The Thom Homomorphism

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Abstract

Notes on the Thom homomorphism which I presented as a seminary for the exam of *Differential Topology*. The course has been taken by professor Riccardo Benedetti at the University of Pisa on the academic year 2015/2016.

These notes are mainly based on the chapter *Cobordism* of the book *Differential Topology*, written by Morris W. Hirsch.

Introduction

Let's denote $\mathfrak{N}_n$ the group of the n-dimensional compact manifolds with the operation of disjoint union. The aim of this study is to find an isomorphism of $\mathfrak{N}_n$ with a homotopy group of a certain space which we will call *Thom space*.

Now consider the following situation. Take a couple (Y^{p+k}, B^p) of manifolds and consider a map $f : S^{n+k} \to Y$ transverse to B, then $f^{-1}(B)$ is a n-dimensional submanifold of S^{n+k}. Thus we can define a function $\pi_{n+k}(Y) \to \mathfrak{N}_n$ such that $[f : S^{n+k} \to Y] \mapsto [f^{-1}(B)]$, which is well defined because if $f, g : S^{n+k} \to Y$ are homotopic then $f^{-1}(B)$ and $g^{-1}(B)$ are easily cobordant.

Therefore, the problem now lies in finding a suitable couple (Y, B) for which this map turns out to be an isomorphism. A first try could be done setting $(Y, B) = (E_{s,k}, G_{s,k})$ with $s > n + k$, where $G_{s,k}$ is the Grassman manifold of the k-planes in $\mathbb{R}^s$ and $E_{s,k}$ is its tautological bundle. Indeed, every compact manifold X^n can be embedded in S^{n+k} and locally there exists a vector bundle $f : U \to E_{s,k}$ where U is a tubular neighborhood of X. Unfortunately, in general this can not be extended to a map $f : S^{n+k} \to E_{s,k}$.

Suppose for example to be in the case $n = 0$, $k = 1$ and $s = 2$, which are the minimum numbers respecting the constraints. In this case $E_{s,k} = E_{2,1}$ is homeomorphic to a Möbius strip. Assume to have taken X as the point $(0, -1)$ in S^1 and the tubular neighborhood U as $S^1 \setminus \{(0, 1)\}$. Furthermore suppose that $f : U \to E_{2,1}$ maps bijectively U to a fibre of the vector bundle $E_{2,1}$. Then it is easy to check that it's not possible to extend f to the point $(0, 1)$.

At this stage, the idea of Thom is to add a point to the space $E_{s,k}$ in order to make possible the extension, which for example becomes clear in the situation just presented.

However, before studying the Thom's construction, we need to recall some technical results, mostly concerning extensions of vector bundle maps and tubular neighborhoods.

1 Preliminaries

As warned in the introduction, we denote $G_{s,k}$ the Grassmannian of the k-planes in $\mathbb{R}^s$ and $E_{s,k}$ its tautological bundle.

1.1 Whitney's embedding theorem

Theorem 1.1 (Whitney's embedding). *Let X^n be a compact manifold, then there exists an embedding of X in $\mathbb{R}^{n+k}$ for every $k > n$.*

Theorem 1.2 (Whitney's extension). *Let $f : X^n \to \mathbb{R}^{n+k}$ be an embedding from a neighborhood of a closed set $A \subseteq X$ with $k > n$. Then there exists an embedding $g : X \to \mathbb{R}^{n+k}$ which coincides with f on A.*

Corollary 1.3 (Whitney's embedding with boundary). *If $k > n+1$, any embedding $\partial W^{n+1} \hookrightarrow S^{n+k}$ extends to an embedding $W^{n+1} \hookrightarrow D^{n+k+1}$.*

Proof. It is easy to construct a function $f : W^{n+1} \to D^{n+k+1}$ which is an embedding from a collar of ∂W^{n+1} and coincides with the given embedding $\partial W^{n+1} \hookrightarrow S^{n+k}$ from the boundary. Then Theorem 1.2 (Whitney's extension) gives the sought embedding from the whole W^{n+1}. □

1.2 Extension theorem for tubular neighborhoods

Theorem 1.4. *Let A be a submanifold of a manifold X. Then every tubular neighborhood of ∂A in ∂X is the intersection with ∂X of a tubular neighborhood for A in X.*

Proof. Call V the tubular neighborhood of ∂A in ∂X, which we want to extend to a tubular neighborhood U of A in X.

Take a collar of ∂X in X, which we can identify with $C_X = \partial X \times [0,1]$. Then $C_A = \partial A \times [0,1] = A \cap \partial(X \times [0,1])$ is a collar of ∂A in A. Now let's take U' a tubular neighborhood of $A \setminus C_A$ in $X \setminus C_X$, which exists for general results on tubular neighborhoods, and consider its intersection with $\partial(X \setminus C_X) = \partial X \times \{1\}$. This gives an other tubular neighborhood V' of ∂A in ∂X.

Consequently, we only need to prove that we can find a tubular neighborhood of C_A in C_X which coincides with $V \cup V'$ on the boundary. However, we now that V and V' are isotopic through a map $H : V \times [0,1] \to \partial X$, which gives a tubular neighborhood of ∂A at every time, then the map $\hat{H} : V \times [0,1] \to C_X$ such that $\hat{H}(x,t) = (H(x,t),t)$ provides the sought tubular neighborhood U'' of C_A in C_X.

Finally, we only need to glue smoothly U' and U'' to obtain the tubular neighborhood U which extends V.

□

1.3 Classification theorem for vector bundles

Lemma 1.5. *Let $\xi = (\pi, E, X)$ be a vector bundle of rank k over the n-manifold X and let U be a neighborhood of a closed set A in X. Suppose to have $\tilde{F} : E|_U \to U \times \mathbb{R}^s$ a vector bundle map injective in every fibre with $s \geq n+k$. Then there exist a vector bundle map $F : E \to X \times \mathbb{R}^s$ injective in every fibre which coincides with $\tilde{F}$ in a neighborhood of A.*

Theorem 1.6. *If $s \geq k+n$, then every bundle $\xi = (\pi, E, X)$ of rank k over an n-manifold X has a classifying map $f_\xi : X \to G_{s,k}$, i.e. $f_\xi^*\ E_{s,k} \cong \xi$. In fact any classifying map $\partial X \to G_{s,k}$ for $\xi|_{\partial X}$ extends to a classifying map for ξ.*

Proof. Let $F : E \to X \times \mathbb{R}^s$ a vector bundle map injective in every fibre given by Lemma 1.5. Thus define $f_\xi : X \to G_{s,k}$ such that $f_\xi(x) = \pi_{\mathbb{R}^s}(F(\xi_x))$, which is a k-plane in $\mathbb{R}^s$ and hence it belongs to $G_{s,k}$. Then f_ξ turns out to be a classifying map for ξ.

Now, suppose to have already a classifying map $f_{\xi|_{\partial X}} : \partial X \to G_{s,k}$ for $\xi|_{\partial X}$. Then take a collar $C_X = \partial X \times [0,1)$ for ∂X in X, so that $E|_{C_X}$ is isomorphic to $E|_{\partial X} \times [0,1)$. Define

$$\begin{array}{rcl} \tilde{F} : E|_{C_X} \cong E|_{\partial X} \times [0,1) & \to & C_X \times \mathbb{R}^s \\ (v,t) & \mapsto & ((\pi(v),t), f_{\xi|_{\partial X}}(v)) \end{array}$$

At this point, always by Lemma 1.5, we can take F coinciding with $\tilde{F}$ on ∂X and do the same construction for f_ξ as before. However, this time f_ξ extends $f_{\xi|_{\partial X}}$.

$\square$

Corollary 1.7. *If $s \geq k+n$, given a vector bundle $\xi = (\pi, E, X)$ of rank k over an n-manifold X, there exists a vector bundle map $f : E \to E_{s,k}$ such that $f \pitchfork G_{s,k}$ and $X = f^{-1}(G_{s,k})$. Furthermore, given a vector bundle map $f|_{\partial X} : E|_{\partial X} \to E_{s,k}$ such that $f|_{\partial X} \pitchfork G_{s,k}$ and $\partial X = (f|_{\partial X})^{-1}(G_{s,k})$, f can be found extending $f|_{\partial X}$.*

Proof. It is sufficient to take $f : E \to E_{s,k}$ such that $f(v) = (f_\xi(x), \pi_{\mathbb{R}^s}(F(v)))$ where $v \in \xi_x$ and f_ξ, F are the maps given in the proof of Theorem 1.6. $\square$

1.4 Homotopy with a vector bundle map

Theorem 1.8. *Let X be a manifold, E a vector bundle over a manifold B and $f : X \to E$ a map such that f and $f|_{\partial X}$ are both transverse to B. Suppose that $A = f^{-1}(B)$ is compact and to have $U \subseteq X$ tubular neighborhood of A in X. Finally take $D \subseteq U$ a disk subbundle.*

Then there exists a homotopy F_t from $f = F_0$ to $F_1 = \phi : X \to E$ such that

- *$\phi|_D$ is the restriction of a vector bundle map $U \to E$ over $f : A \to B$;*
- *$F_t = f$ on $A \cup (X \setminus U)$;*
- *$F_t^{-1}(E \setminus B) = X \setminus A$.*

Proof. Let $\phi : U_x \to E$ be the component of $\mathrm{d}f_x : U_x \subseteq T_xU \to T_{f(x)}E$ along $E_{f(x)}$, for every $x \in A$. By construction, ϕ is a vector bundle map and is homotopic to f as a map from D to E through the homotopy

$$f_t(x) = \begin{cases} \frac{1}{t} f(tx)\,, & 0 < t \leq 1 \\ \phi(x)\,, & t = 0 \end{cases} .$$

This can be extended to a homotopy $F_t : X \to E$ such that $F_t \equiv f_t$ on D and $F_t \equiv f$ on $X \setminus U$ by homotopy extension theorem, which concludes the proof.

$\square$

2 The Thom space of a vector bundle

Let $E \xrightarrow{\pi} B$ be a vector bundle of rank k over the compact manifold B. Consider $E^* = E \cup \{\infty\}$ the one-point compactification of E, which we also call the *Thom space* of the vector bundle.

Proposition 2.1. *With the above notations, the space $E^* \setminus B$ is contractible.*

Proof. The homotopy $h : (E^* \setminus B) \times I \to E^* \setminus B$ such that

$$h(x,t) = \begin{cases} \frac{1+t}{1-t}\, x\,, & \text{if } 0 \le t < 1,\ x \neq \infty \\ \infty\,, & \text{if } t = 1 \text{ or } x = \infty \end{cases}$$

gives a contraction of $E^* \setminus B$ to ∞.

□

Lemma 2.2. *Let A be a closed submanifold of a manifold X. Then two maps $f, g : X \to E^* \setminus B$ which agree on A are homotopic relatively to A.*

Proof. Let $c : (X \setminus A) \times [0,1] \to E^* \setminus B$ be a homotopy from f to g, which exists because $E^* \setminus B$ is contractible, and let $\lambda : X \to \mathbb{R}$ be a map such that $\lambda|_A = 0$ and $\lambda > 0$ otherwise. Then consider $h : X \times [0, \infty] \to E^* \setminus B$ defined by

$$h(x,t) = \begin{cases} c(x, \lambda(x)t)\,, & \text{if } 0 \le t < \infty \\ g(x)\,, & \text{if } t = \infty \end{cases}.$$

Finally, eventually rescaling $[0,\infty]$ to $[0,1]$, h turns out to be a homotopy from f to g relatively to A. □

We say that a map $f : X \to E^*$ is in standard form if there is a submanifold $A \subseteq X$ and a tubular neighborhood $U \subseteq X$ of A such that $U = f^{-1}(E)$, $A = f^{-1}(B)$ and $f|_U$ is a vector bundle map over $f|_A$. In particular, this immediately implies that $f(X \setminus U) = \infty$ and $f \pitchfork B$ if the codimension of A in X is equal to the rank of E over B (which will be our case). This is because the tangent of E could be broken in a part tangent to B and in one tangent to the fibres and the image of the tangent through f surely take the part tangent to the fibres, since f induces isomorphisms in the fibres.

Lemma 2.3. *Let X, B be compact manifolds without boundary, then every map $f : X \to E^*$ is homotopic to a map in standard form.*

Proof. By the transversality theorem, we can assume that $f \pitchfork B$. Then $A := f^{-1}(B)$ is a submanifold of X and we can take U one of its tubular neighborhoods on X.

Because of Theorem 1.8, we can also assume that f agrees on a disk subbundle $D \subseteq U$ with a vector bundle map $\phi : U \to E$. Extend ϕ to a map from X to E^* setting $\phi(X \setminus U) = \infty$, obtaining a map in standard form. Then f and ϕ agree on ∂D and map $X \setminus \operatorname{Int} D$ into $E^* \setminus B$, therefore they are homotopic by Lemma 2.2. □

Proposition 2.4. *Given a structure of CW-complex on B, E^* has a $(k-1)$-connected CW-complex structure having one $(m+k)$-cell for each m-cell of B and one additional 0-cell, corresponding to the point ∞.*

Proof. We denote $\{D^{n_\lambda} \xrightarrow{\sigma_\lambda} B\}_{\lambda\in\Lambda}$ the set of the cells forming the CW-complex structure over B. Now, let's construct the CW-complex structure over E^*, i.e. we have to provide a set $\{D^{n_\mu} \xrightarrow{\sigma_\mu} E^*\}_{\mu\in I}$ such that

1. $\{\sigma_\mu(\text{Int } D^{n_\mu})\}_{\mu\in I}$ is a partition on E^*;
2. $\sigma_\mu|_{\text{Int } D^{n_\mu}}$ is a homeomorphism for every $\mu \in I$;
3. $\sigma_\mu(\partial D^{n_\mu})$ is contained in the union of a finite number of elements of the partition, each having cell dimension less than n_μ, for every $\mu \in I$.

For every $\lambda \in \Lambda$, the cell $c_\lambda = \sigma_\lambda(\text{Int } D^{n_\lambda})$ is homeomorphic to D^{n_λ}, hence it is contractible. Therefore $\pi^{-1}(c_\lambda)$ is homeomorphic to $c_\lambda \times \mathbb{R}^k$. This gives an idea of how to construct the cells partition in E^*.

Indeed, for every $\lambda \in \Lambda$, define $\tilde\sigma_\lambda : D^{n_\lambda} \times D^k \to E^*$ such that $\tilde\sigma_\lambda|_{\text{Int } D^{n_\lambda}\times\text{Int } D^k} \equiv \sigma_\lambda \times \varphi : \text{Int } D^{n_\lambda} \times \text{Int } D^k \to c_\lambda \times \mathbb{R}^k \cong \pi^{-1}(c_\lambda)$, where $\varphi : \text{Int } D^k \to \mathbb{R}^k$ is a chosen homeomorphism. Then define $\tilde\sigma_\lambda$ on $\partial D^{n_\lambda} \times \text{Int } D^k$ by continuity and on $D^{n_\lambda} \times \partial D^k$ going to ∞.

Therefore $\{D^{n_\lambda} \times D^k \xrightarrow{\tilde\sigma_\lambda} E^*\}_{\lambda\in\Lambda} \cup \{D^0 \to \{\infty\}\}$ forms a CW-structure over E^*, composed by a $(m+k)$-cell for each m-cell of B and one additional 0-cell corresponding to the point ∞.

□

3 The Thom homomorphism

For our aims, we are in particular interested in the Thom space over the tautological bundle of the Grassmannian $G_{s,k}$, this is because of the reasons anticipated in the introduction, where we do the one-point compactification to solve the problem of non-extension. Therefore, we are now going to construct our homomorphism from $\pi_{n+k}(E^*_{s,k})$ to $\mathfrak{N}_n$.

Take $\alpha \in \pi_{n+k}(E^*_{s,k})$ represented by a function $f : S^{n+k} \to E^*_{s,k}$, which we can assume transverse to $G_{s,k}$ up to homotopy. Then we define $\tau(\alpha) = [f^{-1}(G_{s,k})] \in \mathfrak{N}_n$, which is a well-defined class of cobordism.

First of all we have to check that τ is addictive. Take $\alpha = [f], \beta = [g] \in \pi_{n+k}(E^*_{s,k})$, where we can suppose that f maps the lower hemisphere to ∞, whereas g maps the upper hemisphere to ∞. Indeed we can suppose f and g in standard form for Lemma 2.3 and then reduce to the sought case using that S^{n+k} minus a point is contractible. Thus $\alpha+\beta$ is the class of the map $S^{n+k} \to E^*_{s,k}$ which coincides with f in the upper hemisphere and with g in the lower one. Therefore we obtain easily that τ is an homomorphism, indeed we have

$$\tau(\alpha+\beta) = [f^{-1}(G_{s,k}) \sqcup g^{-1}(G_{s,k})] = [f^{-1}(G_{s,k})] + [g^{-1}(G_{s,k})] = \tau(\alpha) + \tau(\beta)\,.$$

The theorem we want now to prove states that, for k and s sufficiently big, τ is an isomorphism, which will solve our problem.

Theorem 3.1 (Thom)**.** *The Thom homomorphism $\tau : \pi_{n+k}(E^*_{s,k}) \to \mathfrak{N}_n$ is*

1. *surjective if $k > n$ and $s \geq k+n$;*

2. *injective if $k > n+1$ and $s \geq k+n+1$.*

Proof. Let's prove the two points separately.

1. Let $[X^n] \in \mathfrak{N}_n$, then we can suppose $X^n \subseteq S^{n+k}$ because of Theorem 1.1 (Whitney's embedding), since $k > n$. Furthermore, consider U a tubular neighborhood of X in S^{n+k}.

   Thus, by Corollary 1.7 using $s \geq k+n$, there exists a vector bundle map $f : U \to E_{s,k}$, which we can extend to a map $f : S^{n+k} \to E^*_{s,k}$ mapping $S^{n+k} \setminus U$ to ∞. The application f is transverse to X and $f^{-1}(G_{s,k}) = X$, consequently $\tau([f]) = [X]$, which proves the surjectivity.

2. Take $f : S^{n+k} \to E^*_{s,k}$ such that $\tau([f]) = 0$, then we want to prove that $[f] = 0$, i.e. f is homotopic to a constant.

   First of all we can assume that f is in standard form by Lemma 2.3 and we denote $X^n = f^{-1}(G_{s,k})$ and U the tubular neighborhood of X such that $U = f^{-1}(E_{s,k})$. Then $[X^n] = 0$ and consequently there exists W^{n+1} compact manifold with boundary X. Because of Corollary 1.3 (Whitney's embedding with boundary), the embedding of X in S^{n+k} can be extended to an embedding of W in D^{n+k+1}, since $k > n+1$.

   Now call V the tubular neighborhood of W obtained extending U by Theorem 1.4, then we can extend the bundle map $f : U \to E^*_{s,k}$ to a bundle map $h : V \to E^*_{s,k}$ applying Corollary 1.7, using at this point that $s \geq k+n+1$. Finally let's extend h to all D^{n+k+1} by putting $h(D^{n+k+1} \setminus V) = \infty$. This leads to a homotopy of $f = h|_{S^{n+k}}$ with the constant map $h|_{\{0\}}$, therefore $[f] = 0$ as expected.

$\square$

Example. Let's now do some examples of Thom spaces over Grassmannians and of what Theorem 3.1 (Thom) consequently tells us.

- If $k = 1$, $G_{s,1}$ is diffeomorphic to $\mathbb{P}^{s-1}(\mathbb{R})$, while $E^*_{s,1}$ is diffeomorphic to $\mathbb{P}^s(\mathbb{R})$. Indeed

  $$\begin{array}{rcc} \varphi : \mathbb{P}^s(\mathbb{R}) \setminus \{[0, \dots, 0, 1]\} & \to & \mathbb{P}^{s-1}(\mathbb{R}) \times \mathbb{R}^s \\ [x_0, \dots, x_{s-1}, x_s] & \to & ([x_0, \dots, x_{s-1}], (x_s x_0, \dots, x_s x_{s-1})) \end{array}$$

  is a diffeomorphism between $\mathbb{P}^s(\mathbb{R}) \setminus \{[0, \dots, 0, 1]\}$ and $E_{s,1} \subseteq \mathbb{P}^{s-1}(\mathbb{R}) \times \mathbb{R}^s$, which leads to a diffeomorphism between $\mathbb{P}^s(\mathbb{R})$ and $E^*_{s,1}$.

  Unfortunately, the only case in which Theorem 3.1 (Thom) tells us something is when $n = 0$, from which we obtain that $\tau : \mathbb{Z}/2\mathbb{Z} \cong \pi_1(E^*_{s,1}) \to \mathfrak{N}_0$ is surjective.

- If $k = s$, the Grassmannian $G_{s,s}$ consists in one point, thus $E_{s,s} \cong \mathbb{R}^s$ and consequently $E^*_{s,s} \cong S^s$. As in the previous case, this gives us information only on the surjectivity of $\tau : \mathbb{Z} \cong \pi_s(E^*_{s,s}) \to \mathfrak{N}_0$.

- If $s = 3$ and $k = 2$, we can identify every plane in $\mathbb{R}^2$ with its normal vector, thus $G_{3,2}$ is diffeomorphic to $\mathbb{P}^2(\mathbb{R})$, indeed the antipodal of a vector corresponds to the same plane. It's now easy to see that $E_{3,2}$ is diffeomorphic to the tangent bundle of $\mathbb{P}^2(\mathbb{R})$.

This is the first case in which the full hypothesis of Theorem 3.1 (Thom) are fulfilled, thus we obtain that $\mathfrak{N}_0$ is isomorphic to the second homotopy group of the one-point compactification of $T\mathbb{P}^2(\mathbb{R})$.

4 Homotopy groups of the Thom space

Hence, we have proved that the Thom homomorphism $\tau : \pi_{n+k}(E^*_{s,k}) \to \mathfrak{N}_n$ is an isomorphism if $k > n+1$ and $s \geq k+n+1$.

This drives the study of the cobordism groups to the analysis of the homotopy groups of the Thom space over Grassmannians, which is become an algebraic topology problem.

First of all, let's observe that $E^*_{s,k}$ admits a structure of finite dimensional CW-complex because of Proposition 2.4. Indeed its base space $G_{s,k}$ can be triangulated being a smooth manifold, hence admits a structure of simplicial complex and consequently of finite dimensional CW-complex.

However let's present an explicit structure of CW-complex over $G_{s,k}$ given by the *Schubert cells.*

Definition 4.1. Let's consider the canonical basis $e_1, \ldots, e_s$ of $\mathbb{R}^s$. Then for every $j = (j_1, \ldots, j_k)$ such that $1 \leq j_1 < j_2 < \ldots < j_k \leq s$ define

$$C_j \coloneqq \{W \in G_{s,k} \ : \ \dim(W \cap \operatorname{span}\langle e_1, \ldots, e_{j_l}\rangle\ ) = l \text{ for every } l = 1, \ldots, k\}\,,$$

which are called the Schubert cells of the Grassmannian $G_{s,k}$.

It can be proven the following result, which gives us a CW-complex structure over $G_{s,k}$ composed by a finite number of cells of dimension less or equal then $k(s-k)$.

Proposition 4.2. *The Schubert cells C_j, as $j = (j_1, \ldots, j_k)$ varies over the ones for which $1 \leq j_1 < j_2 < \ldots < j_k \leq s$, form a disjoint partition of $G_{s,k}$ which constitutes a CW-complex structure. In particular C_j is a cell of the CW-complex of dimension* $\dim(C_j) = \sum_{l=1}^{k} j_k - l$.

What just observed, together with Proposition 2.4, says that $E^*_{s,k}$ has a structure of finite $(k-1)$-connected CW-complex. This will give has information about its homotopy groups thanks to the following theorem.

Theorem 4.3 (Serre)**.** *Let X be a finite $(k-1)$-connected CW-complex with $k \geq 2$. Then the Hurewicz homomorphism $\pi_i(X) \to H_i(X, \mathbb{Z})$ such that $[f : S^i \to X] \mapsto f_*(h_i)$, where h_i is the canonical generator of $H_i(S^i)$, has both kernel and cokernel finite abelian groups for $i < 2k-1$.*

Corollary 4.4. *The group $\pi_{n+k}(E^*_{s,k})$, with $k > n+1$ and $s \geq k+n+1$, is finitely generated.*

Proof. It follows from the previous theorem, since the m-element of the complex defining the homology of $E^*_{s,k}$ is generated by the m-cells of the CW-complex structure over $E^*_{s,k}$ and consequently is finite generated. □

5 Final results

From the Corollary 4.4, we obtain directly the following property of the n-th group of cobordism $\mathfrak{N}_n$.

Proposition 5.1. *The group of cobordism of the n-dimensional manifolds $\mathfrak{N}_n$ is finitely generated. This means that there is a finite set of n-manifolds such that every compact n-manifolds is cobordant to the disjoint union of a finite number of copies of these manifolds.*

In particular, since $\mathfrak{N}_n$ is a $\mathbb{Z}/2\mathbb{Z}$ vector space, $\mathfrak{N}_n$ is finite, thus there is $m \in \mathbb{N}$ such that $\mathfrak{N}_n \cong (\mathbb{Z}/2\mathbb{Z})^m$.

More in general, the direct sum $\mathfrak{N}_*$ of the groups $\mathfrak{N}_n$ has been completely characterized. However, we just quote without proof the principal result in the following theorem.

Theorem 5.2 (Thom)**.** *The graded algebra $\mathfrak{N}_* = \sum_{n\geq 0} \mathfrak{N}_n$, where the product is given by the Cartesian product of manifolds and the grading by the dimension, corresponds to the polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by an element $x_n \in \mathfrak{N}_n$ for every $n \neq 2^k - 1$ positive integer. Furthermore, if n is even, it is possible to choose $x_n = [\mathbb{P}^n(\mathbb{R})]$.*

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