Colloquio IV anno - Scuola Normale Superiore

The Positive Mass Theorem

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Contents

1 Asymptotically flat manifolds and their mass **2**

2 The positive mass theorem **3**

3 Easy expansions given by asymptotic flatness **3**

4 Riduction to the theorem of Schoen and Yau **4**

5 Proof of the theorem by Schoen and Yau **5**

5.1 Assuming strictly positive curvature 5

5.2 Existence of a minimizing surface 7

5.3 Properties of the integral of the scalar curvature over the area minimizing surface 8

5.4 Conclusion of the proof proving that we have a contradiction 11

1 Asymptotically flat manifolds and their mass

During all the notes we deal with asymptotically flat oriented three-dimensional manifolds, thus let us start introducing this concept and some notation.

Definition 1.1. A Riemannian manifold (N, g) is said to be asymptotically flat of order $\tau > 0$ if there exists a decomposition $N = N_0 \cup N_\infty$ with N_0 compact and a diffeomorphism $\psi : N_\infty \to \mathbb{R}^n \setminus B_{\rho_0}(0)$ for some $\rho_0 > 0$, such that in the chart inducted by ψ we have

$$g_{ij} = \delta_{ij} + \mathcal{O}_2\left(\frac{1}{r^\tau}\right),$$

where $r = ((x^1)^2+(x^2)^2+(x^3)^2)^{\frac{1}{2}}$. In particular we call N_∞ the *end* of the asymptotically flat manifold N.

Here we denote with $\mathcal{O}_k\left(\frac{1}{r^\tau}\right)$ a function on $\mathbb{R}^n$ whose j^{th} derivative is $\mathcal{O}\left(\frac{1}{r^{\tau+j}}\right)$ for $j = 0, \dots, k$.

Definition 1.2. Given (N, g) an asymptotically flat manifold, we define the mass of its end as

$$m(g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\operatorname{div}_0 g - \,\mathrm{d}\operatorname{tr}_0 g)(\nu)\,\mathrm{dvol}_{S_r} = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\nu_j\,\mathrm{dvol}_{S_r},$$

where the divergence and the trace are computed with respect to the Euclidean metric (the subscript "0" stands for the Euclidean metric) and ν is the outer unit normal of the sphere S_r.

Proposition 1.3. *Given an asymptotically flat manifold (N, g) of order $\tau > \frac{n-2}{2}$, the mass $m(g)$ exists and doesn't depend on the chart.*

Definition 1.4. Given an asymptotically flat manifold (N, g) of order $\tau > \frac{n-2}{2}$, we say that $g \in \mathscr{M}_\tau$ if the scalar curvature of g is integrable and $g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}(N_\infty)$ for some $0 < \alpha < 1$.

We recall that the weighted Hölder space $C^{k,\alpha}_\beta(N_\infty)$ is the set of all C^k functions u such that the norm

$$\|u\|_{C^{k,\alpha}_\beta} = \sum_{i=0}^k \sup_{N_\infty}\left\{r^{-\beta+i}|\nabla^i u|\right\} + \sup_{x,y\in N_\infty}\left\{\min(r(x), r(y))^{-\beta+k+\alpha}\frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}\right\}$$

is finite.

Proposition 1.5. *Given $\tau > \frac{n-2}{2}$, the mass functional is a continuous affine functional on $\mathscr{M}_\tau$.*

2 The positive mass theorem

At this point we have all the definitions to state the positive mass theorem, which substantially says that an asymptotically flat manifold with nonnegative scalar curvature has nonnegative mass. The problem has been solved up to dimension 7 using an inductive argument starting from dimension 3, which exploits the fact that minimal hypersurfaces in low dimensions can't have singularities. Recently a proof in arbitrary dimension has been announced. However here we concentrate on the 3-dimensional case.

Theorem 2.1 (Positive mass)**.** *Let (N, g) be an asymptotically flat manifold of dimension 3 with $g \in \mathscr{M}_\tau$, for $\tau > \frac{1}{2}$, and with scalar curvature $R \geq 0$. Then its mass $m(g)$ is nonnegative.*

The hypothesis of $g \in \mathscr{M}_\tau$ for $\tau > \frac{n-2}{2}$ is added in order to gain integrability of the scalar curvature, existence and indipendence from the chart of the mass and continuity of the mass functional.

The first step of the proof consists in reducing to the following simpler case, in which we know a further term of the asymptotically flat metric.

Theorem 2.2 (Schoen, Yau)**.** *Let (N^3, g) be an asymptotically flat manifold with scalar curvature $R \geq 0$ such that*

$$g_{ij} = \left(1 + \frac{M}{2r}\right)^4 \delta_{ij} + \mathcal{O}_2\left(\frac{1}{r^2}\right) = \left(1 + \frac{2M}{r}\right) \delta_{ij} + \mathcal{O}_2\left(\frac{1}{r^2}\right)$$

in a chart on the end N_∞. In this case we say that (N, g) is a good *asymptotically flat manifold.*

Then the mass of g corresponds to M and it is nonnegative.

Therefore, the next sections are devoted to prove the reduction from Theorem 2.2 (Schoen, Yau) to Theorem 2.1 (Positive mass) and the Theorem 2.2 (Schoen, Yau) itself. However, before that we compute asymptotic expansions of some quantities given by the asymptotic flatness in the form of Theorem 2.2 (Schoen, Yau).

3 Easy expansions given by asymptotic flatness

We are now going to compute some quantities, or their expansions, concerning a good asymptotically flat manifold (N^3, g).

First of all, the expansions of the metric inverse and the square root of the determinant of the metric are very easy to obtain:

$$g^{kl} = \left(1 + \frac{M}{2r}\right)^{-4} \delta^{kl} + \mathcal{O}_2\left(\frac{1}{r^2}\right), \tag{3.1}$$

$$\sqrt{\det(g_{ij})} = \left(1 + \frac{M}{2r}\right)^6 + \mathcal{O}_2\left(\frac{1}{r^2}\right). \tag{3.2}$$

Then we compute the asymptotic expansion of the derivatives of the metric:

$$\partial_k g_{ij} = \partial_k \left(1 + \frac{2M}{r}\right) \delta_{ij} + \mathcal{O}\left(\frac{1}{r^3}\right) = -\frac{2Mx^k}{r^3} \delta_{ij} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{3.3}$$

from which follows directly that the mass coincides with the constant M in a good asymptotically flat manifold, indeed:

$$\begin{aligned} m(g) &= \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\nu_j \,\mathrm{dvol}_{S_r} = \\ &= \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \frac{2M}{r^3} \left(-x^i\delta_{ij} + x^j\delta_{ii}\right) \nu_j \,\mathrm{dvol}_{S_r} = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \frac{4Mx^j}{r^3} \nu_j \,\mathrm{dvol}_{S_r} = \\ &= \frac{M}{4\pi} \lim_{r\to\infty} \int_{S_r} \frac{1}{r^2} \,\mathrm{dvol}_{S_r} = \frac{M}{4\pi} \lim_{r\to\infty} \frac{1}{r^2} \mathrm{vol}(S_r) = M\,. \end{aligned}$$

4 Riduction to the theorem of Schoen and Yau

As already mentioned, in this section we show how Theorem 2.2 (Schoen, Yau) implies Theorem 2.1 (Positive mass).

Fix a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Then define $\eta_\rho : N \to \mathbb{R}$ by

$$\eta_\rho(x) = \eta\left(\frac{r(x)}{\rho}\right) .$$

Now we can consider the flattened metric g_ρ given by

$$(g_\rho)_{ij} = \delta_{ij} + \eta_\rho \phi_{ij} ,$$

where $g_{ij} = \delta_{ij} + \phi_{ij}$ with $\phi_{ij} \in C^{1,\alpha}_{-\tau}(N_\infty)$.

The problem now is that g_ρ could have negative scalar curvature, thus we try to conformally change the metric in order to restore the nonnegative scalar curvature. A first attempt is to impose the scalar curvature to be $\eta_\rho R$ solving the classical Yamabe equation stated just below.

Lemma 4.1 (Yamabe equation)**.** *Given a Riemannian manifold (M^m, g) with scalar curvature R and a positive function $\varphi \in C^\infty(M)$, the conformally equivalent metric $\varphi^{\frac{4}{m-2}} g$ on M has scalar curvature*

$$\tilde{R} = \varphi^{-\frac{m+2}{m-2}} \left(R\varphi - \frac{4(m-1)}{m-2}\, \Delta\varphi \right) = \varphi^{-\frac{m+2}{m-2}}\, \Box\varphi ,$$

where we have defined the operator $\Box := R - \frac{4(m-1)}{m-2}\, \Delta$.

In particular, if M has dimension 3, this becomes $\tilde{R} = \varphi^{-5}\,(R\varphi - 8\Delta\varphi)$.

However it turns out to be convenient to solve the linear equation

$$\Box_\rho \varphi_\rho = \eta_\rho R \varphi_\rho$$

for $\varphi_\rho > 0$ and change the metric by $\tilde{g}_\rho = \varphi_\rho^4 g_\rho$ to obtain

$$\tilde{R}_\rho = \varphi_\rho^{-5}\, \Box_\rho \varphi_\rho = \varphi_\rho^{-4} \eta_\rho R \ge 0\,.$$

Calling $\psi_\rho = \varphi_\rho - 1$ and $\gamma_\rho = R_\rho - \eta_\rho R$, the equation becomes

$$(\gamma_\rho - 8\Delta_\rho)\,\psi_\rho = -\gamma_\rho\,.$$

We omit the proof of the existence of the solution ψ_ρ of this equation such that $\psi_\rho \to 0$ in $C^{2,\alpha}_{-\tau+\varepsilon}$ and the fact that $\tilde{g}_\rho$ fulfills the hypothesis of Theorem 2.2 (Schoen, Yau). Then we have that $m(\tilde{g}_\rho) \geq 0$ and now it sufficient to prove that $\tilde{g}_\rho \to g$ in the topology of $\mathscr{M}_{\tau-\varepsilon}$, from which indeed follows that $m(\tilde{g}_\rho) \to m(g)$ by Proposition 1.5.

We have that

$$g - \tilde{g}_\rho = (g - g_\rho) + \left(1 - \varphi_\rho^{\frac{4}{m-2}}\right) g_\rho = (1-\eta_\rho)\phi + \left(1 - \varphi_\rho^{\frac{4}{m-2}}\right) g_\rho\,.$$

However the two terms of the sum go to zero in $C^{1,\alpha}_{-\tau}$, since $(1-\eta_\rho)\phi$ is bounded and goes to zero and $\varphi_\rho = 1 + \psi_\rho \to 1$ in $C^{1,\alpha}_{-\tau}$. Moreover

$$|R - \tilde{R}_\rho| = |R - \varphi_\rho^{-4}\eta_\rho R| \leq 2(1-\eta_\rho)|R|$$

and consequently $R_\rho \to R$ in $L^1(N)$. We can thus conclude that $\tilde{g}_\rho \to g$ in $\mathscr{M}_{\tau-\varepsilon}$ as expected.

5 Proof of the theorem by Schoen and Yau

In order to show this result, let us assume by contradiction that the total mass is negative, then the proof of the theorem is based on four main steps:

1. Prove that we can assume that $R > 0$ outside a compact subset of N_∞.
2. Find a suitable complete area minimizing surface Σ.
3. Show that the integral over Σ of its sectional curvature is positive.
4. Prove that such a surface can not exist.

We will often identify N_∞ with $\mathbb{R}^3 \setminus \bar{B}_{\rho_0}(0)$.

Remark 5.1. In our proof we will assume that the manifold (N, g) is oriented. However this is not a deep loss of generality, indeed we can assume orientability as long as we consider the double covering and we solve the problem in a manifold with more than one end (the proof is totally the same, working separately on each end, except for the fact that we have to check that the part of the area minimizing surface not in the end lies in a compact subset of the complementary).

5.1 Assuming strictly positive curvature

In this section we prove that:

Given a good oriented asymptotically flat Riemannian manifold (N^3, g) with positive scalar curvature and negative total mass, there exists a conformally equivalent metric $\tilde{g}$ on N such that $(N, \tilde{g})$ fulfills the same hypotheses and, moreover, it has strictly positive scalar curvature outside a compact set.

The main tool to achieve this result is that, under our hypotheses (in particular because of the negative total mass), the Laplacian of $\frac{1}{r}$ is strictly negative outside a compact set.

Therefore, first of all let us compute the asymptotic expansion of the Laplacian of $\frac{1}{r}$ in the end N_∞, using that for a function u on N_∞ we have

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) .$$

By the computations in Section 3 (Easy expansions given by asymptotic flatness), in particular using the Equations (3.1) and (3.2)

$$\begin{aligned}
\Delta\left(\frac{1}{r}\right) &= \left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) \partial_k \left(\left(1 + \frac{M}{2r}\right)^2 \delta^{kl} \partial_l \left(\frac{1}{r}\right) \right) + \mathcal{O}\left(\frac{1}{r^5}\right) = \\
&= \sum_{k=1}^{3} \partial_k \left(\left(1 + \frac{M}{r}\right) \partial_k \left(\frac{1}{r}\right) \right) + \mathcal{O}\left(\frac{1}{r^5}\right) = \\
&= \sum_{k=1}^{3} M \left[\partial_k \left(\frac{1}{r}\right)\right]^2 + \left(1 + \frac{M}{r}\right) \partial_k \partial_k \left(\frac{1}{r}\right) + \mathcal{O}\left(\frac{1}{r^5}\right) = \\
&= \sum_{k=1}^{3} M \frac{(x^k)^2}{r^6} + \left(1 + \frac{M}{r}\right) \partial_k \left(-\frac{x^k}{r^3}\right) + \mathcal{O}\left(\frac{1}{r^5}\right) = \\
&= \frac{M}{r^4} - \left(1 + \frac{M}{r}\right) \sum_{k=1}^{3} \left(\frac{1}{r^3} - \frac{3(x^k)^2}{r^5}\right) + \mathcal{O}\left(\frac{1}{r^5}\right) = \frac{M}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) .
\end{aligned}$$

Notice that we have obtained that $\partial_k \partial_k \frac{1}{r} = 0$, which actually follows from the fact that $\frac{1}{r}$ is the Green's function of the Laplacian in $\mathbb{R}^3$.

Thus, there exists $\rho > \rho_0$ such that $\Delta(\frac{1}{r}) < 0$ for $r \geq \rho$, since $M < 0$ by hypothesis. We will use this fact in order to construct a conformal deformation of the metric g with strictly positive curvature for $r > 2\rho$.

Call $t_0 = \frac{-M}{8\rho_0}$ and define $\beta \in C^\infty(\mathbb{R}^+)$ such that

$$\beta(t) = \begin{cases} t\,, & \text{for } t < t_0 \\ \frac{3t_0}{2}\,, & \text{for } t > 2t_0 \end{cases}$$

and $\beta'(t) \geq 0$, $\beta''(t) \leq 0$ for all $t \in \mathbb{R}^+$. Now we can define $\varphi : N \to \mathbb{R}$ by

$$\varphi = \begin{cases} 1 + \frac{3t_0}{2}\,, & \text{on } N \setminus N_\infty \\ 1 + \beta\left(\frac{-M}{4r}\right)\,, & \text{on } N_\infty \end{cases}$$

in order to have $\Delta\varphi \leq 0$ everywhere and $\Delta\varphi < 0$ for $r > 2\rho$.

We would like to change conformally the metric g through φ, so that the hypotheses of asymptotic flatness and negative total mass don't change and the resulting scalar curvature is positive everywhere and strictly positive for r sufficiently big.

Consequently, let us consider precisely $\tilde{g} = \varphi^4 g$, then

$$\tilde{g}_{ij} = \left(1 - \frac{M}{4r}\right)^4 \left(1 + \frac{M}{2r}\right)^4 \delta_{ij} + \mathcal{O}_2\left(\frac{1}{r^2}\right) = \left(1 + \frac{M}{4r}\right)^4 \delta_{ij} + \mathcal{O}_2\left(\frac{1}{r^2}\right) .$$

However, by Lemma 4.1 (Yamabe equation), it is obvious to conclude that $\tilde{R} \geq 0$ everywhere and $\tilde{R} > 0$ for $r > 2\rho$ on N_∞. Consequently we have proven what we wanted.

5.2 Existence of a minimizing surface

In this section we want to show the following result:

Let (N^3, g) be a good oriented asymptotically flat Riemannian manifold with negative total mass, positive scalar curvature and strictly positive scalar curvature outside a compact set. Then there exists an oriented complete area minimizing surface Σ in N such that $\Sigma \cap N_\infty$ is contained in $E_h \coloneqq \{x = (x^1, x^2, x^3) \in \mathbb{R}^3 \ : \ |x^3| \leq h\}$ for some $h > 0$.

Given $\rho > 2\rho_0$, let C_ρ be the Euclidean circle of radius ρ centered at 0 in the x^1x^2-plane.

Theorem 5.2. *For every $\rho > 2\rho_0$, there exists the surface Σ_ρ of least g-area among those having boundary C_ρ, which is a smooth embedded oriented surface.*

Moreover, there exists $r_0 > 0$ depending only on (N, g) such that, for every point $x_0 \in \Sigma_\rho$ with $B^g_{r_0}(x_0) \cap C_\rho = \emptyset$, $B^g_{r_0}(x_0) \cap \Sigma_\rho$ can be written as the graph of a C^3 function f_ρ over the tangent plane to Σ_ρ in x_0. Furthermore, there exists a constant which bounds all derivatives of f_ρ up to order three in $B^g_{r_0}(x_0)$, always depending only on (N, g).

Here $B^g_{r_0}(x_0)$ stands for the geodesic ball with respect to the metric g on N, to distinguish it to the Euclidean balls in N_∞.

First of all let us prove that there exists $h > \rho_0$ such that $\Sigma_\rho \cap N_\infty \subseteq E_h$ for every $\rho > 2\rho_0$. This will allow us to extract a converging subsequence of the surfaces Σ_ρ to the sought minimal surface Σ.

First of all, let us compute the Hessian of x^3 on N_∞, calling $\{\partial_i\}_{i=1,2,3}$ the frame induced by the standard coordinates x^1, x^2, x^3 on N^∞.

$$\mathrm{Hess}(x^3)(\partial_i, \partial_j) = \partial_i(\partial_j(x^3)) - \nabla_{\partial_i}\partial_j(x^3) = -\Gamma^3_{ij} = -\frac{1}{2}g^{3k}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$$

and, exploiting the Equation (3.3), we obtain

$$\mathrm{Hess}(x^3)(\partial_i, \partial_j) = \frac{Mx^i}{r^3}\delta_{j3} + \frac{Mx^j}{r^3}\delta_{i3} - \frac{Mx^3}{r^3}\delta_{ij} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{5.1}$$

Now fix $\rho > 2\rho_0$ and let $\bar{h}$ be the maximum for x^3 on $\Sigma_\rho \cap N_\infty$, which occurs at a point $\bar{x} \in \Sigma_\rho$. As already mentioned, we want to prove that $\bar{h}$ is bounded indipendently from ρ. Notice that the tangent space to Σ_ρ at $\bar{x}$ is generated by ∂_1, ∂_2.

Take a local orthonormal frame $\{e_i\}_{i=1,2,3}$ around $\bar{x}$ such that $e_i(\bar{x}) = \partial_i(\bar{x})$ for $i = 1, 2$ and $e_3 = \nu$ is a normal unit field for Σ_ρ. Then the hessian of x^3 in the induced metric over Σ_ρ is

$$\begin{aligned}
\mathrm{Hess}^{\Sigma_\rho}(x^3)(e_i, e_j) &= e_i(e_j(x^3)) - \nabla^{\Sigma_\rho}_{e_i} e_j(x^3) = \\
&= e_i(e_j(x^3)) - \nabla_{e_i} e_j(x^3) + \langle \nabla_{e_i} e_j, \nu \rangle \nu(x^3) = \\
&= e_i(e_j(x^3)) - \nabla_{e_i} e_j(x^3) + h_{ij}\nu(x^3),
\end{aligned}$$

where h_{ij} is the second fundamental form with respect to ν. Evaluating in $\bar{x}$ we thus obtain

$$\mathrm{Hess}^{\Sigma_\rho}(x^3)(\partial_i, \partial_j) = \mathrm{Hess}(x^3)(\partial_i, \partial_j) + h_{ij}\nu(x^3)\,.$$

Then, contracting with respect to the induced metric g^{Σ_ρ} on Σ_ρ, we have

$$\sum_{i,j=1}^{2} (g^{\Sigma_\rho})^{ij}\mathrm{Hess}^{\Sigma_\rho}(x^3)(\partial_i, \partial_j) = \sum_{i,j=1}^{2} g^{ij}\mathrm{Hess}^{\Sigma_\rho}(x^3)(\partial_i, \partial_j) =$$
$$= \sum_{i,j=1}^{2} g^{ij}\mathrm{Hess}(x^3)(\partial_i, \partial_j) + H_\nu^{\Sigma_\rho} = -\frac{2M\bar{h}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right)\,,$$

where we have used the Equation (5.1) and the fact that the mean curvature $H_\nu^{\Sigma_\rho}$ of Σ_ρ is zero by minimality.

Since $M < 0$, if $\bar{h}$ is sufficiently large (indipendently from ρ), we see that in the point $\bar{x} \in \Sigma_\rho$ it holds

$$\sum_{i,j=1}^{2} (g^{\Sigma_\rho})^{ij}\mathrm{Hess}^{\Sigma_\rho}(x^3)(\partial_i, \partial_j) > 0\,,$$

which contradicts that x^3 attains a maximum there, because in that case $\mathrm{Hess}^{\Sigma_\rho}(x^3)$ would be negative semidefinite in $\bar{x}$.

Which similar calculations we can also bound the minimum of x^3 and thus we have obtained the existence of h such that $\Sigma_\rho \cap N_\infty \subseteq E_h$ for every $\rho > 2\rho_0$.

At this point, we just have to pass to the limit in order to find our minimal surface Σ. For $\tilde{\rho} > 2\rho_0$ define $A_{\tilde{\rho}} = N \setminus N_\infty \cup \{x \in N_\infty \ : \ (x^1)^2 + (x^2)^2 \leq \tilde{\rho}^2\}$, then we have obtained that $\Sigma_\rho \cap A_{\tilde{\rho}} \subseteq (K \cup E_h) \cap A_{\tilde{\rho}}$, which is a compact subset of N and thus we can apply Ascoli-Arzelà theorem in order to find a converging subsequence of the Σ_ρ.

Indeed this can be done thanks to the Theorem 5.2, which gives us a converging subsequence $\rho_i^{(\tilde{\rho})} \to +\infty$ such that $\{\Sigma_{\rho_i^{(\tilde{\rho})}}\}$ converges in C^2-topology. Then, by a diagonal argument, we find $\rho_i \to +\infty$ so that $\{\Sigma_{\rho_i}\}$ converges uniformly in C^2-topology on compact subsets to an embedded C^2 surface Σ.

Moreover Σ is a properly embedded oriented area minimizing surface on any compact subset of N and obviously $\Sigma \cap N_\infty \subseteq E_h$.

5.3 Properties of the integral of the scalar curvature over the area minimizing surface

In this section we will investigate properties concerning the integral of the section curvature of an area minimizing surface in an asymptotically flat manifold. In particular we will show that:

Let (N, g) be a good oriented asymptotically flat manifold with positive scalar curvature and strictly positive scalar curvature outside a compact set. Moreover let $\Sigma \subseteq N$ be an oriented complete area minimizing surface properly embedded such that $\Sigma \cap N_\infty \subseteq E_h$ for some $h > \rho_0$. Then it holds that

$$\int_\Sigma |K|\,\mathrm{d}V < +\infty \quad \textit{and} \quad \int_\Sigma K\,\mathrm{d}V > 0\,,$$

where K and $\mathrm{d}V$ are the sectional curvature and the volume form of Σ, respectively.

For $\rho > \rho_0$, let us define $\Sigma_{(\rho)} \coloneqq \Sigma \cap [(N \setminus N_\infty) \cup B_\rho(0)]$, then $\mathrm{Area}(\Sigma_{(\rho)}) \leq C_1\rho^2$. Indeed by minimality $\mathrm{Area}(\Sigma_{(\rho)}) \leq \mathrm{Area}(\partial B_\rho(0))$, from which follows the inequality above by asymptotic flatness.

Lemma 5.3. *In the given assumptions for the surface Σ we have that*

$$\int_\Sigma \frac{1}{1+r^a}\,\mathrm{d}V < +\infty\,, \tag{5.2}$$

for every $a > 2$ real. In the case $a = 2$, we have instead that for every $\rho_0 < \rho_1 < \rho_2$ it holds

$$\int_{\Sigma_{(\rho_2)}\setminus\Sigma_{(\rho_1)}} \frac{1}{r^2}\,\mathrm{d}V \leq 2C_1\left(\log\left(\frac{\rho_2}{\rho_1}\right)+1\right)\,. \tag{5.3}$$

Proof. For every $a > 2$ we have that

$$\begin{aligned}
\int_\Sigma \frac{1}{1+r^a}\,\mathrm{d}V &= \int_{\Sigma_{(\rho_0)}} \frac{1}{1+r^a} + \int_{\rho_0}^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Sigma_{(t)}} \frac{1}{1+r^a}\,\mathrm{d}V\right)\mathrm{d}t \leq \\
&\leq \mathrm{Area}(\Sigma_{(\rho_0)}) + \int_{\rho_0}^\infty \frac{1}{1+t^a}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}(\Sigma_{(t)})\right)\mathrm{d}t = \\
&= \mathrm{Area}(\Sigma_{(\rho_0)}) + \left[\frac{1}{1+t^a}\mathrm{Area}(\Sigma_{(t)})\right]_{\rho_0}^\infty + \int_{\rho_0}^\infty \frac{at^{a-1}}{(1+t^a)^2}\mathrm{Area}(\Sigma_{(t)})\,\mathrm{d}t \\
&\leq \mathrm{Area}(\Sigma_{(\rho_0)}) + \int_{\rho_0}^\infty \frac{at^{a-1}}{(1+t^a)^2}\mathrm{Area}(\Sigma_{(t)})\,\mathrm{d}t \leq \\
&\leq C_1\left[\rho_0^2 + a\int_{\rho_0}^\infty \frac{at^{a+1}}{(1+t^a)^2}\,\mathrm{d}t\right] < +\infty\,.
\end{aligned}$$

On the other hand, with similar computations, when $a = 2$ we obtain

$$\begin{aligned}
\int_{\Sigma_{(\rho_2)}\setminus\Sigma_{(\rho_1)}} \frac{1}{r^2}\,\mathrm{d}V &= \int_{\rho_1}^{\rho_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Sigma_{(t)}} \frac{1}{r^2}\,\mathrm{d}V\right)\mathrm{d}t = \int_{\rho_1}^{\rho_2} \frac{1}{t^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}(\Sigma_{(t)})\right)\mathrm{d}t = \\
&= \left[\frac{1}{t^2}\mathrm{Area}(\Sigma_{(t)})\right]_{\rho_1}^{\rho_2} + \int_{\rho_1}^{\rho_2} \frac{2}{t^3}\mathrm{Area}(\Sigma_{(t)})\,\mathrm{d}t \leq \\
&\leq 2C_1 + 2C_1\int_{\rho_1}^{\rho_2} \frac{1}{t}\,\mathrm{d}t = 2C_1\log\left(\frac{\rho_2}{\rho_1}\right)\,.
\end{aligned}$$

□

Take ν a unit normal field to Σ, which is globally defined since Σ is oriented.

Proposition 5.4. *The second variation formula for a variation $u\nu$ with $u \in C_c^\infty(\Sigma)$ is*

$$V''(0) = -\int_\Sigma u[\Delta u + |\mathrm{II}|^2 u + \mathrm{Ricc}(\nu,\nu)u]\,\mathrm{d}V\,,$$

where $V(t)$ is the volume of the variation and $\mathrm{d}V$ is the volume form on Σ with the induced metric from (N, g).

Since Σ is an area minimizing surface, it holds that

$$\int_\Sigma [|\mathrm{II}|^2 + \mathrm{Ricc}(\nu,\nu)]u^2 \,\mathrm{d}V \leq -\int_\Sigma u\Delta u \,\mathrm{d}V = \int_\Sigma |\nabla u|^2 \,\mathrm{d}V$$

for every $u \in C_c^\infty(\Sigma)$. Then, by approximation, for every Lipschitz function u with compact support in Σ we have that

$$\int_\Sigma [|\mathrm{II}|^2 + \mathrm{Ricc}(\nu,\nu)]u^2 \,\mathrm{d}V \leq \int_\Sigma |\nabla u|^2 \,\mathrm{d}V \tag{5.4}$$

Now, for $\rho > \rho_0$, define the function

$$\varphi = \begin{cases} 1\,, & \text{on } \Sigma_{(\rho)} \\ \frac{\log\left(\frac{\rho^2}{r}\right)}{\log\rho}\,, & \text{on } \Sigma_{(\rho^2)} \setminus \Sigma_{(\rho)} \\ 0\,, & \text{outside } \Sigma_{(\rho^2)} \end{cases}.$$

Because of the asymptotic flatness of N, there exists a constant C_2 such that $|\nabla r|^2 \leq C_2$. Thus, taking $u = \varphi$ in the Equation (5.4), we have

$$\begin{aligned}\int_\Sigma [|\mathrm{II}|^2 + \mathrm{Ricc}(\nu,\nu)]\varphi^2 \,\mathrm{d}V &\leq \int_\Sigma |\nabla\varphi|^2 \,\mathrm{d}V \leq \frac{2}{(\log\rho)^2}\int_{\Sigma_{(\rho^2)}\setminus\Sigma_{(\rho)}} \frac{|\nabla r|^2}{r^2}\,\mathrm{d}V \leq \\ &\leq \frac{2C_2}{(\log\rho)^2}\int_{\Sigma_{(\rho^2)}\setminus\Sigma_{(\rho)}} \frac{1}{r^2}\,\mathrm{d}V\,.\end{aligned}$$

Therefore, thanks to the Equation (5.3), we conclude that

$$\int_\Sigma [|\mathrm{II}|^2 + \mathrm{Ricc}(\nu,\nu)]\varphi^2 \,\mathrm{d}V \leq \frac{4C_1C_2}{\log\rho}\left(1 + \frac{1}{\log\rho}\right). \tag{5.5}$$

In particular this implies that

$$\int_{\Sigma_{(\rho)}} |\mathrm{II}|^2 \,\mathrm{d}V \leq \frac{4C_1C_2}{\log\rho}\left(1 + \frac{1}{\log\rho}\right) + \int_\Sigma |\mathrm{Ricc}(\nu,\nu)| \,\mathrm{d}V\,.$$

Thus, letting $\rho \to +\infty$ and using that $\mathrm{Ricc}(\nu,\nu) = \mathcal{O}\left(\frac{1}{r^3}\right)$ thanks to asymptotic flatness, by the Lemma 5.3 it holds

$$\int_\Sigma |\mathrm{II}|^2 \,\mathrm{d}V \leq \int_\Sigma |\mathrm{Ricc}(\nu,\nu)| \,\mathrm{d}V < +\infty\,. \tag{5.6}$$

Take, as in the previous section, a local orthonormal frame $\{e_i\}_{i=1,2,3}$ such that $e_3 = \nu$ is a normal unit field to Σ. Call $h_{ij} = -\langle\nabla_{e_i}\nu, e_j\rangle$ the components of the second fundamental form II, then by definition we have $|\mathrm{II}|^2 = \sum_{i,j=1}^2 h_{ij}^2$.

By minimality of Σ, it holds that $0 = \mathrm{tr}(\mathrm{II}) = h_{11} + h_{22}$ and consequently the Gauss curvature equation tells that

$$K = K_{12} + h_{11}h_{22} - h_{12}^2 = K_{12} - h_{11}^2 - h_{12}^2 = K_{12} - \frac{1}{2}|\mathrm{II}|^2\,. \tag{5.7}$$

Thus, combining this with the Equation (5.6), we have the first part of the proposition

$$\int_\Sigma |K| \, \mathrm{d}V < +\infty \,,$$

since $K_{12} = \mathcal{O}\left(\frac{1}{r^3}\right)$, because N is asymptotically flat.

Even the second sought inequality is an easy consequence of what we have already done. Indeed, plugging the Equation (5.7) in the Equation (5.5), we have that

$$\int_\Sigma \left(K_{12} + \mathrm{Ricc}(\nu,\nu) - K + \frac{1}{2}|\mathrm{II}|^2 \right) \varphi^2 \, \mathrm{d}V \leq \frac{4C_1C_2}{\log\rho}\left(1 + \frac{1}{\log\rho}\right)$$

$$\Longrightarrow \int_\Sigma \left(\frac{R}{2} - K + \frac{1}{2}|\mathrm{II}|^2 \right) \varphi^2 \, \mathrm{d}V \leq \frac{4C_1C_2}{\log\rho}\left(1 + \frac{1}{\log\rho}\right) \,,$$

then, letting $\rho \to +\infty$, we obtain

$$\int_\Sigma \left(\frac{R}{2} - K + \frac{1}{2}|\mathrm{II}|^2 \right) \mathrm{d}V \leq 0 \,.$$

Since $R \geq 0$ and $R > 0$ outside a compact set, it is now obvious to conclude

$$\int_\Sigma K \, \mathrm{d}V > 0 \,.$$

5.4 Conclusion of the proof proving that we have a contradiction

In this last section we want to come to a contradiction proving that at the same time the integral of the section curvature over Σ must be nonpositive, that is:

Let (N, g) be a good oriented asymptotically flat manifold and $\Sigma \subseteq N$ be an oriented complete area minimizing surface properly embedded such that $\Sigma \cap N_\infty \subseteq E_h$ for some $h > \rho_0$. Suppose that

$$\int_\Sigma K \, \mathrm{d}V > 0 \,,$$

then this surface cannot exists, indeed it holds at the same time that

$$\int_\Sigma K \, \mathrm{d}V \leq 0 \,.$$

Notice that we don't require the positivity of the scalar curvature and the negativity of the total mass of N.

Theorem 5.5 (Huber)**.** *Let $(S, \bar{g})$ be a complete Riemannian surface and assume that the Gaussian curvature K of S is integrable. Then*

(i) $\int_S K \, \mathrm{d}V_{\bar{g}} \leq 2\pi\chi(S)$, where χ is the Euler characteristic (Cohn-Vossen's inequality);

(ii) $(S, \bar{g})$ is conformally equivalent to a compact Riemann surface with at most finitely many points removed.

Corollary 5.6. *Under our hypotheses on (N, g) and Σ area minimizing surface in N, Σ is conformally equivalent to $\mathbb{C}$. In particular let us call $F : \mathbb{C} \to \Sigma$ the conformal diffeomorphism.*

Now, let us quote a theorem due to Finn and Huber, which is a generalization of the Gauss-Bonnet theorem for the noncompact case.

Theorem 5.7 (Finn, Huber)**.** *Let D_ρ and C_ρ the disk and the circle of radius ρ in $\mathbb{C}$, respectively. Moreover denote $A_i = \text{Area}(F(D_i))$ and $L_i = \text{Length}(F(C_i))$ for $i \in \mathbb{N}$. Then it holds that*

$$\int_\Sigma K \, \mathrm{d}V = 2\pi - \lim_{i\to\infty} \frac{L_i^2}{2A_i} \, .$$

Thus, in order to prove that $\int_\Sigma K \, \mathrm{d}V \leq 0$, it is sufficient to see that

$$\lim_{i\to\infty} \frac{L_i^2}{4\pi A_i} \geq 1 \, .$$

We will show this inequality using the asymptotic flatness in order to relate lengths and areas with Euclidean lengths and then dealing with the Euclidean quantity thanks to classical estimates.

We call $\tilde{L}_i$ the Euclidean length of $F(C_i)$. Then, by asymptotic flatness of N, we have

$$\tilde{L}_i^2 \leq (1 + \mathrm{o}(1)) L_i^2 \, ,$$

since $F(C_i)$ eventually lies outside every compact set of N. Thus the part concerning the length is easily connected with the Euclidean counterpart and we can focus on the estimates of the areas.

Theorem 5.8. *Let S be a minimal Riemannian surface on $\mathbb{R}^3$ and C be a rectifiable Jordan curve on S of length L which encloses a simply-connected domain D of area A, then $A \leq \frac{L^2}{4\pi}$.*

Denote Σ_i and $\tilde{\Sigma}_i$ respectively the immersed disk and the generic oriented surface of least Euclidean area with boundary $F(C_i)$ and $\tilde{A}$ the Euclidean area. Then, by the theorem stated just above, it holds that

$$\tilde{A}(\tilde{\Sigma}_i) \leq \tilde{A}(\Sigma_i) \leq \frac{\tilde{L}_i^2}{4\pi} \, .$$

Now, let us state one last theorem about minimal surfaces we take for granted.

Theorem 5.9. *Let S be a compact Riemannian minimal surface in $\mathbb{R}^n$, then it lies in the convex hull of its boundary.*

Thanks to this, since $F(C_i) \subseteq E_h$, we have that $\tilde{\Sigma}_i \subseteq E_h$ too. Moreover there exists $x_0 \in \tilde{\Sigma}_i \cap \{(0, 0, x^3) \, : \, x^3 \in \mathbb{R}\}$, because $\tilde{\Sigma}_i$ does not retract onto its boundary, and consequently $\tilde{A}(\tilde{\Sigma}_i \cap B_r(x_0)) \geq \pi r^2$. Since $F(C_i)$ lies outside every compact set for i sufficiently large, we can find a sequence $\rho_i \to \infty$ such that $\tilde{\Sigma}_i \cap B_{\rho_i}(0)$ does not contain boundary terms of Σ_i; then we have

$$\tilde{A}(\tilde{\Sigma}_i \cap B_{\rho_i}(0)) \geq (1 + \mathrm{o}(1)) \pi \rho_i^2 \, ,$$

where in fact we have used that $\tilde{\Sigma}_i \subseteq E_h$.

We now want to compare the area of $\tilde{\Sigma}_i$ with its Euclidean area, thus we slightly modify it near the origin since the metric g is not defined in $B_{\rho_0}(0)$, while $\tilde{\Sigma}_i$ could have a piece there.

Take $\rho_0 \leq \bar{\rho} \leq \rho_0 + 1$ such that $\tilde{\Sigma}_i$ is transverse to $\partial B_{\bar{\rho}}(0)$. Then there exists a domain $\Omega_i \subseteq \partial B_{\bar{\rho}}(0)$ so that $\partial\Omega_i = \tilde{\Sigma}_i \cap \partial B_{\bar{\rho}}(0)$. Thus we define the surface

$$\hat{\Sigma}_i = (\tilde{\Sigma}_i \setminus B_{\bar{\rho}}(0)) \cup \Omega_i \,,$$

which is entirely contained in N_∞.

Noticing that $\tilde{A}(\tilde{\Sigma}_i) \to +\infty$, we conclude that

$$\tilde{A}(\tilde{\Sigma}_i) \leq \tilde{A}(\hat{\Sigma}_i) \leq (1 + \mathrm{o}(1))\tilde{A}(\tilde{\Sigma}_i) \leq (1 + \mathrm{o}(1))\frac{\tilde{L}_i^2}{4\pi}$$

At this point we can estimate the g-area of $\hat{\Sigma}_i$. First of all, by asymptotic flatness, it is possible to choose ρ_i (eventually taking ρ_i smaller) such that

$$A(\hat{\Sigma}_i \cap B_{\rho_i}(0)) \leq \sqrt{\tilde{A}(\hat{\Sigma}_i)}$$

and $A(\hat{\Sigma}_i \cap B_{\rho_i}(0)) \to +\infty$. Moreover, again for the asymptotic flatness, we have that

$$A(\hat{\Sigma}_i \setminus B_{\rho_i}(0)) \leq (1 + \mathrm{o}(1))\tilde{A}(\hat{\Sigma}_i)\,.$$

Therefore, combining the found inequalities, we obtain that

$$A_i \leq A(\hat{\Sigma}_i) \leq (1 + \mathrm{o}(1))\tilde{A}(\hat{\Sigma}_i) \leq (1 + \mathrm{o}(1))\frac{\tilde{L}_i^2}{4\pi} \leq (1 + \mathrm{o}(1))\frac{L_i^2}{4\pi}\,,$$

which leads to the conclusion letting $i \to +\infty$.

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