On the spectral flow for Banach spaces

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Pohang Mathematics Institute, POSTECH

2011, October 22

The spectral flow is an integer associated to a continuous path of operators on a Hilbert space H

$$A\colon [0,1] \to \mathscr{L}(H)$$
$$A(t) \text{ is a Fredholm operator and } A(t) = A(t)^*$$
$$\mathrm{sf}(A) \in \mathbb{Z}.$$

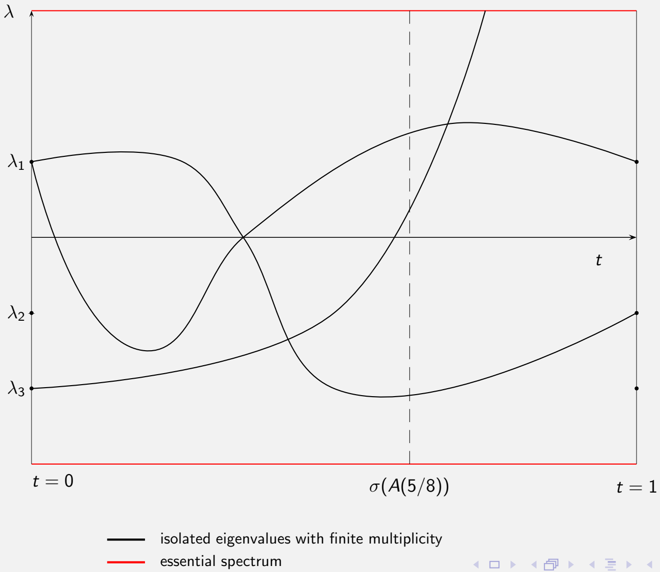
It appeared first in the series of papers of M. F. Atiyah, V. K. Patodi and I. M. Singer (Math. Proc. Cam. Phil. Soc., 1975–1976) and ascribed to a joint study of Atiyah and G. Lusztig.

The spectral flow is described as "net number of eigenvalues that change sign (from $-$ to $+$) while the parameter of the family is completing a period" (M. F. Atiyah, 1976).

In the same paper is defined as the intersection index of the two subsets of $[0,1] \times \mathbb{R}$

$$\mathscr{T} := \{(t,\lambda) \,|\, \lambda \in \sigma(A(t))\}$$
$$\{(t,0) \,|\, 0 \le t \le 1\}.$$

$m_{\lambda_2} = m_{\lambda_3} = 1,\ m_{\lambda_1} = 2,\ \mathrm{sf}(A) = 1$



In literature, the spectral flow is strictly related to

1. K-theory (M. F. Atiyah and I. M. Singer, 1969);
2. the Maslov index in Floer Homology (J. Robbin and S. Salamon, 1995);
3. the winding number of the determinant $\pi_1(U(\infty)) \to \mathbb{Z}$ (J. Phillips, 1996) where
$$U(\infty) := \bigcup_{n=1}^{\infty} U(n, \mathbb{C}).$$
4. bifurcation for Strongly-Indefinite functionals (P. M. Fitzpatrick and J. Pejsachowicz, JFA, JDE).

Fredholm operators

We recall that an operator $A\colon E \to F$ is *Fredholm* if

$$\ker(A) \subset E \text{ has finite dimension,}$$
$$\operatorname{im}(A) \subset F \text{ is closed and has finite co-dimension}$$

and the *Fredholm index* of A is, by definition

$$\operatorname{ind}(A) := \dim\big(\ker(A)\big) - \operatorname{codim}\big(\operatorname{im}(A)\big).$$

We denote with $\mathscr{F}(E,F)$ the set of Fredholm operators.

The notation

$$\mathscr{F}^{sa}(H) = \{A \,|\, A^* = A,\ A \in \mathscr{F}(H)\}.$$

is used.

Theorem (M. F. Atiyah, J. Robbin and D. Salamon (1995), J. Phillips (1996))

The spectral flow

$$\mathrm{sf}\colon \boldsymbol{\Omega}\mathscr{F}^{sa}(H) \to \mathbb{Z}$$

is invariant by fixed-endpoints homotopies.

If H is a separable, there is a unique non simply-connected component of $\mathscr{F}^{sa}(H)$, denoted by $\mathscr{F}^{sa}_*(H)$, where

$$\mathrm{sf}\colon \pi_1(\mathscr{F}^{sa}_*(H)) \to \mathbb{Z}$$

is an isomorphism.

We wish to determine what are the properties of the group homomorphism when H is replaced by an arbitrary Banach space.

We use the definition of spectral flow given by Y. Long and C. Zhu (CAM, 1999) for Banach spaces.

Given $A\colon E \to E$, we define the *essential spectrum*

$$\sigma_e(A) = \{\lambda \in \sigma(A) \,|\, A - \lambda I \notin \mathscr{F}(E)\} \subset \mathbb{C}.$$

An operator $A \in \mathscr{L}(E)$ is called *essentially hyperbolic* if

$$\sigma_e(A) \cap \{\mathrm{re}\,(z) = 0\} = \emptyset.$$

Thus, the spectrum of $\sigma(A)$ can be written as

$$\sigma(A) = \sigma_e(A) \cup \{\lambda_1, \ldots, \lambda_k\}$$

where λ_k are eigenvalues of finite multiplicity

$$e\mathscr{H}(E) = \Big\{A \in \mathscr{L}(E) \,|\, \sigma_e(A) \cap \{\mathrm{re}\,(z) = 0\} = \emptyset\Big\}.$$

Properties of $e\mathcal{H}(E)$

1. $e\mathscr{H}(E) \subset \mathscr{L}(E)$ is an open subset;
2. in every connected component there a symmetry $2P - I$ (that is, P is a projector)
3. given a pair of projectors P and Q, $2P - I$ and $2Q - I$ belong to the same component if and only if there exists an invertible operator $T \in GL(E)$ such that

$$T \text{ is path-connected to } I \text{ and } TPT^{-1} - Q \text{ is compact.}$$

The space of projectors

Given a Banach space E, we say that $P \in \mathscr{L}(E)$ is a *projector* if

$$P^2 = P.$$

We use the notation

$$P(E) := \{P \in \mathscr{L}(E) \,|\, P^2 = P\}.$$

1. It is a closed, locally path-connected subset;
2. inherits a structure of analytical sub-manifold;
3. $\pi_1(P(E), P) \cong \pi_0(GL(\ker(P)), I) \times \pi_0(GL(\mathrm{im}(P)))$.

The relative dimension

Given two projectors P, Q such that $P - Q$ is a compact, the operator

$$Q \in \mathscr{L}(\mathrm{im}(P), \mathrm{im}(Q))$$

is Fredholm. We denote

$$[P - Q] := \mathrm{ind}\big(Q : \mathrm{im}(P) \to \mathrm{im}(Q)\big).$$

If $\mathrm{im}(P), \mathrm{im}(Q)$ have finite dimension

$$[P - Q] = \dim(\mathrm{im}(P)) - \dim(\mathrm{im}(Q)).$$

$$P_c(E; P) := \{Q \in P(E) \,|\, Q - P \text{ is compact}\}.$$

1. Q is path-connected to P in $P_c(E; P)$ if and only if $[P - Q] = 0$.
2. $\pi_1(P_c(E; P), P) \cong \mathbb{Z}_2$.

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The definition of spectral flow

Lemma (G., TMNA, 2010)

Given a path $A \in C([0,1], e\mathscr{H}(E))$, there exists a continuous path

$$P\colon [0,1] \to P(E)$$

such that $P(t) - P^+(A(t))$ is compact.

With $P^+(A(t))$ we denote the spectral projector.

Definition of spectral flow

We define

$$\mathrm{sf}(A) := [P(0) - P^+(A(0)] - [P(1) - P^+(A(1)].$$

If A is a loop $\mathrm{sf}(A) = [P(0) - P(1)]$.

Some remarks on the definition

The s.f. is invariant for fixed-endpoints homotopies and, for every $P \in P(E)$, it induces a group homomorphism

$$\mathrm{sf}_P \colon \pi_1(e\mathscr{H}(E), 2P - I) \to \mathbb{Z}.$$

A path P as in the definition is called *s-section* for $\{P^+(A(t)) \mid t \in \mathbb{R}\}$.

In their paper Y. Long and C. Zhu use s-sections with the additional requirement

$$P(t) = P(A(t), \Omega(t)).$$

Thus,

$$\mathrm{sf}(A) = \sum_{i=1}^{n} \Big([P_i(t_{i-1}) - P_i(A(t_{i-1}))] - [P_i(t_i) - P_i(A(t_i))] \Big).$$

where P_i are s-section on sub-intervals $[t_{i-1}, t_i]$

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A characterisation image of sf_P

Theorem (G.,TMNA, 2010)

Given a projector P and $k \in \mathbb{Z}$, there exists a loop $A \in \mathbf{\Omega}(e\mathscr{H}(E))$ with base point $2P - I$ and

$$\mathrm{sf}(A) = k$$

if and only if P is path-connected to a projector Q such that

$$\mathrm{im}(Q) \subset \mathrm{im}(P), \quad \mathrm{codim}\big(\mathrm{im}(Q)\big) = k.$$

The image of sf_P

- $\mathrm{sf} \neq 0$ Every projector P with infinite-dimensional kernel and image in $L^\infty, L^p, \ell^p, \ell^\infty, c_0, H, \dots,\ 1 \in \mathrm{im}(\mathrm{sf}_P)$;
- $\mathrm{sf} \neq 0$ if $E = X \times X$ and

$$X \cong X^m, \quad \mathrm{codim}(X_m) = m$$

then $\mathrm{im}(\mathrm{sf}_{I\times 0}) \ni m$. In particular, when X is isomorphic to closed subspaces of co-dimension m, but not to subspaces of co-dimension k for $k < m$

$$\mathrm{im}(\mathrm{sf}_{I\times 0}) = m\mathbb{Z}.$$

According to W. T. Gowers and B. Maurey (MA, 1997), such X_m exists at least for $m = 2, 7$;
- $\mathrm{sf}_P = 0$ every projector of finite-dimensional image of kernel. If $E = \mathbb{R}^n$ or E is hereditary undecomposable (W. T. Gowers and B. Maurey, JAMS 1993), $\mathrm{sf}_P = 0$ for every P.

The kernel of sf_P

Theorem (G.,TMNA, 2010)

Given a projector P, there exists an exact sequence

$$\pi_1(P_c(E;P),P) \longrightarrow \pi_1(P(E),P) \longrightarrow \pi_1(e\mathscr{H}(E),2P-I) \xrightarrow{\mathrm{sf}} \mathbb{Z}.$$

The kernel of sf_P

When $GL(E)$ is contractible to a point. For instance, $\ker(\mathrm{sf}_P) = 0$ in the cases

1. an infinite-dimensional Hilbert space (N. Kuiper, Topology, 1965)
2. c_0 (D. Arlt, Invent. Math., 1966)
3. ℓ^p (G. Neubauer, Math. Ann., 1967)
4. L^p, L^∞, $C(K, \mathbb{C})$ for a large class of compact K (B. S. Mityagin, Uspehi Mat. Nauk, 1970).

When $E = (\ell^2 \times \ell^p) \times (\ell^2 \times \ell^p)$, and P is the projector onto the first factor,

$$\pi_1(P(E), P) \supseteq G \cong \mathbb{Z}$$

where P is a projector onto $\ell^2 \times \ell^p \times 0$.

According to A. Douady (Indag. Math., 1968), $GL(E)$ has infinitely many connected components.

Thank you for your attention