d}z = C_{n,P} \int |\hat{\psi}(\xi + z)| \rho(z) \, \mathrm{d}z \, ,$$

where $C_{n,P}$ denote a generic constant depending on n and P. Hereafter we will continue to use $C_{n,P}$, though its value will change.

If we integrate the last inequality with respect to ξ, we get

$$\begin{aligned} |\varphi(0)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int |\hat{\varphi}(\xi)| \, \mathrm{d}\xi \leq C_{n,P} \int \int |\hat{\psi}(\xi + z)| \rho(z) \, \mathrm{d}z \, \mathrm{d}\xi = \\ &= C_{n,P} \int \rho(z) \int |\hat{\psi}(\xi + z)| \, \mathrm{d}\xi \, \mathrm{d}z = C_{n,P} |\hat{\psi}|_{L^1} \, . \end{aligned}$$

Recalling that $(i\xi)^\alpha \hat{\psi} = \widehat{\mathrm{D}^\alpha \psi}$ and taking $|\alpha|$ big enough, we have

$$\begin{aligned} |\hat{\psi}|_{L^1} &= \int_{|\xi|<1} |\hat{\psi}| \, \mathrm{d}\xi + \int_{|\xi| \geq 1} |\hat{\psi}| \, \mathrm{d}\xi \leq C_n |\hat{\psi}|_\infty + \int_{|x| \geq 1} \frac{|\widehat{\mathrm{D}^\alpha \psi}|}{|\xi^\alpha|} \, \mathrm{d}\xi \leq C_n (|\hat{\psi}|_\infty + |\widehat{\mathrm{D}^\alpha \psi}|_\infty) \leq \\ &\leq C_n (|\psi|_{L^1} + |\mathrm{D}^\alpha \psi|_{L^1}) \leq C_n \; |\mathrm{spt}\, \psi| \; |\psi|_{C^{|\alpha|}} \, . \end{aligned}$$

Therefore we have obtained $\varphi(0) \leq C_{n,P} \; |\mathrm{spt}\, \psi| \; |\psi|_{C^{|\alpha|}}$, which shows the continuity of the function $\psi \mapsto \varphi(0)$.

Finally the linearity of F is immediate and this completes the solution. □

We would like to set $G := \mathcal{F}^{-1}\left((2\pi)^{-\frac{n}{2}} P(i\xi)^{-1}\right)$, which would be a fundamental solution for $P(\mathrm{D})$ if $(2\pi)^{-\frac{n}{2}} P(i\xi)^{-1}$ belonged to $\mathcal{S}'(\mathbb{R}^n)$. Unfortunately this is not true in general, thus the following proof solves this problem using a sort of partition of unity called the "Hörmander staircase".

Proof. First of all, we introduce the Hörmander staircase and we prove its existence in the following proposition. We denote P_m the principal part of P, where $m = \deg P$.

Proposition 3. *Let $\eta \in \mathbb{R}^n$ such that $P_m(\eta) \neq 0$. Then there exist characteristic functions χ_k on $\mathbb{R}^n$ for $k = 0, \dots, m$ such that:*

1. *$\chi_k(\xi + \lambda\eta) = \chi_k(\xi)$ for all $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $k = 0, \dots, m$;*
2. *$\sum_{k=0}^m \chi_k \equiv 1$;*
3. *there exists $C > 0$ such that if $\chi_k(\xi) \neq 0$ then $|P(i\xi + k\eta)| > C$ for all $\xi \in \mathbb{R}^n$ and $k = 0, \dots, m$.*

Proof. Fixed $\xi \in \mathbb{R}^n$, we consider the polynomial $z \mapsto P(i\xi + k\eta) = P_m(\eta)\prod_{j=1}^m (z - z_j)$. The function $\omega : \xi \mapsto [z_1, \dots, z_m]$, which takes values on the topological quotient space $X = \mathbb{C}^m / \sim$ of $\mathbb{C}^m$ by permutations of the coordinates, is continuos by well-known results.

For $k = 0, \dots, m$, we define $A_k = \{[z_1, \dots, z_m] \in X \ : \ |\mathrm{Re}\, z_j - k| < \frac{1}{2} \quad \forall j = 1, \dots, m\}$ and consequently we set χ_k as the characteristic function of $\omega^{-1}(A_k) \cap \bigcap_{l=0}^{k-1} \omega^{-1}(X \setminus A_l)$.

Since the sets A_k are closed and cover X, the functions χ_k are measurable and $\sum_{k=0}^m \chi_k \equiv 1$. Furthermore $\omega(\xi + \lambda\eta) = [z_1 - i\lambda, \dots, z_m - i\lambda]$, thus $\xi + \lambda\eta \in \omega^{-1}(B_k)$ if and only if $\xi \in \omega^{-1}(B_k)$, which provides the first requirement.

Finally, if $\xi \in \omega^{-1}(B_k)$, then

$$|P(i\xi + k\eta)| = |P_m(\eta)| \prod_{j=1}^m |k - z_j| \geq |P_m(\eta)| \prod_{j=1}^m |k - \mathrm{Re}\, z_j| > \frac{|P_m(\eta)|}{2^m} \,,$$

which concludes the proof of the proposition. □

At this point, we can construct our fundamental solution as

$$G := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m e^{kx\cdot\eta} \mathcal{F}^{-1}\left(\frac{\chi_k(\xi)}{P(i\xi + k\eta)}\right) \,,$$

which is a well-defined distribution because $\chi_k(\xi) P(i\xi + k\eta)^{-1} \in L^\infty(\mathbb{R}^n)$, since is bounded by $1/C$ thanks to property 3 of Proposition 3, thus we can do the inverse Fourier transform. Moreover there is no problem to multiply by the $C^\infty(\mathbb{R}^n)$-function $e^{kx\cdot\eta}$.

Let's now prove that G is actually a fundamental solution for $P(\mathrm{D})$. We have that

$$\begin{aligned} P(\mathrm{D})G &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m P(\mathrm{D}) \left(e^{kx\cdot\eta} \mathcal{F}^{-1}\left(\frac{\chi_k(\xi)}{P(i\xi + k\eta)}\right)\right) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m e^{kx\cdot\eta} P(\mathrm{D} + k\eta) \mathcal{F}^{-1}\left(\frac{\chi_k(\xi)}{P(i\xi + k\eta)}\right) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m e^{kx\cdot\eta} \mathcal{F}^{-1}\left(P(i\xi + k\eta)\frac{\chi_k(\xi)}{P(i\xi + k\eta)}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m e^{kx\cdot\eta} \check{\chi}_k(\xi) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=0}^m \check{\chi}_k(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \mathcal{F}^{-1}\left(\sum_{k=0}^m \chi_k(\xi)\right) = \frac{1}{(2\pi)^{\frac{n}{2}}} \check{1} = \delta \,, \end{aligned}$$

where we have used the following lemmas to justify the first equality in the last line.

Lemma 4. *Let $u \in L^\infty(\mathbb{R}^n)$ such that there exists $\eta \in \mathbb{R}^n$ for which $u(\xi + \lambda\eta) = u(\xi)$ for every $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then $\frac{\partial u}{\partial \eta} = 0$ as distribution.*

Proof. Given $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle \frac{\partial u}{\partial \eta}, \varphi \rangle &= -\langle u, \frac{\partial \varphi}{\partial \eta} \rangle = -\int u(x) \frac{\partial \varphi}{\partial \eta}(x)\, \mathrm{d}x = -\int u(x) \lim_{h \to 0} \frac{\varphi(x + h\eta) - \varphi(x)}{h}\, \mathrm{d}x = \\
&= -\lim_{h \to 0} \frac{1}{h} \left[\int u(x)\varphi(x + h\eta)\, \mathrm{d}x - \int u(x)\varphi(x)\, \mathrm{d}x \right] = \\
&= -\lim_{h \to 0} \frac{1}{h} \left[\int u(x - h\eta)\varphi(x)\, \mathrm{d}x - \int u(x)\varphi(x)\, \mathrm{d}x \right] = 0\,.
\end{aligned}$$

□

Lemma 5. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ with $\frac{\partial T}{\partial \eta} = 0$ and let $f \in C^\infty(\mathbb{R})$, then $f(\eta \cdot x)\check{T} = f(0)\check{T}$.*

Proof. Without loss of generality, we can suppose $f(0) = 0$, then there exists $g \in C^\infty(\mathbb{R})$ such that $f(y) = yg(y)$. Therefore, given $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle f(\eta \cdot x)\check{T}, \varphi \rangle &= \langle T, \mathcal{F}(\ f(\eta \cdot x)\varphi(x)\ ) \rangle = \langle T, \mathcal{F}(\ (\eta \cdot x) g(\eta \cdot x)\varphi(x)\ ) \rangle = \\
&= \langle T, -i \frac{\partial}{\partial \eta}\ \mathcal{F}(\ g(\eta \cdot x)\varphi(x)\ ) \rangle = \langle \frac{\partial T}{\partial \eta}, i\mathcal{F}(\ g(\eta \cdot x)\varphi(x)\ ) \rangle = 0\,.
\end{aligned}$$

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