SOLUTIONS OF EXERCISES WEEK THREE

Exercise 1. The statement $P \vee Q$ correspond to "P or Q".
Using the symbols $\wedge, \vee$ how can we express "P or Q, but only one of them".

Solution. $(P \vee Q) \wedge \neg(P \wedge Q)$. $\square$

Exercise 2. Suppose that we are dealing only with sets (no proper classes) and that the Class Construction Axiom of Cantor holds. During the lectures, we defined

$$A \in T \Leftrightarrow A \text{ is an infinite set}$$

and showed that $T \in T$. Prove that $T - \{T\} \in T - \{T\}$.

Solution. We set $S := T - \{T\}$. Since T is infinite, S is infinite. Then $S \in T$. Since $T \in T$, we also have $T - \{T\} \subsetneq T$. Therefore, $S \neq T$. Then $S \in T - \{T\}$. $\square$

Exercise 3. Let $f\colon A \to B$ be a function. Given two subclasses $C_1, C_2 \subseteq A$, show that

$$\bar{f}(C_1 \cup C_2) = \bar{f}(C_1) \cup \bar{f}(C_2)$$
$$\bar{f}(C_1 \cap C_2) \subseteq \bar{f}(C_1) \cap \bar{f}(C_2).$$

Solution. Given an element y, we have

$$\begin{aligned} y \in \bar{f}(C_1 \cup C_2) &\Leftrightarrow \exists x \in C_1 \cup C_2 \text{ s.t. } f(x) = y \\ &\Leftrightarrow (\exists x \in C_1 \text{ s.t. } f(x) = y) \vee (\exists x \in C_2 \text{ s.t. } f(x) = y) \\ &\Leftrightarrow (y \in \bar{f}(C_1)) \vee (y \in \bar{f}(C_2)). \end{aligned}$$

Given an element y, we have

$$\begin{aligned} y \in \bar{f}(C_1 \cap C_2) &\Rightarrow \exists x \in C_1 \cap C_2 \text{ s.t. } f(x) = y \\ &\Rightarrow (\exists x \in C_1 \text{ s.t. } f(x) = y) \wedge (\exists x \in C_2 \text{ s.t. } f(x) = y) \\ &\Rightarrow (y \in \bar{f}(C_1)) \cap (y \in \bar{f}(C_2)). \end{aligned}$$

$\square$

Exercise 4. Let $f\colon \mathbf{N} \to \mathbf{N}$ defined as $f(n) = 2^n - 1$. Is the function injective, surjective or bijective?

Solution. f is injective, as $2^n = 2^m \Rightarrow n = m$. f is not surjective because $2^n - 1$ is always an odd number. Then, for instance $2 \notin \operatorname{ran}(f)$. $\square$

Exercise 5. Is $\mathbf{N}$ equipotent to $\mathbf{N} - \{1\}$? is $\mathbf{N}$ equipotent to $\mathbf{N} \cup \{\pi\}$? if the answer is yes, find a bijective function.

Solution. $\mathbf{N} \approx \mathbf{N} - \{1\}$. In fact $g(n) = n + 1$ is a bijective function from $\mathbf{N} - \{1\}$ to $\mathbf{N}$. While

$$h\colon \mathbf{N} \cup \pi \to \mathbf{N}, \quad h(x) = \begin{cases} 1 & \text{if } x = \pi \\ x + 1 & \text{if } x \neq \pi. \end{cases}$$

Date: 2015, March 27.

This function is surjective. In fact, given $n \in \mathbf{N}$, if $n = 1$, then $n = h(\pi)$, so $n \in \operatorname{ran}(h)$. If $n \geq 2$, then $n = h(n+1)$. It is injective, because if $x, y \neq \pi$ we have

$$h(x) = h(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y.$$

If $x = \pi$ and $y \neq \pi$, we have

$$h(x) = h(y) \Rightarrow 1 = y + 1 \Rightarrow y = 0$$

but, by definition of $\mathbf{N}$, the zero is not an element of $\mathbf{N}$. □

Exercise 6. Let f be the function defined from $A = \{0,1,2\}$ to $B = \{3,4\}$ as

$$f(0) = 3,\ f(1) = 3,\ f(2) = 4.$$

Is f injective or surjective? Is there a function $g \colon B \to A$ such that $g \circ f = id_A$? How many of them?

Solution. f is surjective. In fact, $B \subseteq \operatorname{ran}(f)$ as $3 = f(0)$ and $4 = f(1)$. The function is not injective because $f(0) = f(1) = 3$. There is no such function g because, on the contrary,

$$g \circ f(x) = x$$

for every $x \in A$ would imply

$$g(f(0)) = 0, \quad g(f(3)) = 3.$$

However, $f(0) = f(1) = 3$. Then $g(3) = 0 = 3$ which gives a contradiction. Then, there are no functions g such that $g \circ f = id_A$. □

Exercise 7. Let A and B be two classes such that $\#A = n$ and $\#B = m$. How many different functions are $f \colon A \to B$.

Exercise 8. For each element of A we have m different choices in B. Then, there are m^n functions.

Exercise 9. In the example

	x	y	z	D
x	1	0	0	1
y	0	0	1	1
z	0	0	1	1
D	0	0	0	0

What are the sets and proper classes? which of the following classes exist

$$x \cap y, \quad y \cap z, \quad x \cup z, \quad \{x\}, \quad \{y\}, \quad \{D\}, \quad \emptyset, \quad \mathscr{U}?$$

Is Axiom 2 satisfied? is Axiom 3 satisfied?

Solution.

Sets: x, y, z.

Proper classes: D.

$x \cap y = \{x\} \cap \emptyset = \emptyset = y$. So, $x \cap y$ exists because $x \cap y = y$.

$y \cap z = y$.

$x \cup z = \{x\} \cup \{y, z\} = \{x, y, z\}$ which exists because it is equal to D.

$\{x\} = x$.

$\nexists \{y\}$.

$\nexists\{D\}$ because D is a proper class.

$\emptyset = y$.

$\mathscr{U} = D$.

Axiom 2 is not satisfied, because, for example $\{y\}$ does not exist. In order to check Axiom 3, we look at the pairs

Pairs: x, z.

We compare the pairs with the sets, which are x, y, z. Then all the pairs are sets. Then Axiom 3 holds. $\square$

SOLUTIONS OF THE EXERCISES WEEK FIVE

Exercise 1. Le $f\colon A \to B$ be a function such that there are two functions $g_1, g_2\colon B \to A$ such that

$$g_i \circ f = id_A, \quad f \circ g_i = id_B.$$

Prove that $g_1 = g_2$.

Solution. We take the composite function

$$(g_1 \circ f) \circ g_2 = id_A \circ g_2 = g_2.$$

On the other hand

$$(g_1 \circ f) \circ g_2 = g_1 \circ (f \circ g_2) = g_1 \circ id_B = g_1.$$

Then $g_1 = g_2$. □

Exercise 2. Given three classes A, B and x, is it true or false that

(a) $A \subseteq B \Rightarrow A - x \subseteq B - x$
(b) $A \in B \Rightarrow A - x \in B - x$.

Solution. (a). It is true. If $y \in A - x$, then $(y \in A) \wedge (y \notin x)$. Since $A \subseteq B$, we have

$$(y \in A) \wedge (y \notin x) \Rightarrow (y \in B) \wedge (y \notin x).$$

(b). It is false. For instance, $A = \{0, 1\}$ and $B = \{\{0, 1\}\}$, clearly, $A \in B$. If $x = \{0\}$, then

$$A - x = \{1\} \notin B - x = B.$$

□

Exercise 3. Let $x := (a, b)$ be an ordered pair. Find the following classes:

$$\cup(\cap x), \quad \cap(\cup x - \cap x).$$

Solution. We have

$$\cap x = \{a, b\} \cap \{a\} = \{a\}.$$

Then $\cup(\cap x) = a$. If $a \neq b$, then

$$\cup x - \cap x = \{b\}, \quad \cap(\cup x - \cap x) = b.$$

If $a = b$, then $\cup x - \cap x = \emptyset$, so the generalized intersection is not defined. □

Exercise 4 (A2 + A3 + A4). Suppose that the three axioms A2, A3 and A4 hold. Suppose that there exists a set x. Then, there are infinitely many sets.

Solution. From A2, there exists $\emptyset$. From A4, $x \subseteq \emptyset$ implies $\emptyset$ is a set. Now, suppose that there are finitely many sets many sets

$$\emptyset, x_2, \ldots, x_n \tag{1}$$

all different from each other. Still, from A3, we have the sets

$$\{\emptyset\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\} \tag{2}$$

all different from each other. Moreover, there is the emptyset. So, there are $n + 1$ sets. □

Date: 2016, April 11.

Exercise 5. Let A and B be two classes, both non-empty. Are the following implications true or false?

(a) $A \subseteq B \Rightarrow \cup A \subseteq \cup B$
(b) $A \subseteq B \Rightarrow \cap B \subseteq \cap A$.

Are the converse implications true?

Solution.
(a). $x \in \cup A \Rightarrow \exists y \in A$ s.t. $x \in y$. Since $A \subseteq B$, we have

$$\exists y \in A \text{ s.t. } x \in y \Rightarrow \exists y \in B \text{ s.t. } x \in y \Rightarrow x \in \cup B.$$

The converse is not true. For instance, if a, b, c are three sets different from each other, we define

$$A = \{\{a,b\},\{b,c\}\}, \quad B = \{\{a,b,c\}\}, \quad A \nsubseteq B.$$

However,

$$\cup A = \cup B = \{a,b,c\}.$$

(b). $x \in \cap B \Rightarrow (\forall y \in B) x \in y$. Since $A \subseteq B$, we have $x \in y$ for every $y \in A$. Then $x \in \cap A$. The converse implication is not true. Consider the example

$$A = \{\{a\}\}, \quad B = \{\{b\},\{c\}\}.$$

Then $A \nsubseteq B$, but $\cap B = \emptyset \subseteq \cap A$. □

Exercise 6. Is A2 equivalent to any of the following statements?

(a) given a statement $p(x)$ there exists a class R such that $p(R)$ it is true
(b) given a statement $p(x)$ there exists a class R such that

$$x \in R \Leftrightarrow x \text{ is a set and } x \in R.$$

(c) given a statement $p(x)$ there exists a set R such that

$$x \in R \Leftrightarrow x \text{ is a set and } p(x) \text{ is true}$$

(d) given a statement $p(x)$ there exists a class R such that

$$x \in R \Leftrightarrow p(x) \text{ is true.}$$

Solution.
(a). A2 does not imply (a). In fact, here "$p(x)$" has been replaced with "$p(R)$". For instance, if we define $p(x) : x \nsubseteq x$, there is not class R satisfying the property p.

(b). (b) does not imply A2. In the statement "$p(x)$" does not appear at all. Consequently, the new statement, "there exists a class R such that $x \in R \Leftrightarrow x$ is a set and $x \in R$" is always true, regardless whether A2 is satisfied or not.

(c). A2 is equivalent to (c).

(d). A2 does not imply (d). An example is given from the Russell Class. □

Exercise 7. Is it true that $\mathbf{R} \approx \mathbf{R} - \{0\}$? is yes, find a bijective function.

Solution. We can define explicitly a bijective function

$$g(x) := \begin{cases} x+1 & \text{if } x \in \mathbf{N} \cup \{0\} \\ x & \text{if } x \notin \mathbf{N} \cup \{0\}. \end{cases}$$

□

Exercise 8. Let $f\colon A \to B$ be a function. Given two subclasses $D_1, D_2 \subseteq A$, show that

$$\check{f}(D_1 \cup D_2) = \check{f}(D_1) \cup \bar{f}(D_2)$$
$$\check{f}(D_1 \cap D_2) = \check{f}(D_1) \cap \check{f}(D_2).$$

Solution.

$$\begin{aligned} x \in \check{f}(D_1 \cup D_2) &\Leftrightarrow f(x) \in D_1 \cup D_2 \Leftrightarrow (f(x) \in D_1) \vee (f(x) \in D_2) \\ &\Leftrightarrow (x \in \check{f}(D_1)) \vee (x \in \check{f}(D_2)). \\ x \in \check{f}(D_1 \cap D_2) &\Leftrightarrow f(x) \in D_1 \cap D_2 \Leftrightarrow (f(x) \in D_1) \wedge (f(x) \in D_2) \\ &\Leftrightarrow (x \in \check{f}(D_1)) \wedge (x \in \check{f}(D_2)). \end{aligned}$$

□

EXERCISES WEEK SEVEN

Exercise 1 (A1 + A2 + A3)**.** Given a, b be two sets. Prove that $\cup(\cup A \times B) = A \cup B$.

Exercise 2 (A2 + A3)**.** Let $f\colon A \to B$ be a function. As a function f is also a class. Prove that $f \approx A$.

Exercise 3. Prove that $\mathbf{N} \approx \mathbf{N} \times \{0, 1\}$ (that is, find a bijective function).

Exercise 4. In the set of natural numbers, consider the order relation given by $(n, m) \in R$ if and only if $n \mid m$. Find a maximal chain.

Exercise 5 (A1 + A2 + A3 + A4)**.** Let A, B be two classes. Prove that there are two classes $C \approx A$ and $D \approx B$ such that $C \cap D = \emptyset$.

Date: 2015, April 11.

EXERCISES OF WEEK NINE

Exercise 1. Let $C_1, C_2 \subseteq A$ be two chains in the order relation $(A, \leq)$. Is $C_1 \cap C_2$ a chain?

Exercise 2. In $(A, \leq)$ we consider the relation

$$xGy : x \text{ is comparable to } y.$$

Prove that if $(A, \leq)$ is a FOC, then G is an equivalence relation. Is the converse true? that is, if G is an equivalence relation is it true that $(A, \leq)$ a FOC?

Exercise 3. Let A be a non-empty class. Prove that every Choice Function is surjective.

Exercise 4. During the final exam, a student has to solve the following exercise:

"find a Choice Function for the set $A = \{0, 1, 2, 3\}$".

Unfortunately, he did not study Choice Functions. The only thing he remembers is that a Choice Function is a function defined on $\mathscr{P}(A)^*$ to A, but nothing else. Then, he decides to write on his exam sheet a randomly chosen function from $\mathscr{P}(A)^*$ to A, hoping that it will match a Choice Function.

(1). What are the chances that he will give a correct answer to the exercise?

(2). By the way, find a Choice Function for the set $A = \{0, 1, 2, 3\}$.

Exercise 5. In a homework for a course of Set Theory, a professor gives to the students an exercise where they have to find an *injective Choice Function* on a particular set given by the professor. All the students solve the exercise and find the *same* Choice Function! However, they worked independently on the homeworks and never met or send messages to each other during the weekends.

Can you explain how this is possible?

Exercise 6 (A1-A6)**.** Let A and B be two non-empty sets. We define S, as the class of the functions f such that $\text{dom}(f) \subseteq A$ and $\text{ran}(f) \subseteq B$. Prove that S is a set.

Date: 2015, April 29.