Ordinary differential equations in Banach spaces and the spectral flow

Garrisi Daniele

To my parents and my sister

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Contents

Introduction

Given a path $\{A(t) \mid t \in \mathbb{R}\}$, of linear operators on some Banach space E, we consider the differential operator

$$F_A u = \left(\frac{d}{dt} - A(t)\right) u$$

on suitable spaces of curves $u\colon \mathbb{R} \to E$. A classical question is whether the operator F_A is Fredholm and what is its index. If $A(t)$ is a path of unbounded operators the literature is rich. We recall the work of J. Robbin and D. Salamon, [**45**], where A is an asymptotically hyperbolic path of unbounded self-adjoint operators and defined on a common domain $W \subset H$ compactly included in a Hilbert space H. For such paths they prove that the differential operator

$$F_A\colon L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H) \to L^2(\mathbb{R}, H), \quad u \mapsto u' - Au$$

is Fredholm. The index of F_A is minus the *spectral flow* of A, an integer which counts algebraically the eigenvalues of $A(t)$ crossing 0. The result applies to *Cauchy-Riemann* operators and it is widely used in Floer homology. This result has been generalized to Banach spaces with the *unconditional martingale difference* (UMD) property by P. Rabier in [**44**]; the compact inclusion of the domain is still required. In this setting, the identity

$$\operatorname{ind} F_A = -\operatorname{sf}(A). \tag{1}$$

holds. Y. Latushkin and T. Tomilov in [**33**] proved the Fredholmness of the operator F_A for paths A with variable domain $D(A(t)) \subset E$ with E reflexive using *exponential dichotomies*. D. di Giorgio, A. Lunardi and R. Schnaubelt in [**17**] obtained the same results for *sectorial operators* in an arbitrary Banach space and give necessary and sufficient conditions on the *stable* and *unstable spaces* in order to have the Fredholmness of F_A.

For the bounded case the problem has been studied by A. Abbondandolo and P. Majer in [**3**]. This setting is suggested by the Morse Theory on a Hilbert manifold M: given a vector field ξ on M and ϕ_t its flow, x and y hyperbolic zeroes of ξ the stable and unstable manifolds

$$W^s_\xi(x) = \left\{ p \in M \mid \lim_{t \to +\infty} \phi_t(p) = x \right\}$$
$$W^u_\xi(y) = \left\{ p \in M \mid \lim_{t \to -\infty} \phi_t(p) = y \right\}$$

are immersed sub-manifolds of M, in fact they are sub-manifolds if the vector field is the gradient of a Morse function on M. It is not hard to check that the intersection of the stable and unstable manifold of two different zeroes is a sub-manifold if, for every curve $u'(t) = \xi(u(t))$ such that $u(+\infty) = x$ and $u(-\infty) = y$, the differential operator

$$F_A(v) = v' - Av, \quad A(t) = D\xi(u(t))$$

is surjective and $\ker F_A$ splits. Since x and y are hyperbolic zeroes $A(+\infty)$ and $A(-\infty)$ are hyperbolic operators. In [**3**] the study of the Fredholm index of such operator is carried out by considering the stable and unstable spaces

$$W_A^s = \left\{ x \in E \mid \lim_{t \to +\infty} X_A(t)x = 0 \right\}$$
$$W_A^u = \left\{ x \in E \mid \lim_{t \to -\infty} X_A(t)x = 0 \right\},$$

where X_A is the solution of the Cauchy problem $X' = AX$ with $X(0) = I$. If A is an *asymptotically hyperbolic* path on a Hilbert space the following facts hold:

Fact 1. The stable and unstable spaces W_A^s and W_A^u are closed in E and admit topological complements, PROPOSITION 1.2 of [**3**].

Fact 2. The evolution of the stable space $X_A(t)W_A^s$ converges to the negative eigenspace of $A(+\infty)$, and any topological complement of W_A^s converges to the positive eigenspace of $A(+\infty)$, with a suitable topology on the set of closed linear subspaces of a Hilbert, see THEOREM 2.1 of [**3**].

Fact 3. If two paths A and B have compact difference for every $t \in \mathbb{R}$ the stable space W_A^s is compact perturbation of W_B^s, THEOREM 3.6 of [**3**].

Fact 4. The operator F_A is semi-Fredholm if and only if (W_A^s, W_A^u) is a semi-Fredholm pair; in this case $\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u)$, THEOREM 5.1 of [**3**]

In the bounded setting the spectral flow is defined in [**42**] for paths in $\mathcal{F}^{sa}(E)$, the set of Fredholm and self-adjoint bounded operators. Unlike the unbounded case described in [**45**] and in [**44**], given an asymptotically hyperbolic path in $\mathcal{F}^{sa}(E)$ the equality $\operatorname{ind} F_A = -\operatorname{sf}(A)$ does not hold in general. Examples are provided in §7 of [**3**]. Our purpose is to generalize firstly these facts to an arbitrary Banach space E and, secondly, to define the spectral flow for suitable paths and prove that for a class of paths the relation (1) holds.

In the first chapter we define some metrics on the set of closed linear subspaces of E, the *Grassmannian* of E, denoted by $G(E)$, and the subset of complemented subspaces, denoted by $G_s(E)$. This is done in order to have a definition of *convergence of subspaces*. Our main reference is the work of E. Berkson, [**8**]. We also establish which pairs of closed subspaces (X, Y) are *compact perturbation of each other* and the relative dimension for such pairs is defined. In finite-dimensional spaces, the relative dimension is $\dim(X) - \dim(Y)$. A definition of relative dimension exists in Hilbert spaces, refer [**3, 12**], and we know of an existing definition in Banach spaces for pairs of projectors (P, Q) with compact difference, in [**54**]. In this chapter, we extend the definition to arbitrary pairs of closed subspaces in every Banach space E. These definitions allow to state **Fact 1** and **Fact 3**.

We use the notation $\mathcal{P}(E)$ for the space of projectors on a Banach space E and $\mathcal{P}(\mathcal{C}(E))$ for the space of projectors of the Banach algebra $\mathcal{C}(E) = \mathcal{L}(E)/\mathcal{L}_c(E)$. In chapter 2, we prove in Theorem 8.1 that, for every projector P, we can define a group homomorphism, namely φ_P, on the fundamental group of $\mathcal{P}(\mathcal{C}(E))$ at the base point $p(P)$ such that the sequence

$$\pi_1(\mathcal{P}(E), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(\mathcal{C}), p(P)) \xrightarrow{\varphi_P} \mathbb{Z}$$

is exact. We characterize the elements of the image of φ_P and give a sufficient condition making φ_P injective. We have

h1) P is connected to a projector Q such that $Q - P \in \mathcal{L}_c(E)$ and $\dim(Q, P) = m$ if and only if $m \in \operatorname{Im}(\varphi_P)$;

h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected.

We show some examples of Banach space where no projectors fulfills h1) and prove in Proposition 8.4 that if E has a complemented subspace, sum of two subspaces isomorphic to each other and to their closed subspaces of co-dimension m, then a projector on each of the two factors satisfies condition h1). These properties are verified by an orthogonal projection in a Hilbert space with infinite-dimensional kernel and range. The most common Banach spaces such as the measure spaces $L^p(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ and spaces of sequences ℓ^p, m, c_0 (see [**38, 49**] for a richer list and references) have a projector satisfying conditions h1,2). When two projectors are connected by a path of projectors, their ranges are isomorphic, thus, in view of h2), the problem of determining the image of φ_P is strictly related to the question whether $\operatorname{Range} P$ is isomorphic to its subspaces of co-dimension m or not. As a consequence of the counterexamples of W. T. Gowers and B. Maurey in [**26**], of infinite-dimensional Banach space not isomorphic to its hyperplanes, such P might not exits. Using a construction in [**27**] of a space isomorphic to their hypersquares (that is, subspace of co-dimension 2), but not hyperplanes, we can show easily that in some case $\operatorname{Im}(\varphi_P) = 2\mathbb{Z} \subset \mathbb{Z}$. Using a construction of A. Douady in [**20**], we show that φ_P is not injective even if it can be surjective.

In chapter 3 we study basic properties of the Cauchy problem $X'(t) = A(t)X(t), X(0) = 1$. Here $A(t)$ is a continuous and bounded path on $\mathcal{L}(E)$. We define the stable and unstable spaces of A, denote by W^s_A and W^u_A, and prove **Fact 1,2** in Theorem 4.1. The proof differs from the one that the authors of [**3**] used for Hilbert spaces, only for the lack of a scalar product in (63,76) which can be fixed using results of continuous selection, refer Appendix D and [**6**]. Moreover, the stable manifold is well behaved with respect to small, Theorem 5.1 and compact perturbations, Theorem 5.4, that is **Fact 3**.

In chapter 4 we study the Fredholm properties of F_A, defined on the space of continuously differentiable functions vanishing at infinity with their derivatives with values on continuous, vanishing at infinity. In Theorem 2.2 we prove that F_A is a semi-Fredholm operator if and only if the pair (W^s_A, W^u_A) is semi-Fredholm and, if this is the case, the index is the same. The extension is made with slight modification of the argument of Theorem 5.1 of [**3**].

In chapter 5, we give a definition of spectral flow for paths in the space of essentially hyperbolic operators, denoted by $e\mathcal{H}(E)$. Such definition coincides with the one given by C. Zhu and Y. Long in [**54**] and improves it making the spectral flow easier to compute and to produce examples. Their definition, and thus ours, generalizes to Banach spaces the definition known for Hilbert spaces (refer [**42**]). In Theorem 2.3, we prove that, given a projector P, the composition $\operatorname{sf}_{2P-I} \circ \Phi$, where sf_{2P-I} denotes the spectral flow defined on the fundamental group of $(e\mathcal{H}(E), P)$ and Φ is the homotopy equivalence defined in Theorem 1.4, coincides with $-\varphi_P$. Hence, anything holds for the index φ_P is true for the spectral flow as well, including conditions h1) and h2). Thus, in contrast with the behaviour in a separable, infinite-dimensional Hilbert space, where the spectral flow is either trivial or an isomorphism, due to the examples provided in Chapter 2, we have different behaviours. In the last section we prove that for a suitable class of paths in $e\mathcal{H}(E)$, namely the *essentially splitting* and asymptotically hyperbolic ones, there holds

$$\operatorname{ind} F_A = -\operatorname{sf}(A).$$

In [**3**], A. Abbondandolo and P. Majer guessed that the above equality holds in more restricted class of operators, where the positive and negative eigenspaces are fixed

$$E^+(A(t)) = E^+, \quad E^-(A(t)) = E^-.$$

Our result proves that the guess is correct and extends to the class of essentially hyperbolic and essentially splitting operators. We achieve this result in several steps: in Lemma 3.4, we prove that an asymptotically hyperbolic path, A, is essentially splitting if and only if the projectors of the set $\{P^+(A(t)) \mid t \in \mathbb{R}\}$ are compact perturbation of each other. In Theorem 3.5 we compute the spectral flow for an essentially splitting path

$$\operatorname{sf}(A) = -\dim(\operatorname{Range} P^-(A(+(\infty))), \operatorname{Range} P^-(A(-\infty))).$$

For such paths we also compute the Fredholm index of F_A in Theorem 3.3

$$\operatorname{ind} F_A = \dim(\operatorname{Range} P^-(A(+(\infty))), \operatorname{Range} P^-(A(-\infty))).$$

Thus we obtain the desired equality $\operatorname{sf}(A) = -\operatorname{ind} F_A$.

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CHAPTER 1

Topology of the Grassmannian

Given a metric space (X, d), we define the *Hausdörff space*, which is the set of bounded and closed subsets endowed with the distance metric and denoted by $(\mathscr{H}(X), d_{\mathscr{H}})$. We can define a metric on the family of closed, linear subspaces of a given Banach space E through the map that associates a linear space Y with the unit disc of Y. We call this metric space *Grassmannian* and show that equivalent metrics, namely δ_S and δ_1, can be defined on it. We show that the topology is well-behaved with respect to the action of invertible operators of E, to the graph and to the annihilator $Y \mapsto Y^{\perp}$. We show that the subset of the linear, complemented subspaces is open. The last two sections of the chapter deal with the definition of *relative dimension* for pairs of closed linear subspaces, its relation with compact perturbations of projectors and Fredholm operators. We know of an existing definition of relative dimension for pairs of projectors in [**54**]. Our main references are [**8, 39, 30**].

1. The Hausdörff metric

Let (X, d) be a metric space. Given two subsets of $A, B \subseteq X$ it is well defined the distance

$$\operatorname{dist}(a, B) = \inf_{b \in B} d(a, b).$$

We denote by $\mathscr{H}(X)$ the family of closed, nonempty and bounded subsets of X. It is defined a metric on $\mathscr{H}(X)$ as follows: let A, B be two closed and bounded subsets of X. Define

$$\rho_{\mathscr{H}}(A, B) = \sup_{a \in A} \operatorname{dist}(a, B), \;\; \delta_{\mathscr{H}}(A, B) = \max\{\rho_{\mathscr{H}}(A, B), \rho_{\mathscr{H}}(B, A)\};$$

the second is called *Hausdörff metric*. We show that it has all the properties of a metric. It is clearly symmetric; if $\rho_{\mathscr{H}}(A, B) = 0$ $A \subset B$ because B is closed. Thus $\delta_{\mathscr{H}}(A, B) = 0$ if and only if $A = B$. For the triangular inequality let $A, B, C \in \mathscr{H}(X)$ be closed and bounded subsets of X. Given $\varepsilon > 0$ there exists $a_1 \in A$ such that

$$\rho_{\mathscr{H}}(A, C) \leq \varepsilon + \operatorname{dist}(a_1, C) \leq \varepsilon + d(a_1, b) + d(b, c) \tag{2}$$

for any $(b, c) \in B \times C$. Taking $b_1 \in B$ such that $d(a_1, b_1) \leq \varepsilon + \operatorname{dist}(a_1, B)$, (2) becomes

$$\rho_{\mathscr{H}}(A, C) \leq 2\varepsilon + \operatorname{dist}(a_1, B) + d(b_1, c)$$

for any $c \in C$. Taking the infimum over C we find that $\rho_{\mathscr{H}}(A, C) \leq \rho_{\mathscr{H}}(A, B) + \rho_{\mathscr{H}}(B, C)$. Finally, suppose that $\delta_{\mathscr{H}}(A, C) = \rho_{\mathscr{H}}(A, C)$. Therefore

$$\delta_{\mathscr{H}}(A, C) = \rho_{\mathscr{H}}(A, C) \leq \rho_{\mathscr{H}}(A, B) + \rho_{\mathscr{H}}(B, C) \leq \delta_{\mathscr{H}}(A, B) + \delta_{\mathscr{H}}(B, C).$$

The following proposition states a relation between the metric spaces (X, d) and $(\mathscr{H}, \delta_{\mathscr{H}})$. The proof of this can also be found in [**32**].

Proposition 1.1. *The application $\delta_{\mathscr{H}} \colon \mathscr{H} \times \mathscr{H} \to \mathbb{R}^+$ defines a complete metric in $\mathscr{H}(X)$ if and only if (X, d) is complete. Moreover if $\{ A_n \mid n \in \mathbb{N}\}$ is a converging sequence its limit is the set of the limits of sequences $\{a_n\}$ such that $a_n \in A_n$.*

Proof. We have proved that $\delta_{\mathscr{H}}$ is a metric. Given $a, b \in X$ it follows from the definition that $\delta_{\mathscr{H}}(\{a\}, \{b\}) = d(a, b)$; thus, for a Cauchy sequence $\{a_n\} \subset X$, the sequence $\{\{a_n\}\}$ converges to a closed and bounded subset of $S \subset X$. For every element $s \in S$ there holds

$$d(s, a_n) = \operatorname{dist}(s, \{a_n\}) \leq \delta_{\mathscr{H}}(S, \{a_n\})$$

thus s is the limit of the sequence $\{a_n\}$. By uniqueness of the limit S consist of a single point, thus (X, d) is complete. To prove the converse let $\{A_n\}$ be a Cauchy sequence in $\mathscr{H}(X)$ and $\varepsilon > 0$; there exists $n(\varepsilon)$ such that for every $n \geq n(\varepsilon)$

$$\delta_{\mathscr{H}}(A_{n(\varepsilon)}, A_n) < \varepsilon/2;$$

given $a \in A_{n(\varepsilon)}$ using induction we can build a sequence $\{a_k\}$ and $n_k \in \mathbb{N}$ such that

$$a_0 = a,\ a_k \in A_{n_k},\ n_0 = n(\varepsilon),\ n_{k+1} > n_k,\ d(a_{k+1}, a_k) < 2^{-(k+2)}\varepsilon; \tag{3}$$

then $\{a_k\}$ is a Cauchy sequence in X and, since X is complete, converges to a limit, say x. Define L as the set of the elements that are limits of sequences $\{a_k\}$ such that $a_k \in A_{n_k}$. The construction above shows that L is nonempty. We prove now that A_n converges to L; first there exists $a_0 \in A_{n(\varepsilon)}$ such that

$$\rho_{\mathscr{H}}(A_{n(\varepsilon)}, L) < \varepsilon/8 + \operatorname{dist}(a_0, L);$$

let $\{a_k\}$ be as in (3) and call x its limit. Let k be such that $d(a_k, a) < \varepsilon/8$. We have

$$\rho_{\mathscr{H}}(A_{n(\varepsilon)}, L) < \varepsilon/8 + d(a_0, a_k) + d(a_k, x) < \varepsilon/4 + \sum_{j=0}^{k-1} d(a_{j+1}, a_j)$$
$$< \varepsilon/4 + \varepsilon \sum_{j=2}^{\infty} 2^{-j} < \varepsilon/2;$$

thus $\rho_{\mathscr{H}}(A_n, L) \leq \rho_{\mathscr{H}}(A_n, A_{n(\varepsilon)}) + \rho_{\mathscr{H}}(A_{n(\varepsilon)}, L) < \varepsilon$ for every $n \geq n(\varepsilon)$. Similarly there exists $x \in L$ such that

$$\rho_{\mathscr{H}}(L, A_{n(\varepsilon)}) < \varepsilon/8 + \operatorname{dist}(x, A_{n(\varepsilon)});$$

by definition of L there exists a sequence a_k converging to x such that $a_k \in A_{n_k}$. Choose $k(\varepsilon)$ such that, for every $k > k(\varepsilon)$, we have

$$d(x, a_k) < \varepsilon/4,\ n_k > n(\varepsilon);$$

by the triangular inequality, for every $n > n_{k(\varepsilon)}$, we have

$$\rho_{\mathscr{H}}(L, A_{n(\varepsilon)}) < \varepsilon/4 + \operatorname{dist}(x, A_n) + \rho_{\mathscr{H}}(A_n, A_{n(\varepsilon)}) < \varepsilon,$$

thus $\delta_{\mathscr{H}}(L, A_n) < \varepsilon$. This proves the completeness of $\mathscr{H}(X)$. To conclude the proof observe that, since $\rho_{\mathscr{H}}(L, A_n)$ is an infinitesimal sequence, given $x \in L$ there exists an infinitesimal sequence $\{\varepsilon_n\}$ and a_n such that

$$d(x, a_n) - \varepsilon_n < \operatorname{dist}(x, A_n) \leq \rho_{\mathscr{H}}(L, A_n);$$

taking the limit as $n \to \infty$ we prove that $\{a_n\}$ converges to x. □

2. Metrics on the Grassmannian

Let $(E, |\cdot|)$ be a Banach space. We define $G(E)$ as the set of the closed linear subspaces of E, called *Grassmannian*. We define a complete metric on this set. To a linear subspace $Y \subset E$ we have the following subsets of E associated to it:

$$D(Y) = \{y \in E \mid |y| \leq 1\},$$
$$S(Y) = \{y \in E \mid |y| = 1\},\ (Y \neq 0);$$

on $G(E)$ we consider the metric induced by the inclusion of subsets $i\colon G(E) \hookrightarrow \mathscr{H}(E)$, $Y \mapsto D(Y)$. We set

$$\rho(Y, Z) = \rho_{\mathscr{H}}(D(Y), D(Z)),$$
$$\delta(Y, Z) = \delta_{\mathscr{H}}(D(Y), D(Z)).$$

Proposition 2.1. *The subset $i(G(E))$ is closed in $\mathscr{H}(E)$, hence δ is complete.*

Proof. Let Y_n be a sequence in $G(E)$ such that $D_n = D(Y_n)$ converges to $D \subseteq E$, a nonempty, closed and bounded subset of E. Let Z be the linear vector subspace generated by D. We show that D is the unit disc of the space Z. In fact, we have the following properties:

p1) $0 \in D$;
p2) provided $Z \neq \{0\}$ we have $D \supset S(Z)$;
p3) D is *star-shaped* to 0, that is $tx \in D$ for every $t \in [0,1]$ if $x \in D$.

All these properties are consequences of Proposition 1.1. For instance the first follows in that $0 \in D_k$ for every $k \in \mathbb{N}$. For the second let $z \in S(Z)$; since D generates Z there are constants t_i such that

$$z = t_1 y_1 + \cdots + t_n y_n, \quad y_i \in D.$$

Each of these elements are limits of a sequence $y_{i,k} \in D_k$, hence, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} z_k &= t_1 y_{1,k} + \cdots + t_n y_{n,k} \in Y_k \\ z_k/|z_k| &= \hat{t}_1 y_{1,k} + \cdots + \hat{t}_n y_{n,k} \in D_k; \end{aligned}$$

applying the Proposition 1.1 to the second sequence we find $z \in D$. The proof of the third property is similar and we omit it. From p1)–p3) it follows easily that $D \supseteq D(Z)$: given $z \neq 0$ in $D(Z)$ the vector $\hat{z} = z/|z| \in D$ and, since D is star-shaped, $z \in D$. The inclusion $D \subseteq D(Z)$ it is just the definition of Z, hence $D(Z) = D$. To conclude the proof we show that Z is a closed subspace of E. Let $\{z_n\}$ be a sequence converging to $x \in E$; if $x = 0$ clearly $x \in Z$. If $x \neq 0$ for n large each term of the sequence is nonzero. We write

$$z_n = \hat{z}_n \cdot |z_n|, \ \hat{z}_n \in D;$$

since D is closed $\hat{x} \in D$. Thus $z = |x|\hat{x}$ belongs to vector space generated by D, hence $z \in Z$. We have proved that $D = i(Z)$. □

Similarly we can consider the inclusion of spheres given by $j\colon G(E) \setminus \{0\} \hookrightarrow \mathscr{H}(E)$, $Y \mapsto S(Y)$ and define a metric on $G(E) \setminus \{0\}$ as follows

$$\begin{aligned} \rho_S(Y, Z) &= \rho_{\mathscr{H}}(S(Y), S(Z)), \\ \delta_S(Y, Z) &= \delta_{\mathscr{H}}(S(Y), S(Z)), \text{ if } Y, Z \neq 0; \end{aligned}$$

we extend it to a metric on $G(E)$ with $\rho_S(\{0\}, \{0\}) = 0$ and $\rho_S(Y, \{0\}) = \rho_S(\{0\}, Z) = 1$. It is also called *opening metric* (see [**8**], §2). As above we have the following

Proposition 2.2. *The subset $j(G(E) \setminus \{0\})$ is closed in $\mathscr{H}(E)$, hence δ_S is complete.*

The proof is similar to the previous one. It just takes to prove that limits of sequences of spheres is a sphere.

Proposition 2.3. *The metrics δ_S and $\delta_{\mathscr{H}}$ are equivalent. In particular the inequalities*

$$\begin{aligned} \rho_S(Y, Z) &\leq 2\rho(Y, Z) \\ \rho(Y, Z) &\leq \rho_S(Y, Z); \end{aligned}$$

hold.

Proof. To prove the first inequality we will use this fact: for any pair of vectors $x \in S(E)$ and $y \in E \setminus \{0\}$ we have $|x - \hat{y}| \leq 2|x - y|$ where $\hat{y} = y/|y|$. Let $Y, Z \neq \{0\}$ and $\varepsilon > 0$. There exists $y \in S(Y)$ such that, for every $z \in S(Z)$ and $0 < r \leq 1$ there holds

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + |y - z| = \varepsilon + |y - \widehat{rz}| \leq \varepsilon + 2|y - rz|;$$

taking the infimum over $(0, 1] \times S(Z)$ we find

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + 2\text{dist}(y, D(Z) \setminus \{0\});$$

since $\rho_{\mathscr{H}}(S(Y), S(Z)) \leq 1 < 2$ we can write

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq 2 \min\{1, \varepsilon/2 + \text{dist}(y, D(Z) \setminus \{0\})\};$$

since $|y| = 1$ the second member of the inequality becomes

$$\begin{aligned} &2\min\{1, \varepsilon/2 + \operatorname{dist}(y, D(Z) \setminus \{0\})\} \\ \leq &2\min\{\varepsilon/2 + |y|, \varepsilon/2 + \operatorname{dist}(y, D(Z) \setminus \{0\})\}; \end{aligned}$$

the latter is equal to

$$2(\varepsilon/2 + \operatorname{dist}(y, D(Z)) \leq \varepsilon + 2\operatorname{dist}(y, D(Z)).$$

Taking the supremum over $S(Y)$ we obtain

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + 2\rho_{\mathscr{H}}(S(Y), D(Z)) \leq \varepsilon + 2\rho_{\mathscr{H}}(D(Y), D(Z)).$$

If $Y = \{0\}$ and $Z \neq 0$ we have $\rho_{\mathscr{H}}(\{0\}, S(Z)) = 1 = \delta_{\mathscr{H}}(D(Z), \{0\})$, thus we have proved that $\delta_S(Y, Z) \leq 2\delta(Y, Z)$.

We prove the second inequality in the case $Y, Z \neq \{0\}$ first. Suppose $\rho(Y, Z) \neq 0$ and pick $\varepsilon > 0$ such that $0 < 2\varepsilon < \rho(Y, Z)$. There exists $y \in D(Y)$ such that

$$\rho(Y, Z) < \varepsilon/2 + \operatorname{dist}(y, D(Z));$$

in fact this implies $y \neq 0$. Set $\hat{y} = y/|y|$; there exists $\nu \in S(Z)$ such that

$$d(\hat{y}, \nu) < \varepsilon/2 + \operatorname{dist}(\hat{y}, S(Z)).$$

Hence the second term of the first inequality is bounded by $d(y, |y|\nu)$ which is equal to $|y|d(\hat{y}, \nu)$, thus

$$\begin{aligned} \rho(Y, Z) &< \varepsilon/2 + |y|d(\hat{y}, \nu) \leq \varepsilon/2 + d(\hat{y}, \nu) \\ &< \varepsilon + \operatorname{dist}(\hat{y}, S(Z)) \leq \varepsilon + \rho_S(Y, Z). \end{aligned}$$

If one among Y and Z is $\{0\}$ we have $\rho(Y, \{0\}) = 1 = \rho_S(Y, \{0\})$. $\square$

By technical reasons we also define, for two closed subspaces Y, Z

$$\rho_1(Y, Z) = \sup_{y \in D(Y)} \operatorname{dist}(y, Z), \;\; \delta_1(Y, Z) = \max\{\rho_1(Y, Z), \rho_1(Z, Y)\}.$$

The triangular inequality does not hold for ρ_1 (see [**8**], §3 for a counterexample). However the *weakened triangular inequality* holds, that is

$$\rho_1(X, Z) \leq \rho_1(Y, Z)(1 + \rho_1(X, Y)) + \rho_1(X, Y)$$

for every X, Y, Z (see [**30**], Ch. IV, Lemma 2.2) [1].

Proposition 2.4. *The topology generated by the neighbourhoods*

$$\{U(Y, r) \mid Y \in G(E), \; r > 0\}, \; U(Y, r) = \{Z \mid \rho_1(Y, Z) < r\}$$

is equivalent to the one induced by the Hausdörff metric of the discs. More precisely for every Y, Z

$$1/2 \cdot \delta(Y, Z) \leq \delta_1(Y, Z) \leq \delta(Y, Z).$$

Proof. Given $y \in D(Y)$, $\operatorname{dist}(y, Z) \leq \operatorname{dist}(y, D(Z))$, then $\delta_1(Y, Z) \leq \delta(Y, Z)$. In order to prove the lower estimate suppose both Y, Z are different from the null space. Let $y \in S(Y)$; for every $z \in S(Z)$ and $r > 0$ we have

$$\operatorname{dist}(y, S(Z)) \leq |y - z| = |y - \widehat{rz}| \leq 2|y - rz|;$$

taking the infimum over $\mathbb{R}^+ \times S(Z)$ we obtain

$$\operatorname{dist}(y, S(Z)) \leq 2\operatorname{dist}(y, Z \setminus \{0\}).$$

Since $\operatorname{dist}(y, S(Z)) \leq 2$ we can write

$$\begin{aligned} \operatorname{dist}(y, S(Z)) &\leq 2\min\{1, \operatorname{dist}(y, Z \setminus \{0\})\} = 2\min\{|y|, \operatorname{dist}(y, Z \setminus \{0\})\} \\ &= 2\operatorname{dist}(y, Z). \end{aligned}$$

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[1]The inequality allows to consider $d_1(X, Y) = \log(1 + \delta_1(X, Y))$ which is a metric and induces the same topology as the *neighbourhood topology* generated by δ_1.

Then $\delta_S(Y,Z) \le 2\delta_1(Y,Z)$. Since $\delta(Y,Z) \le \delta_S(Y,Z)$ the proof is complete. $\square$

We remark that the quantities introduced in this section such as δ, δ_1 and δ_S induce the same topology on $G(E)$. Since δ, δ_S and δ_1 induce the same topology we will choose time after time the one that most fits our settings.

3. Properties of the Hausdörff topology

Given Banach spaces E and F we denote by $\mathcal{L}(E,F)$ the space of linear and bounded applications and use the abbreviate notation $\mathcal{L}(E)$ to denote $\mathcal{L}(E,E)$. We call *general linear group* the set of invertible bounded operators of E endowed with the topology of the norm and denote it by $GL(E)$. In this section we show that the choice of the Hausdörff metric makes continuous some natural operations on $G(E)$, such as the multiplication by an invertible operator and the annihilator subspace $Y^{\perp}$.

Proposition 3.1. *Consider the set $GL(E) \times G(E)$ with the topology induced by the product metric $\|\cdot\| \times \delta$. The action of $GL(E)$ on $G(E)$ given by*

$$GL(E) \times G(E) \longrightarrow G(E), \quad (T,Y) \longmapsto T \cdot Y$$

is continuous.

Proof. We will prove that this map is locally Lipschitz. Fix $T \in GL(E)$ and let Y, Z be two closed subspaces in $G(E)$. Set $Ty = y' \in D(TY)$ and $r = \|T^{-1}\|$. Hence $|y| \le r$. Thus, by Proposition 2.4, we have

$$\begin{aligned} \mathrm{dist}(y', D(TZ)) &\le 2\mathrm{dist}(y', TZ) = 2r\mathrm{dist}(y'/r, TZ) \le 2r\|T\|\mathrm{dist}(y/r, Z) \\ &\le 2\|T^{-1}\|\|T\|\rho_1(Y,Z) \le 2\|T^{-1}\|\|T\|\rho(Y,Z) \end{aligned}$$

hence

$$\rho(TY, TZ) \le 2\|T^{-1}\|\|T\|\rho(Y,Z). \tag{4}$$

Now fix $Y \in G(E)$, T and S invertible operators and $y' \in D(TY)$. As above $|y| \le r$ and we have

$$\mathrm{dist}(y', D(SY)) \le 2\mathrm{dist}(y', SY) \le 2\|T-S\||y| \le 2\|T-S\|\|T^{-1}\|;$$

taking the supremum over $D(TY)$ and switching T and S we find

$$\delta(TY, SY) \le 2\|T-S\| \max\{\|T^{-1}\|, \|S^{-1}\|\}. \tag{5}$$

Now choose a point $(T_0, Y_0) \in GL(E) \times G(E)$ and set $r_0 = \|T_0^{-1}\|$; given $\alpha < 1$ we claim that in the neighbourhood

$$U = B(T_0, \alpha r_0^{-1}) \times G(E)$$

the map is Lipschitz. It is not hard to prove that for such radius the norm of the inverse of every operator is bounded by a constant that depends only on α and r_0. More precisely, using Von Neumann series, it is simple to find $r_0/(1-\alpha)$ as bound. Let (T,Y) and (S,Z) be two points in U. Hence

$$\begin{aligned} \delta(TY, SZ) &\le \delta(TY, SY) + \delta(SY, SZ) \\ &\le 2\max\{\|T^{-1}\|, \|S^{-1}\|\}\|T-S\| + 2\|S\|\|S\|^{-1}\delta(Y,Z) \\ &\le \frac{2r_0}{1-\alpha}\|T-S\| + 2\alpha r_0^{-1} \cdot \frac{r_0}{1-\alpha}\delta(Y,Z) \\ &\le \frac{2\max\{\alpha, r_0\}}{1-\alpha}\Big(\|T-S\| + \delta(Y,Z)\Big). \end{aligned}$$

$\square$

Proposition 3.2. *If $\rho_1(Y,Z) < 1$ and $Z \subseteq Y$ then $Z = Y$.*

PROOF. If Y is the null space the proof is trivial. Otherwise let $\rho(Y,Z) = 1-\varepsilon_0$ and suppose $S(Y)\setminus Z$ is not empty and contains an element, say y. Let $z \in Z$ be such that

$$\operatorname{dist}(y,Z) \geq |y-z| - \varepsilon_0/2;$$

define $y_0 = z - y$. Since $Z \subseteq Y$, $y_0 \in Y$. Thus $\operatorname{dist}(\hat{y_0}, Z) \geq 1-\varepsilon_0/2$, thus

$$1-\varepsilon_0 = \rho(Y,Z) \geq \operatorname{dist}(\hat{y_0}, Z) \geq 1-\varepsilon_0/2$$

which is impossible, then $Y \subset Z$ and $Y = Z$. □

Definition 3.3. We denote by E^* the space $\mathcal{L}(E,\mathbb{R})$. It is called *topological dual* of E and its elements are called *functionals*. For any subset $S \subset E$ we define

$$S^\perp = \{\xi \in E^* : \langle \xi, s\rangle = 0\, \forall s \in S\},$$

and call it *annihilator* of S.

The annihilator is a linear, closed subspace of E^*, and it is well-behaved with respect to the topology of $G(E)$.

Proposition 3.4. *Given two closed subspaces Y, Z and $Y^\perp$, $Z^\perp$ its annihilators, we have $\rho_1(Y,Z) = \rho_1(Z^\perp, Y^\perp)$.*

PROOF. We prove that, for any closed subspace Y, a functional $\xi \in E^*$ and $x \in E$, the equalities

$$\operatorname{dist}(\xi, Y^\perp) = \sup_{D(Y)} |\langle \xi, y\rangle| = |\xi_{|Y}|, \tag{6}$$

$$\operatorname{dist}(x, Y) = \sup_{D(Y^\perp)} |\langle \eta, x\rangle| \tag{7}$$

hold. The proof of both uses Hahn-Banach theorems of extension of functionals, see [**11**] details. Given ε there exists $\overline{y} \in D(Y)$ such that, for every $\eta \in Y^\perp$, we can write

$$|\xi_{|Y}| < \varepsilon + \langle \xi, \overline{y}\rangle = \varepsilon + \langle \xi - \eta, \overline{y}\rangle \leq \varepsilon + |\xi - \eta|;$$

taking the infimum over $D(Y^\perp)$ we get $|\xi_{|Y}| \leq \operatorname{dist}(\xi, Y^\perp)$. Conversely, given a functional ξ, by Hahn-Banach, there exists an extension ξ_1 of $\xi_{|Y}$ such that $|\xi_1| = |\xi_{|Y}|$. Thus $\eta = \xi - \xi_1$ annihilates Y and we can write

$$\operatorname{dist}(\xi, Y^\perp) \leq |\xi - \eta| = |\xi_1| = |\xi_{|Y}|.$$

We prove the second equality. Let $\varepsilon > 0$. There exists $\eta_1 \in D(Y^\perp)$ such that, for every $y \in Y$

$$\sup_{D(Y^\perp)} |\langle \eta, x\rangle| < \varepsilon + |\langle \eta_1, x\rangle| = \varepsilon + |\langle \eta_1, x - y\rangle| \leq \varepsilon + |x-y|;$$

taking the infimum over Y we find

$$\sup_{D(Y^\perp)} |\langle \eta, x\rangle| \leq \varepsilon + \operatorname{dist}(x, Y).$$

To prove the opposite inequality we distinguish two cases. If $x \in Y$ the proof is trivial, because both terms of (7) are zero. Suppose $x \notin Y$. Let $0 \leq \alpha < 1$. There exists $y_\alpha \in Y$ such that

$$\alpha|x - y_\alpha| < \operatorname{dist}(x, Y) \leq |x - y_\alpha|;$$

since $x - y_\alpha \notin Y$ we can define a functional η_α such that its restriction to Y is zero and $\langle \eta_\alpha, x - y_\alpha\rangle = \alpha|x - y_\alpha|$. By Hahn-Banach for every α there exists an extension $\tilde{\eta}_\alpha$ of η_α such that $|\tilde{\eta}_\alpha| = |\eta_\alpha|$. It is clear by its definition that $\tilde{\eta}_\alpha \in Y^\perp$. Consider $z = \lambda(x - y_\alpha) + y$. We have

$$\begin{aligned} |z| = |\lambda| \left| x - y_\alpha + \frac{y}{\lambda} \right| &\geq |\lambda| \operatorname{dist}(x, Y) \geq \alpha|\lambda||x - y_\alpha| \\ &\geq |\lambda||\langle \eta_\alpha, x - y_\alpha\rangle| = |\langle \eta_\alpha, z\rangle| \end{aligned}$$

then $|\eta_\alpha| \le 1$ and $\tilde\eta_\alpha \in D(Y^\perp)$. Therefore
$$\alpha \mathrm{dist}(x, Y) \le \alpha|x - y_\alpha| = |\langle \tilde\eta_\alpha, x - y_\alpha\rangle| = |\langle \tilde\eta_\alpha, x\rangle| \le \sup_{D(Y^\perp)} |\langle \tilde\eta, x\rangle|.$$
The equality is proved as $\alpha \to 1$. Now we can prove the equality claimed in the statement. We have
$$\rho_1(Y, Z) = \sup_{D(Y)} \mathrm{dist}(y, Z) = \sup_{D(Y)} \sup_{D(Z^\perp)} |\langle \xi, y\rangle|$$
by (7). Here we switch the order of the supremums. By (6) the last term of the equality is
$$\sup_{D(Z^\perp)} \sup_{D(Y)} |\langle \xi, y\rangle| = \sup_{D(Z^\perp)} \mathrm{dist}(\xi, Y^\perp) = \rho(Z^\perp, Y^\perp).$$
□

Corollary 3.5. The map $G(E) \to G(E^*)$ that associates a subspace with its annihilator is continuous.

4. The complemented Grassmannian

We define the *complemented Grassmannian*, $G_s(E)$ as the subset of $G(E)$ of complemented subspaces, and the space of projectors. We prove that the former is an open subset and the latter is homeomorphic to the *splitting* space $\mathrm{Splt}(E)$, defined as the family of pairs (X, Y) which are the complement of each other. We prove that $\mathcal{L}(X, Y)$ is homeomorphic to $G_s(X \times Y)$ for every X, Y Banach spaces.

Definition 4.1. A closed subspace $Y \in G(E)$ is said *complemented* or that *splits* if there exists $Z \in G(E)$ such that $Y \oplus Z = E$, called *complement* of Y. We call *projector* a bounded operator $P \in \mathcal{L}(E)$ such that $P^2 = P$. We introduce the sets
$$G_s(E) = \{ Y \in G(E) \mid Y \text{ is complemented } \};$$
$$\mathcal{P}(E) = \big\{ P \in \mathcal{L}(E) \mid P^2 = P \big\}$$
and call them *complemented Grassmannian* and *space of projectors*, respectively. We will also refer to these spaces as topological subspaces of $G(E)$ and $\mathcal{L}(E)$, respectively.

By the open mapping theorem, $Y \in E$ is complemented if and only if there exists $P \in \mathcal{P}(E)$ such that $\mathrm{Range}\, P = Y$. Moreover, to each complement Z, corresponds a unique bounded $P \in \mathcal{P}(E)$ such that
$$\mathrm{Range}\, P = Y, \quad \ker P = Z.$$

Definition 4.2. Let Y, Z, P as above. We call P the *projector onto* Y *along* Z and denote it by $P(Y, Z)$.

Unless E is an Hilbert space $G_s(E) \subsetneq G(E)$, refer [**34**]. Our aim is to prove that $G_s(E)$ is an open subset of $G(E)$. For this purpose we need to introduce the notion of *minimum gap* between closed spaces (see also [**30**], Ch. IV, §4). We recall that, for any closed subspace $Y \in G(E)$, the quotient space E/Y is endowed with the norm $|x + Y| = \mathrm{dist}(x, Y)$ that makes it a Banach space called *quotient space*. Moreover the projection to the quotient is a bounded operator between two Banach spaces.

Definition 4.3 (The minimum gap)**.** Let Y and Z be two closed subspaces. Set
$$\gamma(Y, Z) = \inf_{Y \setminus Z} \frac{\mathrm{dist}(y, Z)}{\mathrm{dist}(y, Y \cap Z)}$$
if $Y \ne 0$, $\gamma(Y, Z) = 1$ otherwise. We define the *gap* by
$$\hat\gamma(Y, Z) = \min\{\gamma(Y, Z), \gamma(Z, Y)\}.$$

Lemma 4.4. (cf. [**30**], Theorem 4.2, Ch. IV) *Let Y and Z be closed subspaces of E. Then $Y+Z$ is closed in E if and only if $\gamma(Y,Z)>0$.*

Proof. Suppose both spaces are different from $\{0\}$. We prove the statement when $Y\cap Z=\{0\}$. Suppose $X=Y\oplus Z$ is closed and call P the projector onto Y along Z. Since $Y\neq\{0\}$ the projector is not zero. Let $x=y+z$. Then

$$\|P\| = \sup_{y+z\neq 0}\frac{|y|}{|y+z|} = \sup_{y\neq 0}\frac{|y|}{\operatorname{dist}(y,Z)}; \tag{8}$$

taking the inverses in the equation we find then $\|P\|^{-1}=\gamma(Y,Z)$. Suppose, conversely, that $\gamma(Y,Z)>0$. Let $\{y_n\}\subset Y$ and $\{z_n\}\subset Z$ be sequences such that $x_n=y_n+z_n\to x\in E$. If the sequence $\{x_n\}$ has a constant subsequence, then $x\in Z$, since both $\{y_n\}$ and $\{z_n\}$ are constants. Otherwise, up to extracting a subsequence we can suppose that $x_n\neq x_m$ whenever $n\neq m$. Then we can write

$$\begin{aligned}|y_n-y_m| =&\frac{|y_n-y_m|}{|x_n-x_m|}\cdot|x_n-x_m| \leq \frac{|y_n-y_m|}{\operatorname{dist}(y_n-y_m,Z)}\cdot|x_n-x_m|\\ \leq&\frac{|x_n-x_m|}{\gamma(Y,Z)};\end{aligned}$$

since the last term of the inequality is a Cauchy sequence, $\{y_n\}$ (and thus $\{z_n\}$) converges and $x=\lim y_n+\lim z_n\in X$. Since both Y and Z are closed $x\in Y+Z$. For the general case consider the quotient space $E/(Y\cap Z)$ and call π the projection onto the quotient. Let $\tilde{Y}=\pi(Y)$ and $\tilde{Z}=\pi(Z)$; these are closed subspaces of F because π maps closed subspaces of E containing $\ker\pi$ onto closed subspaces. Moreover $\gamma(\tilde{Y},\tilde{Z})=\gamma(Y,Z)$, in fact

$$\operatorname{dist}(\tilde{y},\tilde{Z})=\inf_{z\in Z}\operatorname{dist}(y-z,Y\cap Z)=\operatorname{dist}(y,Z).$$

The proof carries on as follows: suppose $Y+Z$ is a closed subspace. Then $\pi(Y+Z)=\tilde{Y}+\tilde{Z}$ is closed in the quotient space. The space $\pi(Y)$ and $\pi(Z)$ have null intersection thus, by the first part of the proof, $\gamma(\pi(Y),\pi(Z))>0$ hence $\gamma(Y,Z)>0$. The converse is completely similar. □

In the next proposition we prove that $G_s(E)$ is an open subset of $G(E)$. A proof of this is due to E. Berkson, [**8**] Theorem 5.2, when $G(E)$ has the topology induced by the Schäffer metric. However the same proof works for the metric of geometric opening.

Proposition 4.5. *Let $X\in G_s(E)$ be a proper subspace of E. Let $Y\in G_s(E)$ be a complement of X. Denote by P the projector onto X along Y. If $Z\in G(E)$ and*

$$\rho_S(X,Z)<\gamma(X,Y), \tag{9}$$

$$\rho_S(Z,X)<\gamma(Y,X) \tag{10}$$

then $Z\oplus Y=E$. If Q is the projector onto Z along Y the operator $I+Q-P$ is invertible and maps X onto Z. Moreover

$$\|P-Q\|\leq\|I-P\|\frac{\|P\|\rho_S(X,Z)}{1-\|P\|\rho_S(X,Z)} \tag{11}$$

Proof. First we prove that $Z\cap Y=\{0\}$ and $Z+Y$ is closed. In fact, given $y\in Z\cap Y$, $|y|=1$, from (10) we can write

$$\operatorname{dist}(y,X)\leq\operatorname{dist}(y,S(X))\leq\rho_S(Z,X)<\gamma(Y,X)\leq\operatorname{dist}(y,X)$$

which is absurd. To prove that $Y+Z$ is closed it will suffice to show that $\gamma(Z,Y)>0$, by Proposition 4.4. Let $z\in S(Z)$ and $1<\alpha$; there exists $x_\alpha\in S(X)$ such that

$$\alpha\operatorname{dist}(x_\alpha,Z)\geq|x_\alpha-z|;$$

for any $y \in Y$ we can write

$$\begin{aligned}|z-y| &\geq |x_\alpha - y| - |x_\alpha - z| \geq \operatorname{dist}(x_\alpha, Y) - \alpha \operatorname{dist}(x_\alpha, Z)\\ &\geq \gamma(X,Y) - \alpha\rho_S(X,Z);\end{aligned}$$

if $\alpha - 1$ is small the last term is positive. Taking the infimum over Y and $S(Z)$ we get $\gamma(Z,Y) > 0$, hence $Y+Z$ is closed. We prove now that $Z+Y = E$ by showing that $X \subseteq Z+Y$. Let $x \in X$ and $\lambda > 1$; by induction we can build two sequences $\{x_n\} \subset X$, $\{z_n\} \subset S(Z)$, such that

$$x_0 = x,\ |x_n - z_n| \leq \lambda\rho_S(X,Z)|x_n|,\ x_{n+1} = P(x_n - z_n) \tag{12}$$

$$x = \sum_{k=0}^{n}(z_k + y_{k+1}) + x_{n+1} \tag{13}$$

where $y_{k+1} = (I-P)(x_k - z_k)$. For every $k \in \mathbb{N}$ we also have, by induction

$$|x_k| \leq (\lambda\|P\|\rho_S(X,Z))^k|x_0|; \tag{14}$$

by (8) $\|P\| = \gamma(X,Y)^{-1}$ and (9) allows us to choose a positive λ such that

$$\lambda\rho_S(X,Z)\gamma(X,Y)^{-1} < 1.$$

Then $x_k \to 0$. Taking the limit in (13) we find $x \in \overline{Z+Y} = Z+Y$. The operator $I+Q-P$ maps X into Z and fixes Y. Since Q and P project along the same space a direct computation shows that its inverse is $I-Q+P$, thus $(I+Q-P)X = Z$. Choose $\lambda > 1$ and $v \in E$. We apply the construction made above to $x = Pv$. By (12) we have

$$\begin{aligned}|y_{k+1}| &\leq \|I-P\||x_k - y_k| \leq \lambda\|I-P\|\rho_S(X,Z)|x_k|\\ &\leq \frac{\|I-P\|}{\|P\|}(\lambda\|P\|\rho_S(X,Z))^{k+1}|x|\end{aligned}$$

by (13). If $\lambda\|P\|\rho_S(X,Z) < 1$ we have

$$|(P-Q)Pv| \leq \sum_{k=0}^{\infty}|y_{k+1}| \leq \|I-P\|\frac{\lambda\rho_S(X,Z)}{1-\lambda\|P\|\rho_S(X,Z)}|Pv|.$$

Letting $\lambda \to 1$, since $(P-Q)v = (P-Q)Pv$, we obtain (11). □

Corollary 4.6. The subset $G_s(E)$ is open in $G(E)$ with the topology induced by the geometric opening.

Another consequence of Proposition 4.5 is the following

Proposition 4.7. *Let X and Y be Banach spaces. The map $\mathcal{L}(X,Y) \to G_s(X \times Y)$ that associates an operator with its graph is a homeomorphism with the open subset*

$$\{Z \in G_s(X \times Y) \mid Z \oplus \{0\} \times Y = X \times Y\}.$$

Proof. Since S is bounded graph(S) is closed and it is a topological complement of $\{0\}\times Y$, then it is an element of $G_s(X\times Y)$. Hence the map is well defined. For any $S \in \mathcal{L}(X,Y)$ define $\check{S}(x,y) = (x, y+Sx)$; it is an invertible operator. Since graph$(S) = \check{S}(X \times \{0\})$, by Proposition 3.1 the map is continuous and injective. To prove that it is also open let graph(S) be a point in the image. We show that there exists $r > 0$ such that $B(\text{graph}(S), r) \subset \text{Im}(\text{graph})$, with the metric induced by δ_S. We choose

$$r < \hat{\gamma}(\text{graph}(S), \{0\} \times Y);$$

given $Z \in B(\text{graph}(S), r)$, by Proposition 4.5 Z is a topological complement of $\{0\} \times Y$. Thus, for every $x \in X \times \{0\}$ there exists a unique $z \in Z$ such that $Pz = x$. Then P maps isomorphically Z onto X and

$$\text{graph}((I-P)P_{|Z}^{-1}) = Z$$

which concludes the proof. □

Given $X \in G_s(E)$ and Y such that $X \oplus Y$ we can identify X with $X \times \{0\}$, the graph of the null operator. The subset of topological complements of $X \times \{0\}$ is open and homeomorphic to the Banach space $\mathcal{L}(X,Y)$ by Proposition 4.7. Thus we have proved that

Corollary 4.8. $G_s(E)$ is a topological Banach manifold.

Definition 4.9. Define the *space of splittings* the subset

$$\{(X,Y) \in G_s(E) \times G_s(E) \mid X \oplus Y = E\}$$

endowed with the product metric $\delta_S \times \delta_S$ and denote it by $\mathrm{Splt}(E)$.

We can associate to a pair $(X,Y) \in \mathrm{Splt}(E)$ the projector $P(X,Y)$.

Proposition 4.10. *The map $P\colon \mathrm{Splt}(E) \to \mathcal{P}(E)$, $(X,Y) \mapsto P(X,Y)$ is a homeomorphism with its image.*

Proof. First observe that P is a bijection. Its inverse maps P to $(\operatorname{Range} P, \ker P)$. Suppose $(X_0,Y_0) = (\operatorname{Range} P_0, \ker P_0)$ and $\varepsilon > 0$. We prove that there exists $\delta > 0$ such that $P(B((X_0,Y_0),\delta)) \subseteq B(P_0,\varepsilon)$. More precisely, in a suitable neighbourhood of (X_0,Y_0), for every (X,Y) we can choose continuously an invertible operator U that maps X_0 and Y_0 onto X and Y respectively and

$$\|UP_0U^{-1} - P_0\| < \varepsilon. \tag{15}$$

This completes the proof because UP_0U^{-1} is a projector with range X and kernel Y. Thus UP_0U^{-1} is the projector onto X along Y. We construct U and δ as follows: as first step we choose $\delta_0 < \hat{\gamma}(X_0,Y_0)$. If $\delta_S(X_0,X) < \hat{\gamma}(X_0,Y_0)$ the Proposition 4.5 provides us with an operator $T = I + P(X,Y_0) - P_0$ and a positive constant c such that

$$TX_0 = X,\ TY_0 = Y_0,\ \|T - I\| < c\delta_S(X_0,X). \tag{16}$$

As second step we build another invertible operator S that maps Y_0 onto $T^{-1}Y$ and fixes X_0, applying the same Proposition. Hence $U = TS$ fits our request. This can be done if, for instance, $\delta_S(T^{-1}Y,Y_0) < \hat{\gamma}(X_0,Y_0)$. Using the estimate (4) we write

$$\delta_S(T^{-1}Y,Y_0) = \delta_S(T^{-1}Y,T^{-1}Y_0) \le 2\|T\|\|T^{-1}\|\delta_S(Y_0,Y); \tag{17}$$

if $c\delta_S(X,X_0) < 1$, using Von Neumann series, we can estimate $\|T^{-1}\|$ with $1/(1-\|I-T\|)$. Then the (17) becomes

$$\delta_S(T^{-1}Y,Y_0) \le 2\frac{1+c\hat{\gamma}(X_0,Y_0)}{1-c\hat{\gamma}(X_0,Y_0)}\delta_S(Y_0,Y). \tag{18}$$

Then, if we choose

$$\delta_S(Y_0,Y) < \frac{1-c\hat{\gamma}(X_0,Y_0)}{2(1+c\hat{\gamma}(X_0,Y_0))} \tag{19}$$

we have $\delta_S(T^{-1}Y,Y_0) < \hat{\gamma}(X_0,Y_0)$ and it is possible to apply 4.5 and such operator S exists. By (11) and (18) we can write the (19) as

$$\|I - S\| < k\delta_S(Y_0,Y). \tag{20}$$

If we choose $\delta_1 = \min\{\delta_0, 1, 1/8k, 1/4c\}$, using (16) and (20) we can estimate the norm of the operator $U - I$ from above by

$$\begin{aligned} \|T(S-I) + T - I\| &\le k(1 + c\delta_S(X_0,X))\delta_S(Y_0,Y) + c\delta_S(X_0,X) \\ &\le 2k\delta_S(Y_0,Y) + c\delta_S(X_0,X) \le 1/2. \end{aligned} \tag{21}$$

We can write $UP_0U^{-1} - P_0$ as $(U-I)P_0U^{-1} + P_0(U^{-1} - I)$. By (21) the norm of $I - U$ is strictly smaller than 1. Hence $\|U^{-1}\|$ can be estimated by $1/(1-\|I-U\|)$ which is smaller than 2, still by (21). Then

$$\|UP_0U^{-1} - P_0\| \le 4\|P_0\|\|I-U\| \le 4\|P_0\|(2k\delta_S(Y_0,Y) + c\delta_S(X_0,X)).$$

Finally we set

$$\delta = \min\left\{\delta_1, \frac{\varepsilon}{4(2k+c)\|P_0\|}\right\}.$$

The continuity of the inverse follows at once: given $P, Q \in \mathcal{P}(E)$

$$\begin{aligned}\delta_S \times \delta_S((\operatorname{Range} Q, \ker Q), (\operatorname{Range} P, \ker P)) =& \delta_S(\operatorname{Range} Q, \operatorname{Range} P) + \delta_S(\ker Q, \ker P)\\ \leq& 4\|P - Q\|;\end{aligned}$$

in fact is Lipschitz. □

5. Compact perturbation of subspaces

We define a relation of *compact perturbation* for pairs of closed subspaces and an integer that we call *relative dimension*. We prove that it is well-behaved with respect to the Fredholm index of a pair of subspaces and that kernels and images of two operators with compact difference are compact perturbation of each other. When both spaces are complemented, the relation of compact perturbation is equivalent to require that for every pair of projectors (P, Q), the operators $(I - P)Q$ and $(I - Q)P$ are compact. That generalizes an existing definition in [**54**] when $P - Q$ is compact.

We need some preliminary concepts about Fredholm operators and compact operators. We recall some basic definitions and state some useful results about Fredholm operators and Fredholm pairs. For more details we refer to Appendix B.

Given a linear operator $T: E \to F$ we can always consider the vector spaces $\ker T$ and $F/\operatorname{Range} T$. We denote the second by $\operatorname{coker} T$.

Definition 5.1. A bounded operator $T \in \mathcal{L}(E, F)$ is called *semi-Fredholm* if and only if $\operatorname{Range} T$ is closed and either $\ker T$ or $\operatorname{coker} T$ has finite dimension. We define its *index* as

$$\operatorname{ind}(T) = \dim \ker T - \dim \operatorname{coker} T.$$

When only one between $\ker T$ and $\operatorname{coker} T$ has finite dimension we will write, for short, $\operatorname{ind}(T) = \infty$ or $\operatorname{ind}(T) = -\infty$, respectively. If both spaces have finite dimension we say that T is *Fredholm* and the index is a integer.

Definition 5.2. A pair (X, Y) of closed and linear subspaces is said *semi-Fredholm* if and only if $X + Y$ is closed and either $X \cap Y$ or $E/(X + Y)$ has finite dimension. We define its *index* as

$$\operatorname{ind}(X, Y) = \dim X \cap Y - \operatorname{codim} X + Y.$$

When only one between $X \cap Y$ and $X + Y$ has finite dimension we will write $\operatorname{ind}(X, Y) = \infty$ or $\operatorname{ind}(X, Y) = -\infty$, respectively. If both $X \cap Y$ and $X + Y$ have finite dimension the pair is said *Fredholm*.

There is a strict relation between (semi)Fredholm pairs and (semi)Fredholm operators. Precisely, given closed subspaces (X, Y) the operator

$$F_{X,Y}: X \times Y \to E, \quad (x, y) \mapsto x - y \tag{22}$$

is (semi)Fredholm if and only if (X, Y) is (semi)Fredholm and $\operatorname{ind}(X, Y) = \operatorname{ind}(F_{X,Y})$. Given Banach spaces E, F we denote by $\mathcal{L}_c(E, F)$ the set of compact operators.

Definition 5.3. An operator $T: E \to F$ is said *essentially invertible* if and only if there exists $S \in \mathcal{L}(F, E)$ and compact operators $K \in \mathcal{L}_c(E)$, $H \in \mathcal{L}_c(F)$ such that

$$\begin{aligned}S \circ T &= I_E + K\\ T \circ S &= I_F + H.\end{aligned}$$

It is not hard to prove that an operator is Fredholm if and only if is essentially invertible, see Proposition B.3. We end this section with a strong result of perturbation theory.

Theorem 5.4. (cf. [**30**], Ch. IV, §5). *Let (X,Y) be a semi-Fredholm pair. Then there exists $\delta > 0$ such that, $\delta_S(X',X) < \delta$, $\delta_S(Y',Y) < \delta$ implies that (X',Y') is semi-Fredholm and* $\operatorname{ind}(X',Y') = \operatorname{ind}(X,Y)$.

Definition 5.5. (cf. Definition 1.1 of [**2**]). Two closed subspaces X and Y of a Hilbert spaces are compact perturbation one of each other if the orthogonal projections P_X and P_Y have compact difference. This implies that $X\cap Y^\perp$ and $X^\perp\cap Y$ are finite dimensional subspaces and the relative dimension is defined as

$$\dim(X,Y) = \dim(X\cap Y^\perp) - \dim(X^\perp \cap Y).$$

Our first aim is to define the relative dimension for pairs of closed subspaces that do not necessarily split.

Definition 5.6 (commensurability)**.** Let $X, Y \in G(E)$. The pair (X,Y) is said *commensurable* if there are $F, G \in \mathcal{L}(E)$ such that

$$GX \subset Y,\ G_{|X} = (I+H)_{|X}, \tag{23}$$

$$FY \subset X,\ F_{|Y} = (I+K)_{|Y} \tag{24}$$

where H and K are compact operators.

Being commensurable is an equivalence relation. Symmetry and reflectivity are obvious. The proof of transitivity reduces to check that products of compact perturbations of the identity is a compact perturbation of the identity. From now on when X is commensurable to Y we will call the pair (X,Y) commensurable.

Proposition 5.7. *Let (X,Y) be a commensurable pair and (F,G) as above. The restrictions of F and G to Y and X, denoted by f and g respectively, are the essential inverse, one of each other, hence, by Proposition B.3, are Fredholm operators. Moreover, if (F',G') is another pair*

$$\operatorname{ind} f = \operatorname{ind} f',\ \ \operatorname{ind} g = \operatorname{ind} g'. \tag{25}$$

Proof. For every $t\in[0,1]$ consider the convex combinations $F_t = (1-t)F + tF'$, $G_t = (1-t)G + tG'$. It is easy to check that

$$f_t g_t = F_t G_{t|X} = I_X + k(t),$$
$$g_t f_t = G_t F_{t|Y} = I_Y + h(t)$$

where h and k are continuous paths of compact operators on Y and X respectively. Thus, for every t the operators f_t and g_t are the essential inverse one of each other. Taking $t = 0$, we obtain the first part of the statement. By ii) of Proposition B.5, continuous paths of Fredholm operators have constant index. Hence

$$\operatorname{ind} f = \operatorname{ind} f_0 = \operatorname{ind} f_1 = \operatorname{ind} f',$$
$$\operatorname{ind} g = \operatorname{ind} g_0 = \operatorname{ind} g_1 = \operatorname{ind} g'.$$

□

Definition 5.8 (relative dimension)**.** Let (X,Y) and (F,G) be as in the preceding definition. We define the *relative dimension* of the pair $\operatorname{ind} g$ and denote it by $\dim(X,Y)$.

The proposition proved above says that this definition does not depend on the choice of the pair of operators (F,G). Given X,Y,Z such that (X,Y) and (Y,Z) are commensurable the properties

$$\dim(X,X) = 0,$$
$$\dim(X,Y) = -\dim(Y,X),$$
$$\dim(X,Z) = \dim(X,Y) + \dim(Y,Z)$$

follow from the properties of composition of Fredholm operators stated in Proposition B.4. We give now a definition of compact perturbation for pair of splitting subspaces, useful for building examples.

Definition 5.9 (compact perturbation)**.** Let $X, Y \in G_s(E)$. We say that they are *compact perturbation* (one of the each other) if, given two projectors P and Q with ranges X and Y respectively, the operators

$$(I-P)Q,\ (I-Q)P \tag{26}$$

are compact.

When (X, Y) is a pair of elements of the Grassmannian of splitting spaces commensurability and compact perturbation are equivalent.

Proposition 5.10. *Let X and Y closed and complemented subspaces of E. Then (X, Y) is a commensurable pair if and only if X is compact perturbation of Y.*

Proof. Suppose X is compact perturbation of Y and let P and Q be two projectors with ranges X and Y. Clearly $QX \subset Y$ and $PY \subset X$. Moreover,

$$\begin{aligned} Qx &= Qx - x + x = -(I-Q)Px + x \\ Py &= Py - y + y = -(I-P)Qy + y; \end{aligned}$$

we obtain two restrictions of compact perturbation of the identity, as the definition of commensurability requires. Conversely let F and G be as in Definition 5.6 and (P, Q) a pair of projectors with ranges X and Y. We check, for instance, that $(I-P)Q$ is compact.

$$(I-P)Q = (I-P)(Q-FQ) + (I-P)FQ = (I-P)KQ + 0.$$

Similarly $(I-Q)P$ is compact. □

For sake of simplicity we will sometimes use the notation $\dim(P, Q)$ or $[P-Q]$ instead of $\dim(\operatorname{Range} P, \operatorname{Range} Q)$. Let H be a Hilbert space and (X, Y) a pair of two closed subspaces that are compact perturbation one of each other. Call P_X and P_Y the orthogonal projections. By (26) $P_{Y^\perp}P_X$ and $P_{X^\perp}P_Y$ are compact operators. Therefore

$$\begin{aligned} P_X - P_Y &= (P_Y + P_{Y^\perp})P_X - P_Y(P_X + P_{X^\perp}) = \\ &= P_{Y^\perp}P_X - P_Y P_{X^\perp} = P_{Y^\perp}P_X - (P_{X^\perp}P_Y)^* \in \mathcal{L}_c(E). \end{aligned}$$

Hence P_X and P_Y have compact difference and the Definition 5.9 coincides with the one known for Hilbert spaces. The relative dimension can be computed as

$$\dim(X, Y) = \dim\ker P_{Y|X} - \operatorname{coker} P_{Y|X} = \dim(X \cap Y^\perp) - \dim(X^\perp \cap Y)$$

which coincides with the definition of relative dimension in Hilbert spaces. In the following example we compute the relative dimension in some special case.

Example 5.11. Let V_0 and W_0 be finite dimensional subspaces and V_1 and W_1 topological complements of V_0 and W_0 respectively. We prove, using the result of Proposition 5.10, that (V_0, W_0) and (V_1, W_1) are commensurable pairs and compute their relative dimension. Let P and Q be two projectors onto V_0 and W_0. Denote by q the restriction of Q to V_0. It is a linear map between finite dimensional subspaces, hence

$$\dim V_0 = \dim\ker q + \dim \operatorname{Range} q = \dim\ker q + \dim W_0 - \operatorname{coker} q$$

and the Fredholm index of q is the difference of the dimensions of V_0 and W_0. Now consider the pairs (V_1, E) and (E, W_1) and the pairs of projectors $(I-P, I)$, $(I, I-Q)$ Thus

$$\begin{aligned} \dim(V_1, E) &= \operatorname{ind} I_{|V_1} = -\operatorname{codim} V_1 \\ \dim(E, W_1) &= \operatorname{ind} Q = \operatorname{codim} W_1 \end{aligned}$$

hence $\dim(V_1, W_1) = \operatorname{codim} W_1 - \operatorname{codim} V_1$.

Example 5.12. In general it is not true that topological complements of two commensurable subspaces are commensurable. Given two splittings of the space

$$X \oplus X' = E = Y \oplus Y', \quad P = P(X, X'), \quad Q = P(Y, Y')$$

with X and X' compact perturbations of Y and Y' respectively, from the relations (5.9) it follows that

$$P - Q = (I - Q)P + P(I - Q)$$

is a compact operator. This is unlikely to happen even when X and Y are the same space. For instance let $X \subset E$ be a splitting subspace with a topological complement X' such that $\mathcal{L}_c(X', X) \subsetneq \mathcal{L}(X', X)$. For any $L \in \mathcal{L}(X', X) \setminus \mathcal{L}_c(X', X)$ define

$$P(L)(x, y) = (x + Ly, 0);$$

it is easy to check that $P(L)$ is a projector with range X and $P(L) - P$ is not compact. However for a given pair of two commensurable splitting subspaces a pair of projectors with compact difference always exists and we prove it in the next theorem. This is equivalent to find topological commensurable complements.

In the next proposition we describe the relation between the relative dimension and the Fredholm index of Fredholm pairs.

Proposition 5.13. *If X is compact perturbation of Y and (Y, Z) is a Fredholm pair, then (X, Z) is Fredholm and* $\operatorname{ind}(X, Z) = \dim(X, Y) + \operatorname{ind}(Y, Z)$.

Proof. Let P and Q be projectors with ranges X and Y respectively. The restrictions p and q to Y and X are Fredholm operators; we have

$$\begin{aligned} F_{X,Z}(x, z) &= x - z = x - Qx + Qx - z \\ &= (I - Q)Px + Qx - z = (I - Q)Px + F_{Y,Z}(Qx, z) \\ &= ((I - Q)P, 0_Z) \cdot (x, z) + F_{Y,Z} \circ (q, I) \cdot (x, z). \end{aligned} \tag{27}$$

Since $F_{Y,Z}$ and (q, I) are Fredholm their composition is Fredholm; the first summand of the last equation is compact. Hence $F_{X,Z}$ is a compact perturbation of a Fredholm operator and therefore Fredholm by Proposition B.2 and

$$\operatorname{ind} F_{X,Z} = \operatorname{ind} F_{Y,Z} \circ (q, I) = \operatorname{ind} F_{Y,Z} + \operatorname{ind}(q, I) = \operatorname{ind}(Y, Z) + \dim(X, Y)$$

by Proposition B.4. □

Example 5.14. We use Proposition 5.13 with in example that shows that for commensurable pairs there is not a result like the Theorem 5.4, that is, they are not stable by small perturbation: consider a pair (X, Y) such that

(i) X is isomorphic to Y,
(ii) $X \oplus Y = E$ has infinite dimension;

let $f\colon Y \to X$ be an isomorphism and $\operatorname{graph}(f)$ its graph. For every integer n consider the sequence of subspaces

$$Y_n = \operatorname{graph}(nf);$$

since Y_n is graph of a bounded operator $X \oplus Y_n = E$. It is easy to check that Y_n converges to X. Thus there can be no open neighbourhood of X in $G_s(E)$ made of compact perturbations of X. In fact for n large Y_n would be contained in such neighbourhood and (X, Y_n) would be a commensurable pair; since (X, Y_n) is a Fredholm pair also, by Proposition 5.13 we would have proved that (X, X) is a Fredholm pair which happens only if $X \oplus Y$ has finite dimension, in contradiction with hypothesis ii).

The preceding Proposition suggests a definition of the relative dimension that involves the Fredholm index. Precisely, suppose X is compact perturbation of Y. Let Z be a topological complement of Y. Then (Y, Z) is a Fredholm pair. By Proposition 5.13 (X, Z) is a Fredholm pair and

$$\operatorname{ind}(X, Z) = \operatorname{ind}(Y, Z) + \dim(X, Y) = \dim(X, Y). \tag{28}$$

This definition, together with the Theorem 5.4 will allows us to state in the next chapter a stability result of the relative dimension for closed and splitting subspaces.

Theorem 5.15. *Let X be a splitting subspace, compact perturbation of Y. Then there are topological complements X' and Y' that are compact perturbation one of each other and*

$$\dim(X, Y) = -\dim(X', Y')$$

Proof. Let P and Q be projectors with ranges X and Y respectively. As consequence of the Proposition 5.13 the pair $(X, \ker Q)$ is a Fredholm. Let Z be a topological complement of $X \cap \ker Q$ in $\ker Q$ and $R \subset E$ a finite dimensional complement of $X + \ker Q$ in E. Then

$$X \oplus Z \oplus R = E, \quad P_X + P_Z + P_R = I_E;$$

we claim that P_X and Q have compact difference. We write

$$P_X - Q = (I - Q)P_X + (P_X - Q)P_Z + (P_X - Q)P_R;$$

the first term of the right member is compact by definition of compact perturbation, the second is 0, the third has finite rank. Hence

$$Q(I - P_X), \quad P_X(I - Q)$$

are compact operators. It is not hard to prove that for all the pairs of projectors (P', Q') onto X' and Y' respectively, compactness of (26) holds, thus $\ker P_X$ and $\ker Q$ are commensurable spaces. To compute the relative dimension we use restrictions of the operators Q and $I - Q$. We can write

$$\dim(X, Y) + \dim(X', Y') = \operatorname{ind} Q_{|X} + \operatorname{ind}(I - Q)_{|X'} = \operatorname{ind} I_E = 0.$$

□

The next Proposition follows the one known for Hilbert spaces, due to A. Abbondandolo and P. Majer (refer Proposition 2.2 of [**2**]).

Proposition 5.16. *Let $T, S \in \mathcal{L}(E, F)$ be operators with compact difference and closed images. If the kernels and the images split $\ker T$ and $\operatorname{Range} T$ are compact perturbation of $\ker S$ and $\operatorname{Range} T$ respectively and the relation*

$$\dim(\ker T, \ker S) = -\dim(\operatorname{Range} T, \operatorname{Range} S).$$

holds.

Proof. Since kernels and images split we can write

$$\ker T \oplus Y(T) = E = \ker S \oplus Y(S)$$
$$Z(T) \oplus \operatorname{Range} T = F = Z(S) \oplus \operatorname{Range} S$$

Since T and S are isomorphism of $Y(T)$ with $\operatorname{Range} T$ and $Y(S)$ with $\operatorname{Range} S$ respectively, we can define operators T' and S' on F with values in E such that

$$T'T = P(Y(T), \ker T), \quad S'S = P(Y(S), \ker S)$$
$$TT' = P(\operatorname{Range} T, Z(T)), \quad SS' = P(\operatorname{Range} S, Z(S));$$

set $P(T) = P(\ker T, Y(T))$, $P(S) = P(\ker S, Y(S))$ and $K = T - S$. Then

$$(I - P(S))P(T) = S'SP(T) = S'(S - T)P(T) + S'TP(T) = S'KP(T) + 0$$

is a compact operator. Set $Q(T) = P(\operatorname{Range} T, Z(T))$, $Q(S) = P(\operatorname{Range} S, Z(S))$. Then

$$\begin{aligned}(I - Q(S))Q(T) &= (I - Q(S))TT' = (I - Q(S))(T - S)T' + (I - P(S))ST' \\ &= 0 + (I - P(S))KT'\end{aligned}$$

is compact. By Theorem 5.15, up to changing the topological complements of $\ker T$ and $\operatorname{Range} T$, we can suppose that our projectors have compact difference. Hence

$$\begin{aligned}&\dim(\ker T, \ker S) = -\dim(Y(S), Y(T)) \\ &= -\operatorname{ind}(I - P(T))^{Y(T)}_{|Y(S)} = -\operatorname{ind} T(I - P(T))^{\operatorname{Range} T}_{|Y(S)} \\ &\dim(\operatorname{Range} T, \operatorname{Range} S) = \operatorname{ind} Q(T)^{\operatorname{Range} T}_{|\operatorname{Range} S} = \operatorname{ind}(Q(T)S)^{\operatorname{Range} T}_{|Y(S)};\end{aligned}$$

observe that the operator

$$K_1 = T(I - P(T)) - Q(T)S = TT'T - TT'S = TT'(T - S)$$

is compact. Therefore

$$\begin{aligned}\dim(\ker T, \ker S) &= -\operatorname{ind}(Q(T)S + K_1)^{\operatorname{Range} T}_{|Y(S)} \\ &= -\operatorname{ind}(Q(T)S)^{\operatorname{Range} T}_{|Y(S)} = -\dim(\operatorname{Range} T, \operatorname{Range} S).\end{aligned}$$

When $P - Q$ is compact, we know of an existing definition of relative dimension in [**12**] for Hilbert spaces and in [**54**], for Banach spaces and projectors P, Q with compact difference. In the latter, given two projectors, they denote the relative dimension by $[P - Q]$. We will also use this notation in §5. □

The non-complemented case. In the technique used in the proposition above requires that the kernels and images split. We think this restriction can be removed. We also guess that whenever X is complemented and Y is commensurable to Y, then Y also splits.

CHAPTER 2

Homotopy type of Grassmannians

We define the *essentially hyperbolic* operators on a Banach space E, that we will denote by $e\mathcal{H}(E)$, and prove the existence of a group homomorphism

$$\pi_1(e\mathcal{H}(E), 2P - I) \to \mathbb{Z}$$

where P is a projector of E. The construction of such homomorphism is carried out as follows: as first step, in section §2.1, we define the *Calkin algebra*, $\mathcal{C}(E)$, as the quotient of the algebra of bounded operators $\mathcal{L}(E)$ with the closed ideal of compact operators $\mathcal{L}_c(E)$. Then we prove that $e\mathcal{H}(E)$ is homotopically equivalent to $\mathcal{P}(\mathcal{C}(E))$, the space of *idempotent* elements of the Calkin algebra. In section §2.4 we prove that the map

$$p \colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C}), \quad p(P) = P + \mathcal{L}_c(E)$$

is surjective and induces a locally trivial fiber bundle. Using the *Leray-Schauder degree* we prove in section §2.6 that the typical fiber of such bundle has infinite numerable connected components. Hence, for every projector P, we can complete the exact homotopy sequence of the fiber bundle as follows

$$\pi_1(\mathcal{P}(E), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(\mathcal{C}), p(P)) \xrightarrow{\varphi_P} \mathbb{Z};$$

we call φ_P *index* of fiber bundle $(\mathcal{P}(E), \mathcal{P}(\mathcal{C}), p)$ with respect to P or, simply *index* when no ambiguity occurs. Thus, $\varphi_P \circ \Psi_*$ is well-defined on $e\mathcal{H}(E)$, where Ψ is a homotopy equivalence with $\mathcal{P}(\mathcal{C})$. All these facts are proved without making assumptions on the Banach space E. Given a projector P the two conditions

h1) P is connected to a projector Q such that $Q - P \in \mathcal{L}_c(E)$ and $\dim(Q, P) = m$,
h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected,

are sufficient to ensure that $m \in \operatorname{Im}(\varphi_P)$ and φ_P is injective. When $m = 1$, we have an isomorphism. These hypotheses are verified by every projection of a Hilbert space with infinite dimensional range and kernel. In the most common Banach spaces such as L^p spaces and spaces of sequences, we can find such projectors.

In the last section, we give exhibit examples where the homomorphism φ is an isomorphism. This happens, for instance, when E is an infinite-dimensional Hilbert space or L^p for $p \geq 1$ or L^∞ and spaces of sequences ℓ^p, m, c_0.

1. The space of essentially hyperbolic operators

Given a Banach algebra $\mathcal{B}$ with unit 1, we denote by $G(\mathcal{B})$ the set of invertible elements. If $x \in \mathcal{B}$ the *spectrum* of x is defined as the set $\{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin G(\mathcal{B})\}$ and denoted it by $\sigma_{\mathcal{B}}(x)$ or simply $\sigma(x)$. Consider the following subsets endowed with the topology of the norm

$$\mathcal{P}(\mathcal{B}) = \left\{ p \in \mathcal{B} \mid p^2 = p \right\}, \quad \mathcal{Q}(\mathcal{B}) = \left\{ q \in \mathcal{B} \mid q^2 = 1 \right\},$$
$$\mathcal{H}(\mathcal{B}) = \{ x \in \mathcal{B} \mid \sigma(x) \cap \mathrm{i}\mathbb{R} = \emptyset \};$$

We call the elements of these spaces *projectors* (or *idempotents*), *square roots of the unit* and *hyperbolic* respectively. In literature, hyperbolic operators are sometimes defined as those whose spectrum does not intersect the unit circle; in this case *infinitesimally hyperbolic* would be more appropriate for the elements of $\mathcal{H}(\mathcal{B})$. The spaces $\mathcal{P}(\mathcal{B})$ and $\mathcal{Q}(\mathcal{B})$ are analytic, closed, embedded sub-manifolds of $\mathcal{B}$, see [**1**], Lemma 1.4 for a proof; $\mathcal{H}(\mathcal{B})$ is an open subset of $\mathcal{B}$. An analytical diffeomorphism between $\mathcal{P}(\mathcal{B})$ and $\mathcal{Q}(\mathcal{B})$ exists, given by

$$\mathcal{P}(\mathcal{B}) \ni p \mapsto 2p - 1 \in \mathcal{Q}(\mathcal{B}).$$

We prove that these three spaces have the same homotopy type. Since $\mathcal{P}$ and $\mathcal{Q}$ are diffeomorphic they have the same homotopy type; in the next proposition we define a homotopy equivalence between $\mathcal{Q}$ and $\mathcal{H}$. In order to do so, we need some preliminary notations and facts. Let x be an element of the algebra $\mathcal{B}$ and $\{A_i\}$ a finite open cover of the spectrum of pairwise disjoint sets. There are projectors p_i, called *spectral projectors*, such that

$$p_1 + \cdots + p_n = 1, \quad p_i p_j = \delta_{ij} p_j, \quad \sigma_{\mathcal{B}_i}(p_i x p_i) = A_i$$

where $\mathcal{B}_i \subset \mathcal{B}$ is the sub-algebra of the elements $p_i x p_i$ with $x \in \mathcal{B}$. We denote p_i also by $p(x; A_i)$. These projectors can be obtained as integrals

$$p(x; A_i) = \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - x)^{-1} d\lambda$$

where γ_i are closed paths such that each γ_i *surrounds* $A_i \cap \sigma(x)$ in $\mathbb{C} \setminus \cup_{j \neq i} A_j$ in the sense of Definition C.1 of Appendix C.

Proposition 1.1. *The space of roots of the unit is a deformation retract of the space of hyperbolic elements.*

Proof. If q is a square root of the unit its spectrum is contained in $\{-1, +1\}$, hence q is hyperbolic. Call i the inclusion of the space of idempotents in the space of hyperbolic elements. We define a retraction map as follows: let x be a hyperbolic element of the algebra; since,

$$\sigma(x) = (\sigma(x) \cap \{\operatorname{Re} z > 0\}) \cup (\sigma(x) \cap \{\operatorname{Re} z < 0\})$$

the spectrum has an open cover of disjoint subsets. Denote by $p^+(x)$ and $p^-(x)$ the spectral projectors $p(x; (\sigma(x) \cap \{\operatorname{Re} z > 0\}))$ and $p(x; (\sigma(x) \cap \{\operatorname{Re} z < 0\}))$ respectively. We define the map

$$r \colon \mathcal{H} \to \mathcal{B}, \quad r(x) = p^+(x) - p^-(x);$$

r is continuous by Theorem C.3 and $r(x)$ is a square root of unit. We prove that r is a left inverse of the inclusion i. Let q be a square root and $z \in \mathbb{C} \setminus \sigma(q)$, then

$$(z - q)^{-1} = \frac{z}{z^2 - 1} + \frac{q}{z^2 - 1} = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) q;$$

let γ_+ and γ_- be paths that surrounds 1 and -1 in $\mathbb{C} \setminus \{1\}$ and $\mathbb{C} \setminus \{-1\}$ respectively. By integrating both sides of the above equality around γ_+ and γ_- and dividing it by $2\pi i$, we obtain

$$p^+(q) = (1+q)/2, \quad p^-(q) = (1-q)/2, \quad r(q) = p^+(q) - p^-(q) = q;$$

this proves that $\mathcal{Q}$ is a retraction of $\mathcal{H}$. Now, define the continuous map

$$F \colon [0,1] \times \mathcal{H} \to \mathcal{B}, \quad (t, x) \mapsto (1-t)p^+ x p^+ + t p^+ + (1-t) p^- x p^- - t p^-.$$

By Property ii) and iii) of Appendix C, $F(t, x)$ is hyperbolic for every (t, x). We also have $F(0, x) = x$, $F(x, 1) = i \circ r(x)$. Thus $i \circ r$ is homotopically equivalent to $id_{\mathcal{H}}$. $\square$

Definition 1.2. Given an operator $T \in \mathcal{L}(E)$ we call *essential spectrum*, and denote it by $\sigma_e(T)$, the set $\{\lambda \in \mathbb{C} : T - \lambda I : \text{is not Fredholm}\}$.

Definition 1.3. A bounded operator T is called *essentially hyperbolic* if and only if $\sigma_e(T) \cap \mathrm{i}\mathbb{R} = \emptyset$. We denote by $e\mathcal{H}(E)$ the set of essentially hyperbolic operators endowed with the norm topology.

The set of compact operators on a Banach space E is a closed ideal of the algebra of bounded operators. Thus the quotient has a structure of Banach algebra that makes the projection

$$p\colon \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{L}_c(E), \quad T \mapsto T + \mathcal{L}_c(E)$$

an algebra homomorphism. The quotient space is called *Calkin algebra* and we denote it by $\mathcal{C}(E)$ or just $\mathcal{C}$. We characterize the essential spectrum in terms of the Calkin algebra: given $T \in \mathcal{L}(E)$ there holds

$$\sigma_e(T) = \sigma(p(T)). \tag{29}$$

To prove the equality suppose $\lambda \notin \sigma_e(T)$, hence $T - \lambda$ is Fredholm. By Proposition B.3 there exists an essential inverse S such that

$$(T-\lambda)S - I, \quad S(T-\lambda) - I$$

are compact operators. Hence $p(T-\lambda)$ is invertible in the Calkin algebra, $p(S)$ being is its inverse, thus $\lambda \notin \sigma(p(T))$. The prove of the other inclusion also follows from Proposition B.3.

Theorem 1.4. *The space $e\mathcal{H}(E)$ has the homotopy type of $\mathcal{P}(\mathcal{C})$.*

Proof. First we prove that $e\mathcal{H}(E)$ is homotopically equivalent to $\mathcal{H}(C)$. By classical results of continuous selections there exists a continuous right inverse of p, call it s. It is a consequence of Theorem D.1 when the topological space T consists of a point. Using the characterization (29) it is easy to check that $e\mathcal{H}(E) = p^{-1}(\mathcal{H}(\mathcal{C}))$. Moreover, the two continuous maps

$$\begin{aligned} &\mathcal{H}(\mathcal{C}) \times \ker p \to e\mathcal{H}(E), \quad (x, K) \mapsto s(x) + K \\ &e\mathcal{H}(E) \to \mathcal{H}(\mathcal{C}) \times \ker p, \quad A \mapsto \big(p(A), A - s(p(A))\big) \end{aligned}$$

are the inverses of each other, hence $\mathcal{H}(\mathcal{C}) \times \ker p$ is homemorphic to $e\mathcal{H}(E)$. Since $\ker p = \mathcal{L}_c(E)$ is a linear subspace of $\mathcal{L}(E)$, is contractible, thus the two maps are homotopically equivalent to the maps

$$\begin{aligned} &s\colon \mathcal{H}(\mathcal{C}) \to e\mathcal{H}(E) \\ &p\colon e\mathcal{H}(E) \to \mathcal{H}(\mathcal{C}). \end{aligned}$$

Now, by Proposition 1.1, $\mathcal{H}(\mathcal{C})$ has the same homotopy type of $\mathcal{Q}(\mathcal{C})$ which is homeomorphic to $\mathcal{P}(\mathcal{C})$. Taking the composition of all the functions we referred to, we can write explicitly an homotopy equivalence between $e\mathcal{H}(E)$ and $\mathcal{P}(\mathcal{C})$ and its homotopic inverse:

$$\begin{aligned} &\Psi\colon e\mathcal{H}(E) \to \mathcal{P}(\mathcal{C}), \quad A \mapsto p^+(p(A)) \\ &\Phi\colon \mathcal{P}(\mathcal{C}) \to e\mathcal{H}(E), \quad p \mapsto s(2p-1). \end{aligned}$$

□

2. The fiber bundle $(G(\mathcal{B}), \mathcal{P}(\mathcal{B}))$

In this section we define the fiber bundle with total space $G(\mathcal{B})$ and base space $\mathcal{P}(\mathcal{B})$. The exact homotopy sequence associated to the fiber space provides us with some relations between the homotopy groups of the base space and the total space.

Definition 2.1. We say that two projectors p, q are *conjugated* if there exists an invertible element $g \in G(\mathcal{B})$ such that $gp = qg$.

The projector $\overline{p} = 1 - p$ is naturally associated to p.

Proposition 2.2. (cf. [**43**], Proposition 4.2). *In the space of idempotents the following facts hold:*

(i) *if $\|p - q\| < 1$, there exists an invertible element $g \in G_0(\mathcal{B})$ such that $gp = qg$; thus, the space of idempotents is locally arcwise connected;*

(ii) *two idempotents are connected by a continuous path, if and only if there exists $g \in G_0(\mathcal{B})$ such that $gp = qg$.*

Proof. i). Given p, q we define $L(p,q) = pq + (1-p)(1-q)$. As t varies in $[0,1]$, we have

$$(30)\qquad \begin{gathered}(1-t+tL(p,q))(1-t+tL(q,p)) = 1-t(2-t)(p-q)^2\\ L(p,q)L(q,p) = 1-(p-q)^2.\end{gathered}$$

The right term is an invertible operator because $\|p-q\| < 1$ and $t(2-t) \leq 1$. From the second equality it follows that $L(p,q)$ and $L(q,p)$ commute, hence they are invertible too. Moreover, each of them is joint to the unit by the path. From Example C.5 there exists R such that

$$R(p,q) \in G_0(\mathcal{B}),\ \ R(p,q)^2 = \left(1-(p-q)^2\right)^{-1}$$

Thus $L(p,q)R$ and $L(q,p)R$ are the inverse of each other. By multiplying the second of (30) by R on both sides, we obtain

$$(31)\qquad L(q,p)Rp = qL(p,q)R.$$

We define $g(p,q) = L(p,q)R$.

ii). Let α be a continuous path such that $\alpha(0) = p$ and $\alpha(1) = q$. Let $\{t_i : 0 \leq i \leq n\}$ be a partition of the unit interval such that $\|\alpha(t_i) - \alpha(t_{i+1})\| < 1$ for every i. Let

$$g = \prod_{i=0}^{n-1} g(\alpha(t_{n-i}), \alpha(t_{n-i-1}))$$

since g is a product of elements of $G_0(\mathcal{B})$, it also belongs to $G_0(\mathcal{B})$. By applying (31) n times, we obtain $gp = qg$. Conversely, if there exists an element $g \in G_0(\mathcal{B})$ such that $gp = qg$, then the path $g(t)pg(t)^{-1}$ joins q to p. □

Given a projector p, we denote by $\mathcal{P}_p(\mathcal{B})$ the connected component of p and define the following subgroups of $G(\mathcal{B})$:

$$\begin{gathered}G_p(\mathcal{B}) = \{g \in G(\mathcal{B}) : gpg^{-1} \text{ is connected to } p\},\\ F_p = \{g \in G(\mathcal{B}) \mid gp = pg\}.\end{gathered}$$

Clearly $F_p \subset G_p$. We define the map

$$\pi_p \colon G_p \to \mathcal{P}_p, \quad g \mapsto gpg^{-1}.$$

By ii) of Proposition 2.2, π_p is surjective.

Theorem 2.3. (cf. [**43**], §7). *The triple $(G_p, \pi_p, \mathcal{P}_p)$ induces a principal bundle with group F_p acting on itself by multiplication on the left.*

Proof. We prove that there exists an open cover of coordinate neighbourhoods. Fix $q \in \mathcal{P}_p$ and let g as in ii) of Proposition 2.2. On the ball $B(q,1)$ we define a section of the projection map π_p

$$(32)\qquad s_q \colon B(q,1) \to G_p, \quad r \mapsto g(r,q)g.$$

Clearly $\pi_p(s_q(r)) = r$. We define coordinate neighbourhoods

$$\phi \colon B(q,1) \times F_p \to \pi_p^{-1}(B(q,1)), \quad (x,y) \mapsto s_q(x) \cdot y.$$

It is an homeomorphism and $\pi_p(\phi(x,y)) = x$. If two coordinate neighbourhoods, $B(q_1,1)$ and $B(q_2,1)$ intersect, the transition maps are

$$\phi_{2,x}^{-1}\phi_{1,x} \colon F_p \to F_p,\ \ y \mapsto s_2(x)^{-1}s_1(x)y;$$

where s_i are the sections defined on $B(q_1,1)$ and $B(q_2,1)$, respectively. Since $s_2(x)^{-1}s_1(x) \in F_p$ we have defined a *principal bundle* according to [**52**], §8. □

For principal bundles we can write the *exact homotopy sequence*, see [**52**], §17. The sequence

$$\pi_k(F_p,1) \xrightarrow{i_*} \pi_k(G_p,1) \xrightarrow{\pi_{p,*}} \pi_k(\mathcal{P}_p,p) \xrightarrow{\partial} \pi_{k-1}(F_p,1) \tag{33}$$

is exact for every $k \geq 1$.

3. The Grassmannian algebra

Given p, q idempotents of an algebra $\mathcal{B}$ we define the following equivalence relation

$$p \sim q \iff pq = q,\ qp = p. \tag{34}$$

Symmetry is obvious. If (p,q) and (q,r) are equivalent pairs then $pr = p(qr) = (pq)r = qr = r$, similarly $rp = p$, then reflectivity follows.

Definition 3.1. We denote by $Gr(\mathcal{B})$ the set of equivalence classes endowed with the quotient topology.

H. Porta and L. Recht proved in [**43**] that the Grassmannian algebra is a metric space, the quotient projection $\pi\colon \mathcal{P}(\mathcal{B}) \to Gr(\mathcal{B})$ is an open map and there exists a global continuous section of π on $Gr(\mathcal{B})$. In fact any global continuous section is a homotopy inverse of π (see [**43**], §3).

When $\mathcal{B}$ is the algebra of the bounded operators on a Banach space E two projectors are equivalent if and only if they have the same images. In fact the identity $PQ = Q$ means that $\operatorname{Range} Q \subseteq \operatorname{Range} P$. Then we have a well defined bijection

$$Gr(\mathcal{L}(E)) \to G_s(E), \quad \pi(P) \mapsto \operatorname{Range} P.$$

Lemma 3.2 (refer [**25**])**.** *There exists a continuous section of the map that associates a projector with its range. Every section is in fact a homotopy equivalence.*

Proof. Call r the map $\mathcal{P}(E) \ni P \mapsto \operatorname{Range} P$. This is continuous with the opening metric defined in §1. In fact, given $P, Q \in \mathcal{P}(E)$, it can be easily checked that

$$\delta_S(r(P), r(Q)) \leq 2\|P - Q\|.$$

We can build now a continuous section of r using the construction of [**25**] whose idea is the following: fix X a splitting subspace and choose Y a topological complement. By Proposition 4.5, for every $X' \in B(X, \hat{\gamma}(X,Y))$, we have $X' \oplus Y = E$. We define

$$s\colon B(X, \hat{\gamma}(X,Y)) \to \mathcal{P}(E),\ X' \mapsto P(X',Y);$$

by Proposition 4.5 this is a continuous local section of the map r. Since $G_s(E)$ is metric, thus paracompact, there exists a locally finite refinement of the open covering $\{B(X, \hat{\gamma}(X,Y))\}$, say $\mathcal{U} = \{U_i \mid i \in I\}$. Let $\{\varphi_i\}$ be a partition of unit subordinate to $\mathcal{U}$. Thus for every X in $G_s(E)$ define

$$s(X) = \sum_{i \in I} \varphi_i(X) s_i(X),\ s \in C(G_s(E), \mathcal{L}(E)).$$

To prove that $s(X)$ is a projector observe that if $X \in U_i \cap U_j$, then

$$\operatorname{Range} P(X, Y_i) = \operatorname{Range} P(X, Y_j) = X.$$

This is equivalent to

$$s_i(X)s_j(X) = s_j(X),\ s_j(X)s_i(X) = s_i(X);$$

keeping in mind these relations it is easy to prove that $s(X)$ is a projector with range X. In fact

$$\begin{aligned} s(X)^2 &= \sum_i \varphi_i s_i(X) \left(\sum_j \varphi_j s_j(X) \right) = \sum_i \varphi_i \left(\sum_j \varphi_j(X) s_i(X) s_j(X) \right) \\ &= \sum_i \varphi_i s(X) = s(X). \end{aligned}$$

This also proves that $r^{-1}(\{X\})$ is a convex, actually affine, subspace of $\mathcal{P}(E)$. By construction $r \circ s = id$. For every projector P we have

$$r(s \circ r(P)) = r(P), \quad tP + (1-t)s \circ r(P) \in \mathcal{P}(E)$$

for every $t \in [0,1]$. This defines a homotopy between $s \circ r$ and the identity map. $\square$

As application of the preceding Lemma we state a result of stability of the relative dimension defined on Chapter I.

Theorem 3.3. *Let X and Y be continuous functions defined on a topological space M such that $X(t)$ and $Y(t)$ are closed and splitting subspaces and $X(t)$ is compact perturbation of $Y(t)$ for every t in M. Hence $\dim(X(t), Y(t))$ is locally constant.*

Proof. Let s be a continuous section on $G_s(E)$ of the map r defined in the Lemma 3.2. Then it is defined a continuous map

$$\nu \colon G_s(E) \to G_s(E), \quad X \mapsto \ker s(X).$$

By the identity (28) the relative dimension of the pair $(X(t), Y(t))$ is the Fredholm index of the pair $(X(t), \nu(Y(t)))$. Fix t_0 in M; by Theorem 5.4 there exists a open neighbourhood of t_0, say U, such that

$$\operatorname{ind}(X(t), \nu(Y(t))) = \operatorname{ind}(X(t_0), \nu(Y(t_0)))$$

for every $t \in U$. Therefore we conclude with (28). $\square$

Theorem 3.4. *If $\mathcal{B}$ is the algebra of bounded operators on E then $Gr(\mathcal{B})$ with the quotient topology is homeomorphic to $G_s(E)$ with the topology induced by the metric δ_S.*

Proof. Let s and γ be sections on $G_s(E)$ and $Gr(\mathcal{B})$ respectively. We prove that the maps $\pi \circ s$ and $r \circ \gamma$ are inverse one of each other. Let X be a closed splitting subspace. Then

$$\gamma((\pi \circ s)(X)) \sim s(X)$$

then $r(s(X)) = X$. Thus $(r \circ \gamma) \circ (\pi \circ s) = id$. Similarly, we have $(\pi \circ s) \circ (r \circ \gamma) = id$. $\square$

4. Fibrations of spaces of idempotents

Set $\mathcal{B} = \mathcal{L}(E)$; we recall that the Calkin algebra is defined as the quotient algebra $\mathcal{C} = \mathcal{L}(E)/\mathcal{L}_c(E)$ where $\mathcal{L}_c(E)$ is the ideal of compact operators on E. It is a Banach algebra with unit. The projection to the quotient $p \colon \mathcal{B} \to \mathcal{C}$ is a surjective homomorphism. Consider the restrictions

$$p \colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C})$$
$$p_r \colon \mathcal{Q}(E) \to \mathcal{Q}(\mathcal{C}).$$

The purpose of this section is to prove that these maps induce locally trivial bundle, with non-constant fiber. First we need the following

Proposition 4.1. (cf. also [**1**], Proposition 6.1). *The maps p and p_r are surjective.*

Proof. It is enough to prove it for p_r, because the homeomorphism between $\mathcal{P}$ and $\mathcal{Q}$ commutes with the quotient projections. Let q be a square root of identity in the Calkin algebra and let Q be an operator such that $p(Q) = q$. There exists a compact operator K such that $Q^2 = I + K$. The spectrum of $I + K$ is a countable subset of $\mathbb{C}$ with at most 1 as limit point. Let U be a neighbourhood of 1 such that

$$\partial U \cap \sigma(I+K) = \emptyset, \quad U \cap \sigma(I+K) \subset B(1,1).$$

Let P be the spectral projector relative to U. Clearly $I - P$ has finite dimensional range. Let Q_1 and K_1 be the restrictions of Q to the range of P. We have

$$Q_1^2 = I + K_1$$

where K_1 is compact and $\sigma(K_1) \subset B(0,1)$. Thus, Q_1 is invertible. We seek H compact such that

$$(Q_1 - HQ_1)^2 = I, \ [Q_1, H] = 0.$$

The first becomes $(I + K_1)(I - H)^2 = I$. A solution of this equation is given by

$$I - \widehat{f}(K_1), \quad f(z) = \frac{1}{\sqrt{1+z}}$$

where $\widehat{f}(K_1)$ is defined according to Theorem C.4. Since the first coefficient of f in the power series expansion, in a neighbourhood of the origin, is 1, the operator above is compact. Thus, $(I - P) \oplus Q_1(I - \widehat{f}(K_1))$ is a square root of unit and a compact perturbation of Q. □

Theorem 4.2. *The map $p\colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C})$ induces a locally trivial fiber bundle.*

Proof. Let $x_0 \in \mathcal{P}(\mathcal{C})$ and D be its connected component. The tuple $(p^{-1}(D), D, p)$ is a locally trivial bundle with fiber homeomorphic to $p^{-1}(\{x_0\})$. By Theorem 2.3 and Appendix D, there exists a map on a neighbourhood U_{x_0} of x_0

$$T\colon U_{x_0} \to GL(E), \quad (\pi_{x_0} \circ p) \circ T(x) = x.$$

Thus, we can define a coordinate neighbourhood on U_{x_0}, with its inverse, as follows

$$\phi\colon U_{x_0} \times p^{-1}(x_0) \to p^{-1}(U_{x_0}), \quad (x,y) \mapsto T(x)yT(x)^{-1}.$$

There holds $p \circ \phi(x,y) = x$ and is invertible. Given a point $z \in D$, let $g \in G(\mathcal{C})$ and $G \in GL(E)$ such that

$$p(G) = g, \quad gx_0g^{-1} = z.$$

Such g is provided by ii) of Proposition 2.2. The existence of G follows from the surjectivity of $p\colon GL(E) \to G(\mathcal{C})$ (refer Appendix D). We define a trivialization of the neighbourhood $U_z = g^{-1}U_{x_0}g$ as

$$\phi\colon U_z \times p^{-1}(x_0) \to p^{-1}(U_z), \quad (x,y) \mapsto GT(g^{-1}xg)yT(g^{-1}xg)^{-1}G^{-1}.$$

The left composition with p is the projection onto the first factor of the product $U_z \times p^{-1}(x_0)$. In fact,

$$\begin{aligned} p \circ \phi(x,y) &= p(G)p\big(T(g^{-1}xg)yT(g^{-1}xg)^{-1}\big)p(G)^{-1} \\ &= gp\big(T(g^{-1}xg)\big)x_0p\big(T(g^{-1}xg)^{-1}\big)g^{-1} \\ &= gg^{-1}xgg^{-1} = x. \end{aligned}$$

□

5. The essential Grassmannian

In $\mathcal{P}(E)$ and $G_s(E)$ we consider the relation of compact perturbation. We write $X \sim_c Y$ if and only if X is compact perturbation of Y in the sense of Definition 5.9 and $P \sim_c Q$ if and only if they have compact difference. Given $X \in G_s(E)$ and $P \in \mathcal{P}(E)$ we define

$$\mathcal{P}_c(P;E) = \{Q \in \mathcal{P}(E) \mid P \sim_c Q\}$$
$$G_c(X;E) = \{Y \in G_s(E) \mid X \sim_c Y\}$$

endowed with the topology of subspace. We denote by $\mathcal{P}_e(E)$ and $G_e(E)$ the quotient spaces, endowed with the quotient topology. In literature the latter is called *essential Grassmannian*, check, for instance, [**1**], §6. Let Π_e and π_e denote be the projections onto the quotient spaces of $\mathcal{P}(E)$ onto $\mathcal{P}_e(E)$ and $G_s(E)$ onto $G_e(E)$, respectively. By Theorem 4.2, the map

$$p\colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C})$$

has local sections, hence is open. Moreover, two projectors belong to the same class of compact perturbation if and only if their difference is compact, hence the map induced to the quotient

$$p_e\colon \mathcal{P}_e(E) \to \mathcal{P}(\mathcal{C})$$

is a homeomorphism. If $\Pi_e(P) = \Pi_e(Q)$ the operator $P - Q$ is compact. Thus, $\operatorname{Range} P \sim_c \operatorname{Range} Q$ and we have a well defined map

$$r_e \colon \mathcal{P}_e(E) \to G_e(E), \quad \Pi_e(P) \mapsto \pi_e(\operatorname{Range} P).$$

It is quotient map, because obtained as composition of quotient maps.

Proposition 5.1. *There is a homeomorphism between $G_e(E)$ and $Gr(\mathcal{C})$ such that the diagram*

$$\begin{tikzcd}
\mathcal{P}_e(E) \arrow[r, "p_e"] \arrow[d, "r_e"] & \mathcal{P}(\mathcal{C}) \arrow[d, "\pi"] \\
G_e(E) \arrow[r] & Gr(\mathcal{C})
\end{tikzcd}$$

commutes.

Proof. Let P and Q be projectors such that $\pi_e(\Pi_e(P)) = \pi_e(\Pi_e(P))$. Hence $\Pi_e(P) \sim_c \Pi_e(Q)$, that is, $PQ - Q$ and $QP - P$ are compact operators, thus

$$p(P)p(Q) = p(Q), \quad p(Q)p(P) = p(P),$$

hence, $\pi(p(P)) = \pi(p(Q))$. By following each of the steps above in the opposite order, it is easy to check that, if $\pi(p(P)) = \pi(p(Q))$, then $\pi_e(\Pi_e(P)) = \pi_e(\Pi_e(P))$. Thus, given $X \in G_s(E)$ and P such that $\operatorname{Range} P = X$, we have a well defined and injective map

$$g_e \colon G_e(E) \to Gr(\mathcal{C}), \quad g_e(\pi_e(X)) = \pi(p(P)).$$

Since π and p_e are surjective, g_e is also surjective. By definition, $\pi \circ p_e = g_e \circ r_e$. We prove that g_e is continuous. Given $U \subset Gr(\mathcal{C})$, then

$$g_e^{-1}(U) \text{ is open iff. } r_e^{-1}(g_e^{-1}(U))$$

is open, because the quotient topology is the finest making r_e. The latter is $(\pi \circ p_e)^{-1}(U)$, which is open. Since π is also a quotient map, the continuity of the inverse follows. □

Since, by [**43**], §3, π is a homotopy equivalence, r_e is also a homotopy equivalence. A homotopy inverse of r_e is $p_e^{-1} s g_e$ where s is a right inverse of π. We conclude this section by showing that the spaces G_c and $\mathcal{P}_c$ have the same homotopy type.

Proposition 5.2. *Let $X \in G_s(E)$ be a closed complemented subspace and P a projector with range X. The restriction of r to $\mathcal{P}_c(P; E)$ takes values in $G_c(X; E)$ and is a homotopy equivalence.*

Proof. Let r_c be the restriction of r. To achieve this result we follow the same steps of Lemma 3.2. Fix X_0 compact perturbation of X. By Theorem 5.15 there exists a projector P_0 with range X_0 such that $P_0 - P$ is compact. Call Y_0 its kernel and define the local section

$$s_0 \colon B(X_0, \hat{\gamma}(X_0, Y_0)) \to P(E), \; X' \mapsto P(X', Y_0).$$

This is continuous by Proposition 4.5. Since $r(s_0(X')) = X'$, by Proposition 5.10 the operators $(I - s_0(X'))P_0$ and $(I - P_0)s_0(X')$ are compact. The relation $\ker s_0(X') = \ker P_0$ implies $s(X')(I - P_0) = 0$, therefore

$$P_0 - s(X') = (I - s(X'))P_0 + (P_0 - s(X'))(I - P_0) = (I - s(X'))P_0$$

which is compact. Then $s(X') - P$ is compact. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be a locally finite refinement of $\{B(X_0, \hat{\gamma}(X_0, Y_0)) \mid X_0 \in G_c(X; E)\}$ and $\{\varphi_i \mid i \in I\}$ a partition of unit subordinate to $\mathcal{U}$. Then, for any $Y \in G_c(X; E)$

$$s(Y) - P = \sum_{i \in I} \varphi_i(Y)(s_i(Y) - P)$$

is a finite sum of compact operators. The convex combination of $s \circ r_c$ and id is a homotopy map. □

6. The Fredholm group

We call *Fredholm group* the set of invertible operators on a Banach space that can be written as sum of the identity and a compact operator. It is a normal subgroup of $GL(E)$. The Fredholm group is endowed with the norm topology; we denote it by $GL_c(E)$.

Theorem 6.1. *If E is an infinite dimensional Banach space over a field $\mathbb{F}$, that is $\mathbb{R}$ or $\mathbb{C}$, the Fredholm group has the homotopy type of* $\mathrm{Lim}\, GL(n,\mathbb{F})$.

For the proof see, for instance, [**25**]. The homotopy groups of the Fredholm group are, in the real and complex case, respectively

$$\pi_i(GL(\infty,\mathbb{R})) \cong \begin{cases} \mathbb{Z}_2 & i \equiv 0,1 \mod 8 \\ 0 & i \equiv 2,4,5,6 \mod 8 \\ \mathbb{Z} & i \equiv 3,7 \mod 8 \end{cases} \tag{35}$$

$$\pi_i(GL(\infty,\mathbb{C})) \cong \begin{cases} 0 & i \equiv 0 \mod 2 \\ \mathbb{Z} & i \equiv 1 \mod 2 \end{cases} \tag{36}$$

see Theorem II of [**10**]. The spectrum of $T \in GL_c(E)$ is countable, and $\sigma(T) \setminus \{1\}$ is made of eigenvalues of finite multiplicity. When E is a real Banach space it is defined the *Leray-Schauder* degree as

$$\deg(T) = (-1)^{\beta(T)}$$

where $\beta(T)$ is the sum of the algebraic multiplicities of the eigenvalues of T such that $\mathrm{Re}\, z > 1$ eigenvalues. It is well defined on the connected components of $GL_c(E)$ and defines a group isomorphism

$$\deg\colon \pi_0(GL_c(E)) \to \{-1,+1\} \cong \mathbb{Z}_2.$$

See [**35**] for details. The L.S. degree will help us to determine the connected components of $G_c(X;E)$ when E is a real or complex Banach space. We will prove that $G_c(X;E)$ consists of infinitely numerable components; these are

$$G_k(X;E) = \{\, Y \in G_c(X;E) \mid \dim(X,Y) = k \,\}, \; k \in \mathbb{Z}. \tag{37}$$

Lemma 6.2. *The Fredholm group acts transitively on each $G_k(X;E)$ by the left multiplication. Moreover, there are local sections of the action.*

The carrying out of the proof follows the same steps of the Hilbert case outlined in [**1**], §5.

Proof. Let $Y \in G_k$ and $T \in GL_c(E)$. Let t be the restriction of T to Y and $i\colon Y \hookrightarrow E$ the inclusion. Both $t, i \in \mathcal{L}(Y,E)$ are injective and $t - i$ is compact. Hence, by Proposition 5.16 $\mathrm{Range}\, t$ and $\mathrm{Range}\, i$ are compact perturbation of each other and

$$\dim(Y,TY) = \dim(\mathrm{Range}\, i, \mathrm{Range}\, t) = \dim(\ker t, \ker i) = 0.$$

Hence $TY \in G_k$. Let $Y, Z \in G_k(X;E)$, hence $\dim(Y,Z) = 0$. Let s be a continuous right inverse of r_c as in Proposition 5.2. The operator $s(Z) - s(Y)$ is compact, call it K. Observe that the restriction of $s(Z)$ to Y, considered as operator with values in Z, is Fredholm. Similarly we can consider the restriction of $I - s(Z)$ to $Y' := \ker s(Y)$ with values in $\ker s(Z)$. For every y in Y and $y' \in Y'$ we can write

$$s(Z)y = s(Y)y + Ky = (I+K)y$$
$$(I - s(Z))y' = (I - s(Y))y' - Ky' = (I-K)y'.$$

The Fredholm index of these operators is 0 by definition of relative dimension. Fredholm applications of index 0 have a nice property: they are perturbation of an isomorphism by a finite rank operator. Then we can choose R_1 in $\mathcal{L}(Y,Z)$ and R_2 in $\mathcal{L}(Y', \ker s(Z))$ suitable finite rank operators. Call T the operator obtained as direct sum of the two isomorphisms $s(Z)_{|Y} + R_1$ and $(I - s(Z))_{|Y'} + R_2$. It is invertible, maps Y onto Z and can be written as

$$I + (K + R_1)s(Y) - (K - R_2)(I - s(Y))$$

hence belongs to the Fredholm group. This proves that the action is transitive.

Given $Y \in G_k$ we build a local section around Y as follows: let s be a continuous section as in Proposition 5.2. There exists $\varepsilon > 0$ such that, for any $Z \in B(Y, \varepsilon)$ the operator $g(s(Z), s(Y))$ is invertible. By (30) and (31),

$$g(s(Z), s(Y)) \in I + \mathcal{L}_c(E),$$

thus, $g(s(Z), s(Y)) \in GL_c(E)$ and $g(s(Z), s(Y))Y = Z$. Then a local section of the action is defined as

$$B(Y, \varepsilon) \to GL_c(E) \times G_k, \; Z \mapsto (g(s(Z), s(Y)), Y). \tag{38}$$

□

Theorem 6.3. *The connected components of $G_c(X; E)$ are $G_k(X; E)$ with k in $\mathbb{Z}$.*

Proof. Let $Y, Z \in G_c(X; E)$, connected by an arc, $k = \dim(X, Y)$. By Proposition 5.2, there exists a path α in $\mathcal{P}_c(P; E)$ that connects $s(Y)$ to $s(Z)$. Let g be as in ii) of Proposition 2.2, thus $g \in GL_c(E)$. By Lemma 6.2, $g(Y) \in G_k$. Conversely, consider $Y, Z \in G_k$. Hence $\dim(Y, Z) = 0$ and, by Lemma 6.2, there exists $T \in GL_c(E)$ such that $TY = Z$. If E is a complex Banach space, the Fredholm group is arcwise connected. Given a path α that connects I to T the path $\alpha(t)Y$ connects Y to Z. If E is a real Banach space, let $S \in GL(Y) \times GL(Y')$, where $Y \oplus Y' = E$, and $\deg(S) = -\deg(T)$. Then TS maps Y onto Z and is connected to the identity operator and we conclude as in the complex case. □

7. The Stiefel space

In this section we introduce the *Stiefel spaces* and for some $X \in G_s(E)$, we compute its homotopy type. That will help us to determine the homotopy groups of $G_c(X; E)$.

Definition 7.1. Let $X \in G_s(E)$. We define the *Stiefel space*, and denote it by $St(X; E)$, the set of the bounded operators $f \in \mathcal{L}(X, E)$ such that

(i) $f(X)$ is complemented in E;
(ii) f is injective;
(iii) $f - i$ is compact,

where $i\colon X \hookrightarrow E$ is the inclusion. On it we consider the topology of subspace.

The Stiefel space is an analytical manifold because is an open subset of the affine space $I + \mathcal{L}_c(X, E)$. We recall some results on the homotopy type of $St(X; E)$.

Theorem 7.2. (refer [**19**]) *If X is a finite-dimensional subspace of E $St(X; E)$ is contractible.*

Using the techniques of [**25**] it is possible to prove that when X has infinite dimension and infinite co-dimension $St(X; E)$ is contractible. Then, if X has infinite co-dimension $St(X; E)$ is always contractible. The next result is known for Hilbert spaces, see for example [**1**] §5. The generalization to Banach spaces requires, as Lemma 6.2 does, the Proposition 5.2.

Theorem 7.3. *Let $r_{St}\colon St(X; E) \to G_0(X; E)$ be the continuous map defined as $r_{St} f = f(X)$. Then $(St(X; E), r_{St}, G_0(X; E), GL_c(X))$ is a principal fiber bundle. The action of $GL_c(X)$ onto itself is the left multiplication.*

Proof. As first step we build a local section around X. Consider a continuous map as in Proposition 5.2. Let U be an open neighbourhood of X where (38) is defined. Define

$$\gamma_0\colon U \to St(X; E), \; Y \mapsto g(s(Y), s(X))_{|X}.$$

This suffices to build an open cover of coordinate neighbourhoods of G_0. Given $Y \in G_0$, by Lemma 6.2 there exists $T \in GL_c(E)$ such that $TX = Y$. Then a trivialization of $T(U)$ and its inverse are given by

$$\begin{gathered}\phi\colon T(U)\times GL_c(X)\to r_{St}^{-1}(T(U)),\\ (Y',g)\mapsto T\gamma_0(T^{-1}Y')g;\\ \phi^{-1}\colon r_{St}^{-1}(T(U))\to T(U)\times GL_c(X),\\ f\mapsto \big(Tf(X),g(s(X),s(f(X)))\circ T^{-1}f\big).\end{gathered}$$

We have to check that whenever two coordinate neighbourhoods U_i, U_j intersect, for every $Z\in U_i\cap U_j$ the transitions maps are left translations of $GL_c(X)$ onto itself. In fact, given T_i, T_j such that $T_iX=T_jX=Z$ the transition map is

$$\phi_{j,Z}^{-1}\phi_{i,Z}(g)=g(s(X),s(T_jZ))T_j^{-1}T_i\gamma_0(T_i^{-1}Z)\cdot g$$

is the left multiplication by an element of $GL_c(X)$. Then we have $GL_c(X)$ compatibility. □

When $X\subset E$ has infinite co-dimension and infinite dimension the exact sequence of the principal bundle $(St(X;E),r_{St},G_0(X;E),GL_c(X))$ gives isomorphisms

(39) $$\pi_i(G_0(X;E),X)\cong\pi_{i-1}(GL_c(X))\cong\pi_{i-1}(GL(\mathbb{F},\infty)),\ i\geq 1$$

where $\mathbb{F}$ is the real or complex field.

8. The index of the exact sequence

Using exact sequence of the fiber bundle $(\mathcal{P}(E),\mathcal{P}(\mathcal{C}),p)$ we show how to associate an integer to a closed loop in the space of idempotents of $\mathcal{C}(E)$. In fact we define a group homomorphism on $\pi_1(\mathcal{P}(\mathcal{C}))$ denoted by φ. Since $\mathcal{P}(\mathcal{C})$ is homotopically equivalent to the space of essentially hyperbolic operators on E, we definitely have a group homomorphism on $\pi_1(e\mathcal{H}(E))$ obtained as the composition of φ with Ψ, defined in §2.1.

Let P be any projector. By Theorem 4.2 the triple $(\mathcal{P}(E),p,\mathcal{P}(\mathcal{C}))$ is a locally trivial bundle. The typical fiber of $p(P)$ is $\mathcal{P}_c(P;E)$. Then we have an exact sequence

$$\pi_1(\mathcal{P}_c(P;E),P)\xrightarrow{i_*}\pi_1(\mathcal{P}(E),P)\xrightarrow{p_*}\pi_1(\mathcal{P}(\mathcal{C}),p(P))$$

Theorem 8.1. *There exists a group homomorphism $\varphi_P\colon\pi_1(\mathcal{P}(\mathcal{C}),p(P))\to\mathbb{Z}$ such that the sequence of homomorphisms*

$$\pi_1(\mathcal{P}(E),P)\xrightarrow{p_*}\pi_1(\mathcal{P}(\mathcal{C}),p(P))\xrightarrow{\varphi_P}\mathbb{Z}$$

is exact.

Proof. The homomorphism is defined as follows: given a loop $a\in\mathcal{P}(\mathcal{C})$, there exists a path $\beta\in\mathcal{P}(E)$ such that $p\circ\beta=a$. Thus

$$p(\beta(0))=p(\beta(1)),\ \beta(0)-\beta(1)\text{ is cpt.}$$

We define $\varphi(a)=\dim(\beta(1),\beta(0))$. First, we observe that the definition does not depend on the choice of the lifting path. In fact, given β' as above, $\dim(\beta(t),\beta'(t))$ is constant. This follows from the Theorem 3.3. Hence,

(40) $$\dim(\beta'(1),\beta'(0))=\dim(\beta(1),\beta(0))$$

We prove that φ_P is a group homomorphism. Let a,b be two closed paths at the base point $p(P)$. There are two lifting paths α,β such that

$$\begin{gathered}\alpha(0)=P,\ p\circ\alpha=a,\\ \beta(0)=P,\ p\circ\beta=b.\end{gathered}$$

There also exists β' such that $\beta'(0)=\alpha(1)$ and $p\circ\beta'=b$. Define

$$\gamma=\alpha*\beta',\ \gamma(0)=\alpha(1)$$

which is a lifting path for $a * b$. Since β and β' are lifts of the same path b, equality (40) holds. We have

$$\begin{aligned}\varphi_P(a * b) &= \dim(\beta'(1), \alpha(0)) = \dim(\beta'(1), \beta'(0)) + \dim(\alpha(1), \alpha(0)) \\ &= \dim(\beta(1), \beta(0)) + \dim(\alpha(1), \alpha(0)) = \varphi_P(a) + \varphi_P(b).\end{aligned}$$

Finally, we prove that the sequence above is exact.
$\ker \varphi_P \subseteq \mathrm{Im}(p_*)$. Suppose $\varphi_P(a) = 0$. Hence, $\dim(\beta(1), P) = 0$. By Theorem 6.3, there exists a path $\gamma \in \mathcal{P}_c(E; P)$ such that

$$\gamma(0) = \beta(1),\ \gamma(1) = P,\ p_*(\beta * \gamma) = a.$$

$\mathrm{Im}(p_*) \subseteq \ker \varphi_P$. Given a loop in $\alpha \in \mathcal{P}_c(E; P)$, we have $\varphi_P(p_*(a)) = \dim(\alpha(1), P) = 0$. □

Corollary 8.2. Given $P \in \mathcal{P}(E)$, we have the following properties of the kernel and image of φ_P:

- h1) $m \in \mathrm{Im}(\varphi_P)$ if and only if there exists a projector Q in the same connected component of P such that $Q - P$ is compact and $\dim(Q, P) = m$;
- h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected.

Property h2) follows straightforwardly from the exactness of the sequence. Let $m \in \mathrm{Im}(\varphi_P)$. Hence, there exists a path β of projectors such that $\dim(\beta(1), P) = m$. Thus, we choose $Q = \beta(1)$. Conversely, let Q be a projector in the same connected component of P such that $Q - P$ is compact. If β joins P to Q, $p \circ \beta$ is a loop and $\varphi_P(p \circ \beta) = m$.

When the index is trivial. When P is a projector whose image has finite dimension or finite co-dimension its component in $\mathcal{P}(\mathcal{C})$ consists of a single point, hence φ_P is the null homomorphism. There are infinite-dimensional spaces, called *undecomposable*, where the only complemented subspaces have finite dimension or finite-co-dimension; an example of such space was described by W. T. Gowers and B. Maurey in [**26**]. In that case $\mathcal{P}(\mathcal{C})$ consists of two points. We observe that in h2), $\mathrm{Range}\, Q \cong \mathrm{Range}\, P$, by ii) of Proposition 2.2. In [**26**] W. T. Gowers and B. Maurey showed that there are infinite-dimensional Banach spaces which are not isomorphic to any of their proper subspaces. In this, case φ_P is the null homomorphism.

The next lemma is needed in order to exhibit a wide class of examples where h1) conditions holds. By sake of completeness we exhibit a proof of it. Such proof follows also from [**43**].

Lemma 8.3. *Let E be a Banach space, and $X, Y \subset E$ closed subspaces such that $X \cong Y$ and $X \oplus Y = E$. Two projectors P_X, P_Y with ranges X and Y, are connected by a continuous path on $\mathcal{P}(E)$.*

Proof. It is enough to prove it when P_X is the projector onto X along Y and P_Y is $I - P_X$, because the set of projectors having a fixed range is a convex subset of the space of projectors. Check, for instance Lemma 3.2.

Let σ be an isomorphism of Y with X. We define the path

$$G_{\sigma,\theta}(x + y) = (\cos\theta\, x + \sin\theta\, \sigma y) + (-\sin\theta\, \sigma^{-1} x + \cos\theta\, y)$$

of invertible operators of E. Direct computations show that $G_\sigma(-\theta)$ is its inverse. Moreover $G_\sigma(0)$ is the identity and $G_\sigma(\pi/2)$ conjugates the projector P to P_Y. Then the path

$$P_\theta = G_{\sigma,\theta} P G_{\sigma,-\theta}, \quad 0 \le \theta \le \pi/2$$

has the required properties. □

Here is a concrete example where the first condition hold.

Proposition 8.4. *Let $E = X \oplus Y$ be a Banach space and X a closed complemented subspace isomorphic to its closed subspaces of co-dimension m. Let P be the projector onto X along Y. Then P satisfies the condition* h1) *with the integer m.*

Proof. Since X is isomorphic to its hyperplanes, we can choose subspaces $X^m, R_m \subset X$, where R has dimension m, X^m is closed and $X^m \cong X$. We have the decomposition and isomorphism

$$E = R_m \oplus X^m \oplus Y, \; X^m \cong Y.$$

By applying Lemma 8.3 with $E = X^m \oplus Y$, we obtain that P_{X^m} is connected to P_Y. By applying it a second time to E and subspaces $X = R_m \oplus X^m$ and Y, we obtain that P_X is connected to P_Y. Hence, P_X is connected to P_{X^m}. □

The argument used to connect the two projectors P and P_X is a modification of the one used for Hilbert spaces by J. Phillips in Proposition 6 of [**42**] when $m = 1$: given the decomposition

$$E = X^1 \oplus R_1 \oplus Y$$

a shift operator s maps X^1 and $R_1 \oplus Y$ isomorphically onto $X^1 \oplus R_1$ and Y respectively. Since the general linear group of a Hilbert space is contractible the projectors are connected. The isomorphism G_σ used in the proof is connected to the identity regardless of whether $GL(E)$ is connected or not.

Example 8.5. Given a Banach space X which is isomorphic to its subspaces of co-dimension two, but not to the subspaces of co-dimension one, let P be the projector onto the first factor of $E = X \oplus X$. Then, by Proposition 8.4, $2 \in \mathrm{Im}(\varphi_P)$. However, $1 \notin \mathrm{Im}(\varphi_P)$, because condition h2) with $m = 1$ implies the existence of an isomorphism of X with a hyperplane. Thus,

$$\mathrm{Im}(\varphi_P) = 2\mathbb{Z} \subset \mathbb{Z}.$$

An example of such space X was showed by W. T. Gowers and B. Maurey in [**27**]. Thus, there are projectors such that the homomorphism is not surjective, but not trivial.

When the index is an isomorphism. When E is one of the following Banach spaces, an infinite-dimensional Hilbert space, spaces $L^p(\Omega, \mu)$ for $p \geq 1$ and $L^\infty(\Omega, \mu)$, or spaces of sequences ℓ^p, m, c_0, the following three conditions hold:

(i) $E \cong E \times E$;
(ii) E is isomorphic to its hyperplanes;
(iii) $GL(E)$ is contractible to a point.

From (i), we can write $E = X \oplus Y$ where $X \cong Y$. By (ii), X is isomorphic to a hyperplane, thus $1 \in \mathrm{Im}(\varphi_{P(X,Y)})$ by Proposition 8.4. By (33) with $k = 1$, we obtain the condition h1), because F_P and G_P are contractible, by (iii). Hence, $\varphi_{P(X,Y)}$ is a group isomorphism.

The Douady space. We exhibit an example of Banach space E with a projector P of infinite dimensional range and kernel and a loop a in $\mathcal{P}(\mathcal{C})$ with base point $p(P)$ such that $\varphi(a) = 0$ but not homotopically equivalent to the constant path.

Proposition 8.6. *Let $X \subset E$ be a complemented subspace isomorphic to its complement and P a projector such that $P(E) = X$. If $GL(X)$ is not connected, the component of P in $\mathcal{P}(E)$ is not simply connected.*

Proof. Choose a topological complement Y and let $T \in GL(X)$ be such that there exists no path joining T to the identity. Let σ be an isomorphism of Y with X. Hence the invertible operator

$$T_1 = \begin{pmatrix} T & 0 \\ 0 & \sigma T^{-1} \sigma^{-1} \end{pmatrix}$$

lies in the connected component of $GL(E)$ of the identity. A path can be defined as $G_{\sigma,\theta} T_1 G_{\sigma,-\theta}$ where $G_{\sigma,\theta}$ is the operator defined Lemma 8.3. Call S such path and define $\alpha = SPS^{-1}$. Since T_1 commutes with P the path α is a loop with base point P. The group homomorphism

$$\Delta \colon \pi_1(\mathcal{P}(E), P) \to \pi_0(GL(X)) \times \pi_0(GL(Y))$$

induced by the fiber bundle $(GL(E), \pi_P, \mathcal{P}(E))$ maps α to T_1. Thus $\Delta\alpha \neq 0$, hence $\alpha \neq 0$. □

In order to find non-contractible loops with vanishing index we need some projector P such that the inclusion

$$j_*\colon \pi_1(\mathcal{P}_c(P;E)) \to \pi_1(\mathcal{P}(E),P)$$

is not surjective. We will prove that for some spaces the second group contains infinitely many distinct elements, while the first is a finite group, according to (39). Let F and G be such that

(i) every bounded map $G \to F$ is compact,
(ii) both F and G are isomorphic to their hyperplanes;

let $X = F \oplus G$. Let $T \in GL(X)$. We can write it block-wise using the projectors on F and G

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Since C is compact, it is possible to prove that A and B are Fredholm operators such that $\operatorname{ind}(A) + \operatorname{ind}(D) = 0$ (refer [**38**]). We define $f(T) = \operatorname{ind}(A)$. We have the following

Lemma 8.7. (refer [**20**]). *The map $f\colon GL(X) \to \mathbb{Z}$ is continuous and surjective.*

We define $E = X \oplus X$. By the lemma, we have a surjective homomorphism, obtained by composition

$$(f \times 0) \circ \Delta\colon \pi_1(\mathcal{P}(E),P) \to \mathbb{Z}.$$

Hence, given a loop $\alpha \notin j_*(\pi_1(\mathcal{P}_c(P;E)))$, the element $a = p_*(\alpha)$ is non trivial and, since the sequence in Theorem (8.1) is exact, $\varphi_P(a) = 0$.

A pair of spaces with the properties i) and ii) is given by (ℓ^p, ℓ^q) with $p > q > 1$; refer Theorem 4.23 of [**47**] for property i). Isomorphisms with hyperplanes can be defined using the operators $(sx)_1 = 0$, $(sx)_i = x_{i-1}, i \geq 2$ for every $x \in \ell^p$. Thus, in the space

$$E = (\ell^p \oplus \ell^2) \oplus (\ell^p \oplus \ell^2),$$

if we call P the projector onto the first factor, then φ_P is not injective. Finally, we observe that the image of P is isomorphic to a hyperplane. Thus, by Proposition 8.4, $\operatorname{Im}(\varphi_P) = \mathbb{Z}$.

CHAPTER 3

Linear equations in Banach spaces

We state and prove some general results about differential equations on a Banach algebra with unit, usually denoted by 1. We are mainly concerned of the Cauchy problem

$$u'(t) = A(t)u(t), \; u(0) = 1 \tag{41}$$

where A is a continuous path in a Banach algebra $\mathcal{B}$. Local existence and uniqueness hold. In fact these solutions admit a prolongation to the whole real line $\mathbb{R}$. Denote by X_A the solution of (41). Using local uniqueness we prove some properties of the solution X_A. When $\mathcal{B}$ is the algebra of bounded operators on a Banach space E two linear subspaces, the *stable* and *unstable space*, are defined

$$W_A^s = \left\{ x \in E \mid \lim_{t \to +\infty} X_A(t)x = 0 \right\}$$
$$W_A^u = \left\{ x \in E \mid \lim_{t \to -\infty} X_A(t)x = 0 \right\}.$$

If A is a bounded and asymptotically hyperbolic these are closed linear subspaces, admit a topological complement, and have the asymptotic behaviour

$$\lim_{t \to +\infty} X_A(t)W_A^s = E^-(A_0(+\infty)),$$
$$\lim_{t \to +\infty} X_A(t)Y = E^+(A_0(+\infty))$$

where $W_A^s \oplus Y = E$. The limits are taken in the topology of $G(E)$. In the last section we look at the effects of perturbation of an asymptotically hyperbolic path on its stable space. Precisely the stable space varies continuously in the topology of $G_s(E)$. If $A - B$ is a path of compact operators then W_A^s and W_B^s are compact perturbation one of each other.

1. The Cauchy problem

Let $\mathcal{B}$ be a Banach algebra and A a continuous path defined on the real line. Given $u, v \in \mathcal{B}$ we can always consider two *Cauchy problems*

$$X_{A,u}{}'(t) = A(t)X_{A,u}(t), \; X_{A,u}(0) = u \tag{42}$$

$$X^{A,v\prime}(t) = X^{A,v}(t)A(t), \; X^{A,v}(0) = v. \tag{43}$$

By Theorem A.1 unique local solutions always exist and the maximal solutions can extended, by Proposition A.6, to $\mathbb{R}$.

Proposition 1.1. *Let $u, v \in \mathcal{B}$. We have*

$$X^{-A,v}(t) \cdot X_{A,u}(t) = vu,$$
$$X_{A,u}(t) \cdot X^{-A,v}(t) = X_{A,1}(t) \cdot uv \cdot X^{-A,1}(t)$$

for every $t \in \mathbb{R}$. Moreover $X_{A,1}$ is invertible and its inverse is $X^{-A,1}$.

Proof. To prove the first equality consider the C^1 path $X^{-A,v} \cdot X_{A,u}$. By hypothesis the path is vu at $t = 0$ and its derivative is

$$X^{-A,v\prime}X_{A,u} + X^{-A,v}X'_{A,u} = -X^{-A,v}AX_{A,u} + X^{-A,v}AX_{A,u} = 0;$$

then $X^{-A,v}X_{A,u}(t) = vu$ for every t. To prove the second we argue similarly. The path $X_{A,v}(t) \cdot X^{-A,u}(t)$ has derivative

$$X_{A,v}{}'X^{-A,u} + X_{A,v}X^{-A,u}{}' = [A, X_{A,v} \cdot X^{-A,u}]$$

and is therefore solution of the Cauchy problem $X' = [A, X]$ with starting point at uv. By direct computation $X_{A,1} \cdot uv \cdot X^{-A,1}$ solves the same equation. By uniqueness the second equality holds. The first equality applied to $u = v = 1$ gives $X^{-A,1} \cdot X_{A,1} = 1$. Since $X_{A,1} \cdot X^{-A,1}$ and the constant path 1 solve the same equation, $X_{A,1}$ is invertible. □

Definition 1.2. An element $u \in \mathcal{B}$ is a *left inverse* if there exists v, called *right inverse* for u, such that $uv = 1$. We denote the subsets of left and right inverses by $\mathcal{B}_l$ and $\mathcal{B}_r$ respectively.

Proposition 1.3. *If $u \in \mathcal{B}_r$ (resp. $\mathcal{B}_l$) then $X_{A,u} \subset \mathcal{B}_r$ (resp. $\mathcal{B}_l$). If u is invertible then $X_{A,u}(t)^{-1} = X^{-A,u^{-1}}(t)$.*

Proof. Let $u \in \mathcal{B}_r$ and v be such that $vu = 1$. By the first equality of Proposition 1.1 $X_{A,u} \subset \mathcal{B}_r$. If $u \in G(\mathcal{B})$ let v be its inverse. The first and the second of 1.1 give $X^{-A,v} \cdot X_{A,u} = X_{A,u} \cdot X^{-A,v} = 1$. □

We will abbreviate the notation for the rest of this section: for curves with starting point 1 we write X_A instead of $X_{A,1}$.

Proposition 1.4. *$\mathcal{B}_r$ and $\mathcal{B}_l$ are open subsets of $\mathcal{B}$.*

Proof. We will prove that $\mathcal{B}_r$ is open. Let $u \in \mathcal{B}_r$ and v be such that $v \cdot u = 1$. Let $r_0 = 1/\|v\|$ and $h \in \mathcal{B}$. Then

$$v(u + h) = vu + vh = 1 + vh.$$

If $h \in B(u, r_0)$, by the Von Neumann series, $1 + vh$ is invertible. Then $(1 + vh)^{-1}v$ is a left inverse of $u + h$. Actually, in a neighbourhood of u, we have defined a smooth function

$$B(u, r_0) \to \mathcal{B}_l,\ u' \mapsto L_{u,r_0}(u') = [1 + v(u' - u)]^{-1}v \in \mathcal{B}_l. \tag{44}$$

such that $L_{u,r_0}(u') \cdot u = 1$. The same conclusions hold for $\mathcal{B}_l$. □

Proposition 1.5. *Let $X \in C^1(\mathbb{R}, \mathcal{B}_r)$. There exists $A \in C(\mathbb{R}, \mathcal{B})$ such that $X_{A,X(0)} = X$.*

Proof. As first step we prove that there exists a path Y with values in $\mathcal{B}_l$ such that $YX \equiv 1$. Let $t_0 \in \mathbb{R}$. Since $X(t_0) \in \mathcal{B}_r$ there exists $Y(t_0)$ such that $Y(t_0)X(t_0) = 1$ and the (44) provides us with a differentiable map defined in a neighbourhood $B(t_0, \varepsilon(t_0))$, namely $L_{X(t_0),\varepsilon(t_0)}$. By paracompactness of $\mathbb{R}$ we can extract a locally finite sub-covering of $\{B(t, \varepsilon(t)) \mid t \in \mathbb{R}\}$, say $\mathcal{U} = \{U_i \mid i \in I\}$. Let $\sigma\colon I \to \mathbb{R}$ be a choice function and $\{\varphi_i \mid \operatorname{supp}\varphi_i \subseteq U_i\}$ a partition of unity subordinate to $\mathcal{U}$. Then set

$$Y = \sum_i \varphi_i Y_{\sigma(i)}.$$

Actually Y is infinitely differentiable. Its image lies in $\mathcal{B}_l$, in fact

$$Y(t)X(t) = \sum_i \varphi_i Y_{t_i}(t)X(t) = \sum_i \varphi_i(t)1 = 1.$$

Now, in the chain of equalities $X' = X' \cdot 1 = X' \cdot YX = (X'Y)X$ set $A = X'Y$ and obtain $X' = AX$. By uniqueness, $X = X_{A,X(0)}$ q.e.d. □

This proposition gives us a characterization of the solutions of $X' = AX$ when the starting point lies in $\mathcal{B}_r$ (resp. $\mathcal{B}_l$). They are just C^1 curves on $\mathcal{B}_r$ (resp. $\mathcal{B}_l$).

Proposition 1.6. *$G(\mathcal{B})$ is union of connected components of $\mathcal{B}_r$.*

Proof. Let $\mathcal{B}'_r$ be a connected component of $\mathcal{B}_r$ such that $G(\mathcal{B}) \cap \mathcal{B}'_r \neq \emptyset$. Let $x \in \mathcal{B}'_r$. Since $\mathcal{B}'_r$ is an open set we may choose a path $\Gamma \in C^1([0,1], \mathcal{B}'_r)$ such that $\Gamma(0) = g \in G(\mathcal{B})$ and $\Gamma(1) = x$. Then, by Proposition 1.5, $\Gamma = X_{A,\Gamma(0)}$ for some $A \in C([0,1], \mathcal{B})$. Since $\Gamma(0)$ is an invertible element of $\mathcal{B}$ Proposition 1.1 states that $\Gamma(t) \in G(\mathcal{B})$ for any $t \in [0,1]$. In particular $\Gamma(1) \in G(\mathcal{B})$ thus $\mathcal{B}'_r \subset G(\mathcal{B})$. □

The proofs of the following equalities are consequence of the uniqueness of the solutions of Cauchy problems. Given a path $A \in C(\mathbb{R}, \mathcal{B})$, $\tau \in \mathbb{R}$ we denote by A_τ the path $A(\cdot + \tau) = A(t + \tau)$.

Proposition 1.7. *Let A and B be two continuous paths. Then*

$$\begin{aligned} X_{A+B} &= X_A \cdot X_{X_A{}^{-1} B X_A} \\ X_{A(\cdot+s)}(t) X_A(s) &= X_A(t+s) \end{aligned}$$

for any $t, s \in \mathbb{R}$.

Proof. Let $X = X_A X_{X_A{}^{-1} B X_A}$. Differentiating

$$\begin{aligned} X' &= X'_A \cdot X_{X_A{}^{-1} B X_A} + X_A \cdot X'_{X_A{}^{-1} B X_A} \\ &= (A+B) X_A \cdot X_{X_A{}^{-1} B X_A} = (A+B) X \end{aligned}$$

hence $X = X_{A+B}$. To prove the second equality let $Y = X_{A(\cdot+s)}(t) X_A(s)$. Differentiating we find that $Y'(t) = A(t+s) Y(t)$, $Y(0) = X_A(s)$. Since the same holds for $Z(t) = X_A(t+s)$ the second equality is proved. □

Proposition 1.8. *Let $A, B \in C(\mathbb{R}, \mathcal{B})$*

$$X_B(t) = X_A(t) + \int_0^t X_A(t) X_A(\tau)^{-1} (B - A) X_B(\tau) d\tau \tag{45}$$

Proof. Call X and Y respectively the left and right members of (45). We have $X(0) = Y(0) = 1$ at $t = 0$. We prove that both solve the Cauchy problem $u' = Au + (B - A) X_B$ with starting point 1. In fact

$$\begin{aligned} X' &= B X_B = A X_B + (B - A) X_B = A X + (B - A) X_B \\ Y' &= A X_A + A \int_0^t X_A(t) X_A(\tau)^{-1} (B - A) X_B(\tau) + (B - A) X_B \\ &= A Y + (B - A) X_B. \end{aligned}$$

□

When $\mathcal{B}$ is the algebra of bounded operators on a Banach space E, given a path A in $\mathcal{L}(E)$ we can always consider the adjoint $A^* \in C(\mathbb{R}, \mathcal{L}(E^*))$. The relation

$$(X_A^{-1})^* = X_{-A^*} \tag{46}$$

holds. In fact the derivative of the left member is

$$(-(X_A)^{-1} A X_A X_A^{-1})^* = -A^* (X_A^{-1})^* = X'_{-A^*}.$$

2. Exponential estimate of X_A

In this section we denote by $C_b(\mathbb{R}, \mathcal{B})$ the space of bounded functions in $\mathcal{B}$. This space is endowed with the norm $\|A\|_\infty = \sup_{t \in \mathbb{R}} \|A(t)\|$ that makes it a Banach space.

Proposition 2.1. *If A is bounded $X_A(t)$ satisfies the exponential estimate*

$$\|X_A(t) X_A(s)^{-1}\| \leq c e^{\lambda |t-s|} \tag{47}$$

for some $c > 0$, $\lambda \in \mathbb{R}$ and any $t, s \in \mathbb{R}$.

Proof. Let $r = t - s$. By the Proposition 1.7 it is enough to prove that

$$\|X_{A(\cdot+s)}(r)\| \leq ce^{\lambda|r|}$$

for every $r \in \mathbb{R}$. To achieve this inequality we apply the Gronwall's lemma to the function $\alpha(r) = \|X_{A(\cdot+s)}(r)\|$. In fact since

$$\alpha(r) \leq 1 + \int_0^r \|A_{(\cdot+s)}(\tau)\|\alpha(\tau)d\tau$$

by the Gronwall's lemma (see Lemma A.5)

$$\alpha(r) \leq 1 + \int_0^r e^{\|A\|_\infty(r-\tau)}d\tau.$$

Easy computations show that $c = 2\max\{1, 1 - 1/\|A\|_\infty\}$ and $\lambda = \|A\|_\infty$ fit our request. Repeating the same argument for $t < 0$ we complete the proof. $\square$

Proposition 2.2. *Let $A, H \in C_b(\mathbb{R}, \mathcal{B})$. If $\|X_A(t)X_A(s)^{-1}\| \leq ce^{\lambda(t-s)}$ for any $t \geq s \geq 0$ we have $\|X_{A+H}(t)X_{A+H}(s)^{-1}\| \leq ce^{\mu(t-s)}$ where $\mu = \lambda + c\|H\|_\infty$.*

Proof. Applying the first equality of Proposition 1.7 to A and μ it easy to check that X_A satisfies the exponential estimate for any $t \geq s \geq 0$ with constants (c, λ) if and only if $A + \mu$ does the same with $(c, \lambda - \mu)$. In fact

$$X_{A+\mu I}(t) = X_A \cdot X_{X_A \cdot \mu I \cdot X_A^{-1}}(t) = X_A \cdot X_{\mu I}(t) = e^{\mu t}X_A(t).$$

Set $B = A + \mu I$. Hence we just have to prove that if $(c, \lambda - \mu)$ works with X_B then $(c, 0)$ works with X_{B+H}. Now fix $s \geq 0$. By the second of Proposition 1.7 $X_B(t)X_B(s)^{-1} = X_B(s + t - s)X_B(s)^{-1} = X_{B(\cdot+s)}(t - s)$ and the statement reduces to prove that

$$X_{B_s}(r) \leq ce^{(\lambda-\mu)r} \Rightarrow X_{B_s+H_s}(r) \leq c, \; r > 0, \tag{48}$$

where $B_s = B_{(\cdot+s)}$, $H = H_{(\cdot+s)}$. To prove (48) fix $t \in \mathbb{R}^+$ and consider the following map of $C_b([0, t], \mathcal{B})$ into itself

$$X \mapsto (fX)(r) = X_{B_s}(r)\left[1 + \int_0^r X_{B_s}(\tau)^{-1}H_s(\tau)Y(\tau)d\tau\right].$$

By (45) $X_{B_s+H_s}$ is a fixed point of f. We will prove that f is a contraction and that $\overline{B}(0, c)$ is invariant for f. Since every nonempty closed invariant subset for a contraction contains its fixed point this will conclude the proof. It is enough to prove that the linear application $L = f - X_{B_s}$ is bounded and $\|L\| < 1$. This will suffice to prove that L is a contraction, hence the affine map $L + X_B$ is also a contraction. Let X in $C([0, t], \mathcal{B})$

$$\|LX\|_\infty \leq \frac{c\|H\|_\infty}{\mu - \lambda}\left(1 - e^{-(\mu-\lambda)t}\right)\|X\|_\infty,$$

hence f is a contraction. To prove that $\overline{B}(0, c)$ is invariant for f let $X \in \overline{B}(0, c)$ thus

$$\begin{aligned}
\|(fX)(t)\| &= \left\|X_{B_s}(t)\left[1 + \int_0^t X_{B_s}(\tau)^{-1}H_s(\tau)Y(\tau)d\tau\right]\right\| \\
&\leq ce^{(\lambda-\mu)t} + c^2\|H\|_\infty \int_0^t \|X_B(t)X_B(\tau)^{-1}\|d\tau \\
&\leq ce^{(\lambda-\mu)t}\left(1 - \frac{c\|H\|_\infty}{\mu - \lambda}\right) + \frac{c^2\|H\|_\infty}{\mu - \lambda} = c.
\end{aligned}$$

Then $\|fX\|_\infty \leq c$ and the proof is complete. $\square$

3. Asymptotically hyperbolic paths

For the remainder of this chapter we restrict our attention to the algebra of bounded operators on a Banach space E. Given a continuous path A in the space of bounded operators, defined on $\mathbb{R}^+$ we define the *stable space* as

$$W^s_A = \left\{ x \in X \,\middle|\, \lim_{t\to+\infty} X_A(t)x = 0 \right\}.$$

Similarly, if A is a path defined on $\mathbb{R}^-$ we define the unstable space

$$W^u_A = \left\{ x \in X \,\middle|\, \lim_{t\to-\infty} X_A(t)x = 0 \right\}.$$

Using the equalities of Proposition 1.7, for every $t \geq 0$ and $s \leq 0$ we have

$$X_A(t)W^s_A = W^s_{A(\cdot+t)},\ X_A(s)W^u_A = W^u_{A(\cdot+s)}. \tag{49}$$

We denote by $\mathbb{H}^+$ and $\mathbb{H}^-$ the semi-planes of $\mathbb{C}$ with positive and negative real part, respectively. Let A_0 be a hyperbolic operator, that is $\sigma(A_0) \cap \mathrm{i}\mathbb{R} = \emptyset$. Thus we have a decomposition of the spectrum

$$\sigma(A_0) = \sigma^+(A_0) \cup \sigma^-(A_0)$$

where $\sigma^\pm(A_0) = \sigma(A_0) \cap \mathbb{H}^\pm$. Let P^+, P^- be the spectral projectors of the decomposition, E^+ and E^- their range respectively. It is clear that the stable and unstable spaces of the constant path A_0 are E^- and E^+. In the following theorem we prove that if $A = A_0 + H$ is a small perturbation of A_0 the stable and unstable spaces of A are closed and admit a topological complement.

Proposition 3.1. (cf. [**3**], Proposition 1.2). Let A_0 be a hyperbolic operator, with $\sigma^-(A_0)$ and $\sigma^+(A_0)$ nonempty, and a pair (c, λ), $\lambda > 0$ such that, for any $t \geq 0$

$$\|e^{tA_0}|_{E^-}\| \leq ce^{-\lambda t},\ \|e^{-tA_0}|_{E^+}\| \leq ce^{-\lambda t}. \tag{50}$$

Let $M := \max\{\|P^+\|, \|P^-\|\}$. There are positive constants h, ν, b depending only on c and λ such that if

$$\|H\|_\infty \leq \frac{\lambda}{Mc(1+\sqrt{c})}$$

the following facts hold:

(i) for every $t \geq 0$, $X_A(t)W^s_A$ is the graph of a bounded operator $S(t) \in \mathcal{L}(E^-, E^+)$,

(ii) $\|S(t)\| \leq c^2 \displaystyle\int_t^\infty e^{-\nu(\tau-t)}\|H(\tau)\|d\tau$,

(iii) the function S has much differentiability as X_A,

(iv) for every $u_0 \in W^s_A$ and every $t \geq s \geq 0$ there holds

$$|X_A(t)u_0| \leq be^{-\nu(t-s)}|X_A(s)u_0|.$$

Proof. First we check what kind of differential equation satisfies $u = X_A \cdot u_0$, for any $u_0 \in E^- \oplus E^+$, in terms of the projectors $P^\pm$. Let $u = x + y$. Differentiating both sides we find that

$$\begin{cases} x' = A_- x + A_\mp y \\ y' = A_\pm x + A_+ y \end{cases} \tag{51}$$

where $A_\pm = P^+AP^-$, $A_- = P^-AP^-$ and so on. For every $r \geq t \geq s$ the system above can be rewritten as

$$\begin{aligned} x(t) &= X_{A_-}(t)X_{A_-}(s)^{-1}x(s) + \int_s^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_\mp(\tau)y(\tau)d\tau \\ y(t) &= X_{A_+}(t)X_{A_+}(r)^{-1}y(r) - \int_t^r X_{A_+}(t)X_{A_+}(\tau)^{-1}A_\pm(\tau)x(\tau)d\tau. \end{aligned} \tag{52}$$

By hypothesis $A_{0,-}$ fulfills the exponential estimate (47) with constants $(c,-\lambda)$. Thus A_-, by Proposition 2.2, also does it with constants c and $-\mu_- = -\lambda + c\|H_-\|$. Similarly, by (50) $-A_+^*$ fulfills the estimate (47) with constants c and $-\mu_+ = -\lambda + c\|H_+^*\| = -\lambda + c\|H_+\|$. By the equality (46) we have

$$
\begin{aligned}
\|X_{A_+}(t)X_{A_+}(r)^{-1}\| &= \|(X_{A_+}(t)X_{A_+}(r)^{-1})^*\| \\
=&\|X_{A_+}(r)^{-1^*}X_{A_+}(t)^*\| = \|X_{-A_+^*}(r)X_{-A_+^*}(t)^{-1}\|
\end{aligned} \tag{53}
$$

for $r \geq t \geq 0$. The first equation of (52) gives inequalities

$$
\begin{aligned}
\left|\int_s^t X_{A_-}(t)X_{A_-}(\tau)^{-1}H_{\mp}(\tau)y(\tau)d\tau\right| &\leq c\int_s^t e^{-\mu_-(t-s)}\|H_{\mp}(\tau)\||y(\tau)|d\tau \\
&\leq \frac{c\|H_{\mp}\|}{\mu_-}\left(1-e^{-\mu_-(t-s)}\right)\|y\|_{\infty,[s,t]}
\end{aligned} \tag{54}
$$

and the second gives

$$
\begin{aligned}
\left|\int_t^r X_{A_+}(t)X_{A_+}(\tau)^{-1}H_{\pm}(\tau)x(\tau)d\tau\right| &\leq c\int_t^r e^{-\mu_+(\tau-t)}\|H_{\pm}(\tau)\|dt\,\|x\|_{\infty,[t,r]} \\
&\leq \frac{c\|H_{\pm}\|}{\mu_+}\left(1-e^{-\mu_+(r-t)}\right)\|x\|_{\infty,[t,r]}.
\end{aligned} \tag{55}
$$

Since μ_+ and μ_- are positive, in the second of (52) we can take the limit as $r \to +\infty$. Set $s = 0$ in the first of (52). Therefore the equations (54) and (55) permit to define a continuous map on the Banach space $C_b(\mathbb{R}^+, E^- \oplus E^+)$

$$
\varphi_{A,x_0} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = L_A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} X_{A_-}(\cdot)x_0 \\ 0 \end{pmatrix} \tag{56}
$$

where

$$
L_A\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \int_0^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_{\mp}(\tau)y(\tau)d\tau \\ -\int_t^{\infty} X_{A_+}(t)X_{A_+}(\tau)^{-1}A_{\pm}(\tau)x(\tau)d\tau \end{pmatrix}
$$

By (54) and (55), the operator L_A is bounded. A bounded solution u of (51), with $P^-u(0) = x_0$ is a fixed point of φ_{A,x_0}. The estimate of $\|H\|_\infty$ in the hypothesis gives

$$
(2c^3)^{1/2}\|H_{\mp}\| < \mu_-,\ (2c^3)^{1/2}\|H_{\pm}\| < \mu_+ \tag{57}
$$

hence φ_{A,x_0} is a contraction. Clearly if $u_0 \in W_A^s$ the curve $X_A(t)u_0$ is a fixed point of φ_{A,x_0}. Using (54) and (55) we prove that if u is fixed point then $u(0) \in W_A^s$, hence u is not just bounded, but infinitesimal also. If $u(0) = 0$ it is clear. Suppose $u(0) \neq 0$. For any $t \geq s$

$$
\begin{aligned}
|x(t)| \leq ce^{-\mu_-(t-s)}|x(s)| + \frac{c\|H_{\mp}\|}{\mu_-}\left(1-e^{-\mu_-(t-s)}\right)\|y\|_{\infty,[s,t)} \leq \\
\leq \max\{c|x(s)|, \frac{c\|H_{\mp}\|}{\mu_-}\|y\|_{\infty,[s,\infty)}\},
\end{aligned} \tag{58}
$$

the supremum on the real axis is allowed since we know that both x and y are bounded. From (55)

$$
|y(s)| \leq \frac{c\|H_{\pm}\|}{\mu_+}\|x\|_{\infty,[s,\infty)} \tag{59}
$$

and, taking the sup on $[s,\infty)$

$$
\|y\|_{\infty,[s,\infty)} \leq \frac{c\|H_{\pm}\|}{\mu_+}\|x\|_{\infty,[s,\infty)} \tag{60}
$$

and we get

$$\|x\|_{\infty,[s,\infty)} \le \max\{c|x(s)|, \frac{c^2\|H_\pm\|\|H_\mp\|}{\mu_-\mu_+}\|x\|_{\infty,[s,\infty)}\}; \tag{61}$$

the estimate of $\|H\|$ also implies that $c^2\|H_\pm\|\|H_\mp\| < \mu_-\mu_+$, therefore (61) allows to write

$$\|x\|_{\infty,[s,\infty)} \le c|x(s)|, \tag{62}$$

and, by (59) we get the final estimate

$$|y(s)| \le \frac{c^2\|H_\pm\|}{\mu_+}|x(s)|. \tag{63}$$

It is easy to check that x does not vanish at any point of $\mathbb{R}^+$ for, if such $t \in \mathbb{R}^+$ exists (63) implies $y(t) = 0$, thus $0 = x(t)+y(t) = u(t) = X_A(t)u_0$. Since $X_A(t)$ is invertible we had $u_0 = 0$ in contradiction with the hypothesis. If E is a Hilbert space it is easy to build a continuous path $U(t)$ of operators in $\mathcal{L}(E^-, E^+)$ that maps $x(t)$ to $y(t)$ and $\|U(t)\| = |y(t)|/|x(t)|$. Just define

$$U(t)z = \frac{(x(t), z)}{|x(t)|^2}y(t)$$

where $(\cdot,\cdot)$ denotes the scalar product of the Hilbert space. For Banach spaces we need some results of continuous selection such as Theorem 4 of [7]. By Corollary D.2 of Appendix D there exists a path U_ε continuous and bounded in $\mathcal{L}(E^-, E^+))$ such that

$$U_\varepsilon(t)x(t) = y(t), \ \|U_\varepsilon(t)\| \le (1+\varepsilon)\frac{c^2\|H_\pm\|}{\mu_+} + \varepsilon$$

for every $\varepsilon > 0$. Then we can write the first of (51) as

$$x' = [A_-(t) + A_\mp(t)U_\varepsilon(t)]x$$

Since $A_\mp(t)U_\varepsilon(t)$ is a bounded operator in $\mathcal{L}(E^-)$ we can apply the Proposition 2.2: in fact A_- satisfies an exponential estimate with constants $(c, -\mu_-)$, then the path $A_-(t) + A_\mp(t)U_\varepsilon(t)$ does it with constants $(c, -\nu_\varepsilon)$ where

$$-\nu_\varepsilon = -\mu_- + c\|H_\mp U_\varepsilon\| \le -\mu_- + c\|H_\mp\| \cdot (1+\varepsilon)\frac{c^2\|H_\pm\|}{\mu_+} + c\varepsilon\|H_\mp\|.$$

Let $\nu = \nu_0$. We have $-\mu_+\nu = -\mu_-\mu_+ + c^3\|H_\pm\|\|H_\mp\|$. By (57) $-\mu_+\nu < 0$, hence $-\nu < 0$. Then, if we choose ε small enough $-\nu_\varepsilon < 0$ and

$$|x(t)| \le e^{-\nu_\varepsilon(t-s)}|x(s)|, \quad t \ge s \ge 0.$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} |x(t)| &\le ce^{-\nu(t-s)}|x(s)| \\ |y(t)| &\le \frac{c^3\|H_\pm\|}{\mu_+}e^{-\nu(t-s)}|x(s)| \end{aligned} \tag{64}$$

and x and y vanish at infinity. Thus the fixed point u of φ_{A,x_0} can be characterized as a curve that solves (51) such that

$$u(+\infty) = 0, \quad P^-u(0) = x_0. \tag{65}$$

An application S in $\mathcal{L}(E^-, E^+)$ whose graph is W^s_A is defined as follows: given x_0 in E^- there exists a unique fixed point of φ_{A,x_0}, call it u. Thus $u(0) \in W^s_A$. We define $Sx_0 = P^+u(0)$ and we have

$$u(0) = P^-u(0) + P^+u(0) = x_0 + Sx_0$$

hence graph$(S) \subseteq W_A^s$. Conversely, given $u_0 \in W_A^s$ the curve $v(t) = X_A(t) \cdot u_0$, by the characterization in (65), is the fixed point of $\varphi_{A,P^- u_0}$, hence $P^+ u_0 = SP^- u_0$. Then graph$(S) = W_A^s$. We can write explicitly S

$$S = P^+ \circ ev_0 \circ (I - L_A)^{-1} \cdot \begin{pmatrix} X_{A_-}(\cdot)x_0 \\ 0 \end{pmatrix}, \tag{66}$$

where ev_0 is defined on C_b as the evaluation at $t = 0$. Then S is bounded and W_A^s is closed and $E = E^+ \oplus W_A^s$. Since $A_t = A_0 + H(\cdot + t)$ the same constants work to show that $W_{A_t}^s = X_A(t) W_A^s$ is graph of an unique bounded operator, say $S(t)$ and i) is proved. By direct computation

$$S(t) = P^+ X_A(t)(I_{E_-} + S) \cdot [P^- X_A(t)(I_{E_-} + S)]^{-1}. \tag{67}$$

Hence $S \in C(\mathbb{R}, \mathcal{L}(E^-, E^+))$ inherits the regularity of X_A and ii) follows.

Taking the limit as $r \to +\infty$ and $t = 0$ in (55)

$$\begin{aligned} |Sx_0| = |y_0| &\leq c \left(\int_0^\infty e^{-\mu_+ \tau} \|H_\pm(\tau)\| d\tau \right) \|x\|_\infty \\ &\leq c^2 \left(\int_0^\infty e^{-\nu \tau} \|H(\tau)\| d\tau \right) |x_0| \end{aligned}$$

since $\nu < \mu_+$. For the general case consider the shifted path $A(\cdot + t)$. Then

$$\begin{aligned} |S(t)x_0| &\leq c^2 \left(\int_0^\infty e^{-\mu_+ \tau'} \|H_{(\cdot+t)\pm}(\tau')\| d\tau' \right) |x_0| \\ &= c^2 \left(\int_t^\infty e^{-\mu(\tau - t)} \|H_\pm(\tau)\| d\tau \right) |x_0| \end{aligned}$$

where $\tau = t + \tau'$. This proves iii). Finally let $u_0 \in W_A^s$. By (64) and (57) we can write

$$\begin{aligned} |X_A(t)u_0| = |x(t) + y(t)| \leq |x(t)| + |y(t)| &\leq ce^{-\nu(t-s)} \left(1 + \frac{c^2 \|H_\pm\|}{\mu_+} \right) |x(s)| \\ &\leq (c + c^2)\|P^-\| e^{-\nu(t-s)} |u(s)| \leq be^{-\nu(t-s)} |X_A(s)u_0|. \end{aligned}$$

where $b = c(1 + c)\|P^-\|\|P^+\|$. The proof is complete. □

Proposition 3.2. (cf. [**3**], Proposition 1.2). With the same hypotheses of the preceding statement we have

(i) for every $t \geq 0$, $X_A(t)E^+$ is the graph of an operator $T(t) \in \mathcal{L}(E^+, E^-)$,
(ii) $\|T(t)\| \leq c^2 \int_0^t e^{-\nu(t-\tau)} \|H(\tau)\| \, d\tau$,
(iii) T is as much differentiable as X_A,
(iv) for every $y_0 \in E^+$, $t \geq s \geq 0$ the inequality

$$|X_A(t)y_0| \geq b^{-1} e^{\nu(t-s)} |X_A(s)y_0|$$

holds.

Proof. Let $\bar{t} \in \mathbb{R}^+$ and $\bar{y} \in E^+$. In (52) let $r = \bar{t}$ and $s = 0$. Then we have a continuous map, on $C([0, \bar{t}], E^- \oplus E^+)$ into itself

$$\psi_{A,\bar{y}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = R_A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ X_{A^+}(\cdot)X_{A^+}(\bar{t})^{-1}\bar{y}. \end{pmatrix} \tag{68}$$

where R_A is a bounded operator defined as

$$R_A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_0^t X_{A_-}(t) X_{A_-}(\tau)^{-1} A_\mp(\tau) y(\tau) d\tau \\ -\int_t^{\bar{t}} X_{A_+}(t) X_{A_+}(\tau)^{-1} A_\pm(\tau) x(\tau) d\tau \end{pmatrix}$$

The map is continuous because R_A is bounded. If $\|H\|_\infty$ is estimated by the same constant of the preceding proposition $\|R_A\| < 1$, hence $\psi_{A,\overline{y}}$ is a contraction. The fixed point v is a solution of (51) characterized by the property

$$P^- v(0) = 0, \quad P^+ v(\bar{t}) = \overline{y} \tag{69}$$

Let $\overline{y} \in E^+$ and let u be the fixed point of (68). We define $T(\bar{t})\cdot\overline{y} = P^- u(\bar{t})$. By (68) $u(0) \in E^+$ and $P^+ u(\bar{t}) = \overline{y}$, hence

$$\overline{y} + T(\bar{t})\overline{y} = P^+ u(\bar{t}) + P^- u(\bar{t}) = u(\bar{t}) = X_A(\bar{t})u(0)$$

thus $\mathrm{graph}\,(T(\bar{t})) \subset X_A(\bar{t})E^+$. Conversely, let $z \in X_A(\bar{t})E^+$ and $y \in E^+$ be such that $z = X_A(\bar{t})y$. The curve $u = X_A(\cdot)y$ has the property (69), thus coincides with the fixed point of ψ_{A,P^+z}. Hence

$$z = P^+ z + P^- z = P^+ z + P^- u(\bar{t}) = P^+ z + P^- v(\bar{t}) = P^+ z + T(\bar{t})P^+ z.$$

Hence $X_A(\bar{t})E^+ = \mathrm{graph}\,(T(\bar{t}))$. The map can also be written as

$$T(\bar{t})y = P^- \circ ev_0 (I - R_A)^{-1} \cdot \begin{pmatrix} 0 \\ X_{A^+}(\cdot)X_{A^+}(\bar{t})^{-1}\overline{y}. \end{pmatrix}$$

and i) is proved. For every $t \geq 0$

$$T(t) = P_-(P_+|_{X_A(t)E^+})^{-1} = P_- X_A(t)[P_+ X_A(t)]^{-1} \tag{70}$$

and iii) follows. Now let $0 \leq t \leq r \leq \bar{t}$. If (x, y) is the fixed point of ψ_A we find

$$|x(t)| \leq \frac{c\|H_\mp\|}{\mu_-}\|y\|_{\infty,[0,t]} \tag{71}$$

still from (54) and (55) we can write

$$\begin{aligned} |y(t)| &\leq ce^{-\mu_+(r-t)}|y(r)| + \frac{c\|H_\pm\|}{\mu_+}\left(1 - e^{-\mu_+(r-t)}\right)\|x\|_{\infty,[0,r)} \leq \\ &\leq \max\{c|y(r)|, \frac{c\|H_\pm\|}{\mu_+}\|x\|_{\infty,[0,r]}\}; \end{aligned} \tag{72}$$

from (71) we write

$$\|x\|_{\infty,[0,t]} \leq \frac{c\|H_\mp\|}{\mu_-}\|y\|_{\infty,[0,t]} \tag{73}$$

for any $0 \leq t \leq r$. By (72) and (73)

$$\|y\|_{\infty,[0,r]} \leq \max\{c|y(r)|, \frac{c^2\|H_\pm\|\|H_\mp\|}{\mu_-\mu_+}\|y\|_{\infty,[0,r]}\}. \tag{74}$$

Since $c^2\|H_\pm\|\|H_\mp\| < \mu_-\mu_+$ we have

$$\|y\|_{\infty,[0,r]} \leq c|y(r)|. \tag{75}$$

Setting $t = r$ in (71) from (75) it follows that

$$|x(r)| \leq \frac{c^2\|H_\mp\|}{\mu_-}|y(r)| \tag{76}$$

As we have done for the preceding Proposition for every $\varepsilon > 0$ the Corollary D.2 provides us with $V_\varepsilon \in C([0,\bar{t}], \mathcal{B}(E^+, E^-))$ such that

$$V_\varepsilon(r)y(r) = x(r), \;\; \|V_\varepsilon\| \leq (1+\varepsilon)\frac{c^2\|H_\mp\|}{\mu_-} + \varepsilon$$

hence $y' = (A_+ + H_\pm V_\varepsilon)y$. Applying the Proposition 2.2 to $-A_+^*$ for every $\varepsilon > 0$ and $r \geq t \geq 0$ there holds $|y(r)| \geq c^{-1}e^{\nu_\varepsilon(r-t)}|y(t)|$. Taking the limit as $\varepsilon \to 0$

$$|y(r)| \geq c^{-1}e^{\nu(r-t)}|y(t)|. \tag{77}$$

for every $r \geq t \geq 0$. By (76)

$$|y(s)| = \frac{1}{1+c}\Big(c|y(s)| + |y(s)|\Big) \geq \frac{1}{1+c}\Big(|x(s)| + |y(s)|\Big) \geq \frac{1}{1+c}|u(s)|. \tag{78}$$

Given $u_0 \in E^+$, using (78) and the fact that the norm of a projector is at least 1 we can write

$$\begin{aligned} |X_A(r)u_0| &\geq |y(r)| \|P^+\|^{-1} \geq (c\|P^+\|)^{-1} e^{\nu(r-t)}|y(s)| \\ &\geq \frac{e^{\nu(r-t)}|u(t)|}{c(1+c)\|P^+\|} \geq b^{-1}e^{\nu(r-t)}|X_A(t)u_0|. \end{aligned} \tag{79}$$

and (iv) follows. Finally

$$\begin{aligned} |T(\bar{t})\bar{y}| = |x(\bar{t})| &\leq c\left(\int_0^{\bar{t}} e^{-\mu_+(\bar{t}-\tau)}\|H_\mp\|\right)\|y\|_{\infty,[0,\bar{t}]} \\ &\leq c^2\left(\int_0^{\bar{t}} e^{-\nu(\bar{t}-\tau)}\|H_\mp\|\right)|\bar{y}|; \end{aligned}$$

the last estimate follows from (75) with $r = \bar{t}$ and (ii) is proved. □

Remark 3.3. The statements and the proofs of the two theorems regard only the stable space. To obtain the same conclusions for the unstable space defined on the negative real line just set $\check{A}(t) = A(-t)$. Using argument of uniqueness of Cauchy problems we obtain

$$X_A(-t) = X_{-\check{A}}(t),\ W^u_A = W^s_{-\check{A}},\ -\check{A}(+\infty) = -A(-\infty).$$

Thus we can apply Propositions 3.1 and 3.2 to $-\check{A}$ on the positive real line.

4. Properties of W^s_A and W^u_A

In the preceding section it has been proved that W^s_A (as W^u_A) is a splitting space if A is close, in the uniform topology, to a constant hyperbolic path A_0. We prove that it is true for any asymptotically hyperbolic path. Conversely we provide, for any pair (X, Y) in $G_s(E)$, a path A such that $(W^s_A, W^u_A) = (X, Y)$.

Theorem 4.1. (cf. [**3**], Theorem 2.1). Let A be an asymptotically hyperbolic path of operators defined on $\mathbb{R}^+$. Let $A_0 = A(+\infty)$, $E^+ \oplus E^-$ the spectral decomposition. Then W^s_A is a splits

(i) W^s_A is the only closed subspace W such that $X_A(t)W \to E^-$,
(ii) $\|X_A(t)_{|W^s_A}\| \leq ce^{-\lambda(t-s)}\|X_A(s)_{|W^s_A}\|$ for suitable $c, \lambda > 0$ and every $t \geq s \geq 0$,
(iii) for every $V \in G_s(E)$ such that $V \oplus W^s_A = E$ $\rho(X_A(t)V, E^+) \to 0$,
(iv) $\inf_{\substack{v\in V\\|v|=1}} |X_A(t)v|$ grows at exponential rate,
(v) $W^s_{-A^*} = (W^s_A)^\perp$.

Proof. Let $A(+\infty) = A_0$. Since A_0 is a hyperbolic operator there exist c and λ such that the condition (50) holds. Let $H = A_0 - A$. If τ is large enough $\|H_{(\cdot+\tau)}\|$ is smaller than the constant of Proposition 3.1 then

$$W^s_{A(\cdot+\tau)} = X_A(\tau)W^s_A$$

is a topological complement of E^+ and, since $X_A(\tau)$ is invertible, W^s_A is closed too and

$$X_A(\tau)W^s_A \oplus E^+ = E = W^s_A \oplus X_A(\tau)^{-1}E^+.$$

Now for $t \geq \tau$ the Proposition 3.1 says that $X_{A(\cdot+\tau)}(t)W^s_{A(\cdot+\tau)}$ is the graph of a bounded linear map $S(t)\colon E^- \to E^+$ and

$$\|S(t)\| \leq c^2 \int_t^\infty e^{-\nu(t-\tau')}\|H(\tau+\tau')\|d\tau'.$$

This implies that $S(t)$ converges to the null operator as $t \to +\infty$. By Proposition 3.1, graph$(S(t))$ converges to graph$(0) = E^-$, hence $X_A(t+\tau)W_A^s = X_{A(\cdot+\tau)}(t)W_{A(\cdot+\tau)}^s \to E^-$.

The ii) follows from iv) of Proposition 3.1 taking the supremum over the unit sphere of W_A^s on both sides of the inequality.

Let V be a closed subspace of E. Up to a time shift we can suppose that V is graph of a bounded operator $L \in \mathcal{L}(E^+, W_A^s)$. First we prove that $\rho(X_A(t)E^+, X_A(t)V)$ converges to 0. Let $v \in X_A(t)V$ and $y \in E^+$ be such that $v = X_A(t)\cdot(y + Ly)$. Set $u = X_A(t)y$. Then

$$\begin{aligned}|v-u| = |X_A(t)Ly| &\le be^{-\nu t}\|L\||y| \le b^2e^{-2\nu t}\|L\||X_A(t)y|\\ &= b^2e^{-2\nu t}\|L\||u| \le b^2e^{-2\nu t}\|L\|(|v| + |v-u|)\end{aligned}$$

since $\alpha(t) := b^2e^{-2\nu t}\|L\|$ is an infinitesimal sequence, for $t \ge \bar{t}$ we have $\alpha(t) < 1$ and the above inequality becomes

$$|v-u| \le \alpha(t)(|v| + |v-u|) \Rightarrow |v-u| \le \frac{\alpha(t)}{1-\alpha(t)}|v|$$

and we conclude that $\rho(X_A(t)Y, X_A(t)E^+) \to 0$ as $t \to +\infty$. On other hand

$$\begin{aligned}|u-v| = |X_A(t)Ly| &\le be^{-\nu t}\|L\||y| \le b^2e^{-2\nu t}\|L\||X_A(t)y|\\ &= b^2e^{-2\nu t}\|L\||u| = \alpha(t)|u|\end{aligned}$$

and $\rho(X_A(t)E^+, X_A(t)V) \le \alpha(t)$. The proof is complete using the fact that $\rho(X_A(t)E^+, E^+) \to 0$ which follows from i) and ii) of Theorem 3.2.

To prove the converse of i) let $W \subseteq E$ be a closed subspace such that $X_A(t)W \to E^-$. By iii) for every topological complement of W_A^s, say V, we have $V \cap W = \{0\}$, hence $W \subset W_A^s$. There exists $t_0 > 0$ such that, $\rho(X_A(t_0)W, X_A(t_0)W_A^s) < 1$ and, by Proposition 3.2, $X_A(t_0)W = X_A(t_0)W_A^s$ hence $W = W_A^s$ and i) is proved.

In order to prove the iv) we can suppose, up to a time shift, that $V \oplus W_A^s = E = W_A^s \oplus E^+$. Again $V = $ graph(L), $L \in \mathcal{L}(E^+, W_A^s)$. Then

$$\begin{aligned}|X_A(t)v| = |X_A(t)y + X_A(t)Ly| &\ge |X_A(t)y| - |X_A(t)Ly|\\ &\ge b^{-1}e^{\nu t}|y| - be^{-\nu t}\|L\||y| = (b^{-1}e^{\nu t} - be^{-\nu t}\|L\|)|y|\\ &\ge 1/(1+\|L\|)\,(b^{-1}e^{\nu t} - be^{-\nu t}\|L\|)|v|\end{aligned}$$

and iv) follows by taking the infimum over $S(V)$. By (46) we have the chain of equalities

$$X_{-A^*}(t)(W_A^s)^\perp = (X_A(t)^{-1})^*(W_A^s)^\perp = (X_A(t)W_A^s)^\perp. \tag{80}$$

Since $X_A(t)W_A^s$ converges to E^- and E^- splits the Proposition 3.4 allows us to take the limit in (80) which is $(E^-)^\perp$. Since $(E^-)^\perp = E^-(-A^*)$, by i)

$$X_{-A^*}(t)(W_A^s)^\perp \to E^-(-A^*)$$

implies $(W_A^s)^\perp = W_{-A^*}^s$. □

Analogous statements hold for the unstable space W_A^u by considering the path $-\check{A}$.

Lemma 4.2. *Let A be an asymptotically hyperbolic path of on $\mathbb{R}^+$. Then $X_A(t)W_A^s = E^-$ for every $t \ge 0$ if and only if $A(t)E^- \subseteq E^-$*

Proof. For any $W \subseteq E$ such that $X_A(t)W = E^-(A(+\infty))$ we can set $t = 0$ to get $W = E^-(A(+\infty))$, hence

$$X_A(t)E^- = E^- \tag{81}$$

for any $t \ge 0$. Now, fix $\bar{t} \in \mathbb{R}^+$ and let $x \in E^-$, $\bar{x} = X_A(\bar{t})^{-1}x$. By the (81) the curve $u(t) = X_A(t)\bar{x}$ is C^1 and takes values in E^-, therefore $u'(t) \in E^-$ for any $t \in \mathbb{R}^+$. Hence

$$E^- \ni u'(\bar{t}) = A(\bar{t})X_A(\bar{t})\bar{x} = A(\bar{t})X_A(\bar{t})X_A(\bar{t})^{-1}x = A(\bar{t})x.$$

Conversely, assume that the second condition is true for any $t \in \mathbb{R}^+$. First we prove that $X_A(t)E^- \subseteq E^-$. Let $x \in E^-$ and let $u(t) = X_A(t)x$. In the second of (51) we have $A_\pm x = 0$ by hypothesis, thus $y' = A_+ y$. Hence

$$P^+ u(t) = X_{A_+}(t)P^+u(0);$$

since $P^+u(0) = 0$ we have $P^+u = 0$ and from the first of (51) we obtain $u(t) \in E^-$. Now, X_A sets a continuous path of semi-Fredholm operators on E^-. By Proposition B.5 these operators have the same index for any $t \in \mathbb{R}^+$. Since $X_A(0) = Id$ the index of these operators is zero. Since every $X_A(t)$ is restriction of an invertible operator they are injective, thus surjective, that is $X_A(t)E^- = E^-$. In particular $X_A(t)E^-$ converges to E^-. By i) of Theorem 4.1 $E^- = W^s_A$. □

Proposition 4.3. *Given a pair of splitting subspaces (X, Y) in E there exists a path A, continuous and asymptotically hyperbolic on $\mathbb{R}$, such that $W^s_A = X$, $W^u_A = Y$.*

Proof. Let P, Q be two projectors on X and Y respectively. We build first a path A^s on $\mathbb{R}^+$ such that $W^s_{A^s} = X$. Let A^s be the constant path $I - 2P$ which is hyperbolic because $(I-2P)^2 = I$. The spectral projector on the negative and positive eigenprojectors are, respectively, P and $I - P$. A solution $x + y$ of (51) satisfies

$$\begin{aligned} x' &= A^s_- x + A^s_\mp y = -x \\ y' &= A^s_+ y + A^s_\pm x = y. \end{aligned}$$

Thus $X_{A^s}(t) = e^{-t}P + e^t(I - P)$ and the stable space is X. Similarly we can define $A^u(t) = 2Q - I$ for $t < 0$. The joint path $A^u \# A^s$ is piecewise continuous. In order to find a smooth path consider a smooth function φ such that $\varphi([-1/2, 1/2]) = 1$ and $\varphi({}^c(-1,1)) = -1$. Thus the path

$$A = \begin{cases} \varphi(t)P + (I-P) & t \geq 0 \\ \varphi(t)(I-Q) + Q & t < 0 \end{cases}$$

is smooth. The solution of (51) with starting point $x(0) + y(0)$ is

$$\begin{cases} x(t) &= e^{\Phi(t)}x(0) \\ y(t) &= e^t y(0) \end{cases}$$

where Φ is the smooth function such that $\Phi(0) = 0$ and $\Phi'(t) = \varphi(t)$. Since Φ diverges to $-\infty$ as $t \to +\infty$ the stable space is X. Since Φ diverges to $+\infty$ as $t \to -\infty$, hence the unstable space is Y. □

5. Perturbation of the stable space

In the previous sections we have defined the stable (and unstable) space and proved that is an element of $G_s(E)$, the Grassmannian of splitting subspaces. Thus, in the set

$$C_h(\mathbb{R}^+, \mathcal{L}(E)) = \left\{ A \in C(\overline{\mathbb{R}}^+, \mathcal{L}(E)) \mid \sigma(A(+\infty)) \cap i\mathbb{R} = \emptyset \right\}$$

endowed with the uniform topology it is defined an application that maps A to W^s_A. In the next two theorems we prove that it is continuous and that if two paths differ by a path of compact operators then the stable spaces are compact perturbation one of each other.

Theorem 5.1. (cf. [**3**], Theorem 3.1). The map $A \mapsto W^s_A$ is continuous.

Proof. Since $C_h(\mathbb{R}^+, \mathcal{L}(E))$ is a metric space it is enough to prove that the map is sequentially continuous. Let $\{A_n \mid n \in \mathbb{N}\}$ be a sequence in $C_h(\mathbb{R}^+, \mathcal{L}(E))$ converging to an asymptotically hyperbolic path A. Let $A(+\infty) = A_0$. Call $P^\pm$ the spectral projectors on $E^-(A_0)$ and $E^+(A_0)$ respectively. Since A_0 is hyperbolic, there exist a pair (c, λ) such that

$$\|e^{tA_0}P^-\| \leq ce^{-\lambda t}, \quad \|e^{-tA_0}P^+\| \leq ce^{-\lambda t}, \quad t \geq 0.$$

The sequence $\{A_n\}$ converges to A_0 uniformly as $n \to \infty$. Moreover $A(t)$ converges to A_0, as $t \to +\infty$. Using triangular inequalities we can find $\tau \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that, for every $t \geq \tau$ and $n \geq N$

$$\|A_n(t) - A_0\| \leq \frac{\lambda}{Mc(1+\sqrt{c})}. \tag{82}$$

where $M = \max\{\|P^+\|, \|P^-\|\}$. Therefore for every $n \geq N$ the paths $A_{n,\tau}$, together with A_τ, fulfill the conditions of Proposition 3.1. In particular there are $S_n, S \in \mathcal{L}(E^-, E^+)$ such that

$$X_{A_n}(\tau)W^s_{A_n} = W^s_{A_{n,\tau}} = \operatorname{graph}(S_n), \quad X_A(\tau)W^s_A = \operatorname{graph}(S).$$

It is enough to prove that S_n converges to S. In fact, by Proposition 4.7, this implies that $X_{A_n}(\tau)W^s_{A_n}$ converges to $X_A(\tau)W^s_A$ and the conclusion follows because X_{A_n} converges to X_A point-wise. For the remainder of the proof we omit the subscript τ from the paths. We recall that, by (66), given $x \in E^-$

$$S_n x = P^+ ev_0(I - L_{A_n})^{-1}(X_{A_{n-}}(\cdot)x) = P^+ \sum_{k=0}^{\infty} ev_0[L^k_{A_n}(X_{A_{n-}}(\cdot)x)]. \tag{83}$$

Since the estimate (82) holds for every $n \geq N$ we can apply the Proposition 2.2 to A_{n-} and A_{n+} in order to obtain uniform exponential estimates

$$\|X_{A_{n-}}(t)X_{A_{n-}}(s)^{-1}x\| \leq ce^{-\mu_-(t-s)}|x|$$
$$\|X_{A_{n+}}(t)X_{A_{n+}}(r)^{-1}x\| \leq ce^{\mu_+(t-r)}|x|$$

where μ_- and μ_+ are the same constants defined in Proposition 3.1. By (54) and (55) there exists $0 < \alpha < 1$ such that $\|L_{A_n}\| \leq \alpha$ for every $n \geq N$. Then

$$|[L^k_{A_n}X_{A_{n-}}(\cdot)x]| \leq c\alpha^k|x|. \tag{84}$$

In order to prove that S_n converges to S we show, by induction on $k \in \mathbb{N}$, that $L^k_{A_n}X_{A_{n-}}(\cdot)x$ converges to $L^k_A X_{A_-}(\cdot)x$ point-wise. Therefore the series

$$\sum_{k=0}^{\infty} ev_0[L^k_{A_n}(X_{A_{n-}}(\cdot)x)]$$

converges point-wise and, by (84), is dominated uniformly on $\mathbb{N}$ by the series of the sequence $\{\alpha^k\}$. This is enough to obtain the convergence of series to the point-wise limit. We claim that for every $t \geq 0$

$$\lim_{n\to\infty} L^k_{A_n}X_{A_{n-}}(t)x = L^k_{A_0}X_{A_{0-}}(t)x,$$
$$L^k_{A_n}X_{A_{n-}}(t)x \in E^-, \text{ if } k \text{ is even},$$
$$L^k_{A_n}X_{A_{n-}}(t)x \in E^+, \text{ if } k \text{ is odd}.$$

If $k = 0$ the thesis follows since $x \in E^-$ by hypothesis. Suppose it is true for $k \in \mathbb{N}$. If k is odd, by (3)

$$L^{k+1}_{A_n}X_{A_{n-}}(t)x = \int_0^t X_{A_{n-}}(t)X_{A_{n-}}(\tau)^{-1}A_{n\mp}(\tau)L^k_{A_n}X_{A_{n-}}(\tau)x d\tau \tag{85}$$

which belongs to E^-. The last term converges to $L^k_{A_n}X_{A_{n-}}(t)x$ by inductive hypothesis. The other converges by Proposition 1.8 and the fact that A_n converges to A. The integrand of (85) is bounded in $[0, t]$ by

$$c^2e^{-\mu_-(t-\tau)} \sup_n \|A_n\|_\infty \alpha^k|x|.$$

Then, by the dominate convergence theorem, the left member of (85) converges point-wise. If k is even, by (3)

$$L_{A_n}^{k+1} X_{A_{n-}}(t)x = -\int_t^\infty X_{A_{n+}}(t) X_{A_{n+}}(\tau)^{-1} A_{n\pm}(\tau) L_{A_n}^k X_{A_{n-}}(\tau) x d\tau. \tag{86}$$

Similarly the integrand converges point-wise and is dominated by

$$c^2 \alpha^k e^{\mu_+(t-\tau)} |x| \sup_n \|A_n\|_\infty \in L^1(\mathbb{R}^+).$$

Again, by the dominate convergence theorem, we clinch the point-wise convergence of (86) and the inductive step is concluded. Thus

$$\lim_{n\to\infty} ev_0[L_{A_n}^k X_{A_{n-}}(\cdot)x] = ev_0[L_{A_0}^k X_{A_{0-}}(\cdot)x],$$
$$|ev_0[L_{A_n}^k X_{A_{n-}}(\cdot)x]| \le c\alpha^k |x|$$

for every $k \in \mathbb{N}$ we have convergence of the series. □

We state without proof a couple of facts on compactness useful for the next theorem.

Lemma 5.2. *Let J be an interval of the real line, $K \in L^1(J, \mathcal{L}(E))$ such that $K(t) \in \mathcal{L}_c(E)$ almost everywhere. Then the map*

$$C_b(J, E) \ni u \longmapsto \int_J K(\tau) u(\tau) d\tau \in E$$

is a compact operator.

Proof. When K is constant the map is obtained by composition on the left with a compact operator. If K is a characteristic function on J it is sum of compact operators. We conclude with the density of characteristic functions in $L^1(J, \mathcal{L}(E))$ and closeness of compact operators. □

Theorem 5.3 (Ascoli-Arzelà)**.** *Let X be a compact metric space, E a Banach space. A bounded subset $\mathcal{W} \subset C(X, E)$ is relatively compact if and only is equicontinuous and, for every $x \in X$, the set $\mathcal{W}(x) = \{ f(x) \mid f \in \mathcal{W} \}$ is relatively compact in E.*

For a proof see [**18**], pp. 142–143.

Theorem 5.4. (cf. [**3**], Theorem 3.6). Let $A, B \in C_h(\mathbb{R}^+, \mathcal{L}(E))$ be such that $K = B - A$ is a compact operator for every t. Then W_A^s is a compact perturbation of W_B^s and

$$\dim(W_A^s, W_B^s) = \dim(E^-(A(+\infty)), E^-(B(+\infty))).$$

Proof. Up to a time shift we can assume that A and B satisfy the conditions of the Proposition 3.1. Then W_A^s and W_B^s are graph of operators

$$S_A \in \mathcal{L}(E^-(A(+\infty)), E^+(A(+\infty))),$$
$$S_B \in \mathcal{L}(E^-(B(+\infty)), E^+(B(+\infty))).$$

Let $P^-(A)$ and $P^-(B)$ be the spectral projectors of the negative eigenspaces. Observe that

$$W_A^s = \ker(P^+(A) - S_A P^-(A)), \; W_B^s = \ker(P^+(B) - S_B P^-(B)).$$

The differences $P^\pm(A) - P^\pm(B)$ are compact operators; we wish to prove that $S_A P^-(A) - S_B P^-(B)$ is also compact. Therefore W_A^s is a compact perturbation of W_B^s and, by Proposition 5.16,

$$\begin{aligned}
\dim(W_A^s, W_B^s) &= \dim(\ker(P^+(A) - S_A P^-(A)), \ker(P^+(B) - S_B P^-(B))) \\
&= \dim(\mathrm{Range}\,(P^+(B) - S_B P^-(B)), \mathrm{Range}\,((P^+(A) - S_A P^-(A))) \\
&= \dim(E^+(B(+\infty)), E^+(A(+\infty))) \\
&= \dim(E^-(A(+\infty)), E^-(B(+\infty))),
\end{aligned}$$

which is the thesis when W^s_A and W^s_B are graphs. In the general case there exists a real τ such that $A(\cdot+\tau)$ and $B(\cdot+\tau)$ satisfy the conditions of Proposition 3.1. Then

$$\begin{aligned}\dim(W^s_{A(\cdot+\tau)}, W^s_{B(\cdot+\tau)}) &= \dim(X_A(\tau)W^s_A, X_B(\tau)W^s_B)\\ &= \dim(W^s_A, X_A(\tau)^{-1}X_B(\tau)W^s_B)\\ &= \dim(W^s_A, W^s_B) + \dim(W^s_B, X_A(\tau)^{-1}X_B(\tau)W^s_B)\end{aligned}$$

The last term of the equality is 0 because $X_A(\tau)^{-1}X_B(\tau)$ can be written as $I + (X_A(\tau)^{-1} - X_B(\tau)^{-1})X_B(\tau)$ which is an invertible operator of the Fredholm group. Then the conclusion follows from Proposition 5.16. Now we write, by (66)

$$\begin{aligned}S_A P^-(A) &= P^+(A)ev_0[(I-L_A)^{-1}X_{A_-}(\cdot)P^-(A)],\\ S_B P^-(B) &= P^+(B)ev_0[(I-L_B)^{-1}X_{B_-}(\cdot)P^-(B)].\end{aligned}$$

Using the Theorem of Ascoli–Arzelà we prove first that $L_A - L_B$ is a compact operator on $C_b(\mathbb{R}^+, E)$. In fact let $\mathcal{W}$ be a bounded subset of $C_b(\mathbb{R}^+, E)$. Given $u \in \mathcal{W}$ for every $t \in \mathbb{R}^+$ we have

$$\begin{aligned}(L_A u)'(t) &= [P^+(A)A(t)P^+(A) + P^-(A)A(t)P^-(A)](L_A - I)u(t) + A(t)u(t)\\ (L_B u)'(t) &= [P^+(B)B(t)P^+(B) + P^-(B)B(t)P^-(B)](L_B - I)u(t) + B(t)u(t).\end{aligned}$$

Since A and B are bounded the set $\{(L_A - L_B)u(t) \mid u \in \mathcal{W}\}$ is bounded by a constant that depends on t at most. Then $(L_A - L_B)\mathcal{W}$ is equicontinuous. Now we prove that the set

$$\{(L_A - L_B)u(t) \mid u \in \mathcal{W}\}$$

is relatively compact. The prove is carried on interpolating L_A and L_B and applying Lemma 5.2 to the differences as follows

$$\begin{aligned}(L_A - L_B)u(t) = P^-(A)&\int_0^t X_{A_-}(t)X_{A_-}(\tau)^{-1}P^-(A)A(\tau)P^+(A)u(\tau)d\tau\\ - P^-(B)&\int_0^t X_{B_-}(t)X_{B_-}(\tau)^{-1}P^-(B)B(\tau)P^+(B)u(\tau)d\tau\\ - P^+(A)&\int_t^\infty X_{A_+}(t)X_{A_+}(\tau)^{-1}P^+(A)A(\tau)P^-(A)u(\tau)d\tau\\ + P^+(B)&\int_t^\infty X_{B_+}(t)X_{B_+}(\tau)^{-1}P^+(B)B(\tau)P^-(B)u(\tau)d\tau.\end{aligned}$$

Since $X_A(t) - X_B(t)$ and $A(t) - B(t)$ are compact by interpolation we obtain the sum of two integrals on $[0,t]$ and $[t,+\infty)$ with compact integrands. We conclude by applying Lemma 5.2 to the two integrands. By composition $S_A P^-(A) - S_B P^-(B)$ is compact. □

CHAPTER 4

Ordinary differential operators on Banach spaces

Given a path $A \in C(\mathbb{R}, \mathcal{L}(E))$ we study the properties of the differential operator $F_A u = u' - Au$. When E is a Hilbert space the operator can be defined in $H^1(\mathbb{R}, E)$ with values in $L^2(\mathbb{R}, E)$. By THEOREM 5.1 of [**3**] the operator F_A is Fredholm if and only if the pair (W^s_A, W^u_A) is a Fredholm pair and

$$\operatorname{ind} F_A = \operatorname{ind}(W^s_A, W^u_A).$$

In this chapter we prove the same result when E is a Banach space and the operator F_A is defined on $C^1_0(\mathbb{R}, E)$ and takes values in $C_0(\mathbb{R}, E)$, where

$$\begin{aligned} C_0(\mathbb{R}, E) &= \left\{ u \in C(\mathbb{R}, E) \mid \lim_{t \to \pm\infty} u(t) = 0 \right\} \\ C^1_0(\mathbb{R}, E) &= \left\{ u \in C^1(\mathbb{R}, E) \mid \lim_{t \to \pm\infty} u(t) = 0, \ \lim_{t \to \pm\infty} u'(t) = 0 \right\}. \end{aligned}$$

We remark that the result also holds when F_A is defined on the Sobolev space $W^{1,p}(\mathbb{R}, E)$ with values in $L^p(\mathbb{R}, E)$ with $p \geq 1$.

1. The operators F^+_A and F^-_A

Consider the spaces

$$\begin{aligned} C_0(\mathbb{R}^+, E) &= \left\{ u \in C^1(\mathbb{R}^+, E) \mid \lim_{t \to +\infty} u(t) = 0 \right\} \\ C^1_0(\mathbb{R}^+, E) &= \left\{ u \in C^1(\mathbb{R}^+, E) \mid \lim_{t \to +\infty} u(t) = 0, \ \lim_{t \to +\infty} u'(t) = 0 \right\}; \end{aligned}$$

we define the operator

$$F^+_A \colon C^1_0(\mathbb{R}^+, E) \to C_0(\mathbb{R}^+, E), \ u \mapsto u' - Au$$

and similarly F^-_A on $C^1_0(\mathbb{R}^-, E)$. We wish to prove that when A is asymptotically hyperbolic F^+_A has a right inverse. First observe that in special case $A \equiv A_0$ the operator F_{A_0} is invertible and its inverse is given by

$$R_{A_0} h = G_{A_0} * h \tag{87}$$

for any $h \in C^1_0(\mathbb{R}, E)$, where

$$G_{A_0}(t) = e^{tA_0} \left[P^-(A_0) 1_{\mathbb{R}^+} - P^+(A_0) 1_{\mathbb{R}^-} \right] \tag{88}$$

where $P^-(A_0)$ and $P^+(A_0)$ are the spectral projectors of A_0 relative to decomposition $\sigma(A_0) = \sigma^+ \cup \sigma^-$ and $1_{\mathbb{R}^+}$ and $1_{\mathbb{R}^-}$ are the characteristic functions of the subsets $\mathbb{R}^+$ and $\mathbb{R}^-$. Exponential estimates of G_{A_0} makes $G_{A_0} * h$ a continuously differentiable function in $C^1_0(\mathbb{R}, E)$. Moreover

$$\begin{aligned} F_{A_0}(G_{A_0} * h)(t) &= (G_{A_0} * h)' - A_0(G_{A_0} * h) \\ &= A_0(G_{A_0} * h) + P^- h(t) + P^+ h(t) - A_0(G_{A_0} * h) = h; \end{aligned}$$

hence R_{A_0} is a right inverse of F_{A_0}. Otherwise

$$G_{A_0} * F_{A_0} u = \int_{-\infty}^{t} e^{(t-\tau)A_0} P^-(u' - A_0 u) d\tau - \int_{t}^{+\infty} e^{(t-\tau)A_0} P^+(u' - A_0 u) d\tau$$

integration by parts lead to

$$\int_{-\infty}^{t} e^{(t-\tau)A_0} P^-(u' - A_0 u)d\tau = P^- u(t)$$

$$-\int_{t}^{+\infty} e^{(t-\tau)A_0} P^+(u' - A_0 u)d\tau = P^+ u(t)$$

taking the sum we conclude. If A is a asymptotically hyperbolic path we know that W_A^s and W_A^u are closed and have topological complements. Choose X_s and X_u such that $X_s \oplus W_A^s = E = X_u \oplus W_A^u$ and let $P_s = P(W_A^s, X_s)$, $P_u = P(W_A^u, X_u)$. Define

$$G^+_{A,P_s}(t,\tau) = X_A(t)\left[P_s 1_{\mathbb{R}^+} - (I - P_s)1_{\mathbb{R}^-}\right] X_A(\tau)^{-1} \tag{89}$$

$$G^-_{A,P_u}(t,\tau) = X_A(t)\left[(I - P_u)1_{\mathbb{R}^+} - P_u 1_{\mathbb{R}^-}\right] X_A(\tau)^{-1} \tag{90}$$

Proposition 1.1. *If A is an asymptotically hyperbolic path there are positive constants (c, λ) such that*

$$\|G^+_{A,P_s}(t,\tau)\| \le c e^{-\lambda|t-\tau|} \tag{91}$$

for every $(t,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Proof. By the Theorem 4.1, if P_s is a projector on W_A^s, $I - P_s^*$ is a projector on $(W_A^s)^\perp = W^s_{-A^*}$. Hence $(G^+_{A,P_s}(t,\tau))^* = G_{-A^*,I-P_s^*}(\tau,t)$ and it's enough to prove the statement for $t \ge \tau \ge 0$. We have

$$\begin{aligned} \|G^+_{A,P_s}(t,\tau)\| &\le \|X_A(t)P_s X_A(t)^{-1}\| \cdot \|X_A(t)X_A(\tau)^{-1}\| \\ &\le c' e^{-\lambda(t-\tau)} \|X_A(t)P_s X_A(t)^{-1}\|. \end{aligned} \tag{92}$$

For every $t \in \mathbb{R}^+$ $P(t) = X_A(t)P_s X_A(t)^{-1}$ is a projector onto $X_s(t) = X_A(t)W_A^s$ and $I - P(t)$ onto $X_u(t) = X_A(t)X_u$. By Theorem 4.1, i) and iii), $X_s(t)$ converges to $E^-(A(+\infty))$, and $X_u(t)$ to $E^+(A(+\infty))$. Then by Proposition 4.10 the $P(t)$ is bounded (in fact converges to a projector). Then the last term of (92) is estimated by $Mc'e^{-\lambda(t-\tau)}$. □

This allows us to prove the following

Proposition 1.2. *Let A be a bounded continuous path on $\mathbb{R}^+$. Then $F_{A,+}$ is a bounded operator. Moreover if A is asymptotically hyperbolic F_A^+ has right inverse also and one is given by*

$$R^+_{A,P_s} h(t) = \int_{\mathbb{R}} G^+_{A,P_s}(t,\tau) h(\tau) 1_{\mathbb{R}^+}(\tau) d\tau. \tag{93}$$

where P_s is a projector onto the stable space.

Proof. That F_A^+ is bounded it's clear from the definition. We prove that R^+_{A,P_s} maps $C_0(\mathbb{R}^+, E)$ in $C_0^1(\mathbb{R}^+, E)$. In fact if $h \in C_0$ then $R^+_{A,P_s} h(t)$ is

$$\int_0^t X_A(t)P_s X_A(\tau)^{-1} h(\tau)d\tau - \int_t^{+\infty} X_A(t)(I - P_s)X_A(\tau)^{-1} h(\tau)d\tau$$

hence is continuous and continuously differentiable. By the (91) we have

$$\|R^+_{A,P_s} h(t)\| \le \int_{\mathbb{R}^+} c e^{-\lambda|t-\tau|} |h(\tau)| d\tau \le \|h\|_\infty \int_{\mathbb{R}^+} e^{-\lambda|t-\tau|} d\tau \le \frac{\|h\|_\infty}{\lambda} e^{-\lambda t} \tag{94}$$

hence $R^+_{A,P_s} h \in C_0(\mathbb{R}^+, E)$. Since its derivative is

$$(R^+_{A,P_s} h)' = A R^+_{A,P_s} h + h \tag{95}$$

and A is bounded, we have $R^+_{A,P_s} h \in C_0^1(\mathbb{R}^+, E)$. Actually (94) and (95) say that R_{A,P_s} is a bounded operator. Still from (95)

$$F_A^+ R^+_{A,P_s} h = (R^+_{A,P_s} h)' - A R^+_{A,P_s} h = A R^+_{A,P_s} h + h - A R^+_{A,P_s} h = h.$$

Then R^+_{A,P_s} is a right inverse of F_A^+. □

Similarly we have

Proposition 1.3. *If A is a bounded continuous path on $\mathbb{R}^-$ the operator $F_{A,-}$ is bounded and admits a right inverse if A is asymptotically hyperbolic. One is given by*

$$R^-_{A,P_u}h(t) = \int_{\mathbb{R}} G^-_{A,P_u}(t,\tau)h(\tau)1_{\mathbb{R}^-}(\tau)d\tau.$$

where P_u is a projector onto the unstable space.

The proof is completely similar and we omit it.

Example 1.4. Notice that if A_0 is invertible but not hyperbolic these operators can be non surjective. For example let E be the Euclidean space $\mathbb{R}^2$ and define

$$A_0 = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \; e^{A_0} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta.$$

First observe that $F^+_{A_0}$ is injective: given u in $C^1_0(\mathbb{R}^+,E)$ such that $F^+_{A_0}u = 0$. We have $u(t) = R_{t\theta}u(0)$ by uniqueness of the solutions of (42). Since R_θ is an isometry $|u(t)| = |u(0)|$ for every $t \geq 0$. Taking the limit as $t \to +\infty$ we obtain $u(0) = 0$, hence u is zero. Now let h be a continuous function on $\mathbb{R}^+$ that vanishes at $+\infty$ and u in $C^1_0(\mathbb{R}^+,E)$ such that $F^+_{A_0}u = h$. Since $F^+_{A_0}$ is injective

$$u(t) = e^{tA_0}\left(\int_0^t e^{-sA_0}h(s)ds + u(0)\right) \tag{96}$$

is the only solution of the problem. Fix v_0 in $E \setminus \{0\}$ and α in $C_0(\mathbb{R}^+,\mathbb{R}^+)$ not integrable. Let $h(s) = \alpha(s)R_{s\theta}v_0$. Since R_θ is an isometry, the norm of $u(t)$ is equal to the one of

$$\int_0^t R_{-s\theta}h(s)ds + u(0) = \int_0^t \alpha(s)R_{-s\theta}(R_{s\theta})v_0 ds + u(0) = \int_0^t \alpha(s)ds\, v_0 + u(0). \tag{97}$$

Since the last term of (97) does not converge to 0 as $t \to +\infty$ the function h is not in the image of $F^+_{A_0}$.

Given a continuous function h in $C_0(\mathbb{R}^+,E)$ evaluating $R^+_{A,P_s}h$ at $t = 0$ we obtain a vector of $\ker P_s$. Similarly we can evaluate $R^-_{A,P_u}h$ and we have a continuous functions

$$r^+_{A,P_s} : C_0(\mathbb{R}^+,E) \to X_s, \;\; h \mapsto ev_0 R^+_{A,P^s}h$$
$$r^-_{A,P_u} : C_0(\mathbb{R}^-,E) \to X_u, \;\; h \mapsto ev_0 R^-_{A,P^u}h.$$

When no ambiguity occurs on the choice of the path A and the projectors we simply denote them by r^+ and r^- respectively. We have the following

Proposition 1.5. (cf. [**3**], Lemma 4.2). The functions r^+ and r^- are linear and continuous applications and map $C^\infty_c((0,+\infty),E)$ onto X_s and $C^\infty_c((-\infty,0),E)$ onto X_u.

Proof. We prove the assertion for r^+. Since R^+_{A,P_s} is bounded, r^+ is bounded. Let v be a vector of E and $\varphi \in C^\infty_c((0,+\infty),\mathbb{R})$ a smooth function such that

$$U = -\int_{\mathbb{R}} \varphi(\tau)X_A(\tau)^{-1}d\tau$$

is an invertible operator on E. We choose $h = \varphi \cdot U^{-1}v$

$$\begin{aligned} r^+h &= -(I-P_s)\int_0^{+\infty} X_A(\tau)^{-1}\varphi(\tau)U^{-1}v d\tau \\ &= -(I-P_s)\int_0^{+\infty} X_A(\tau)^{-1}\varphi(\tau)d\tau U^{-1}v = (I-P_s)v. \end{aligned}$$

□

In the above proof one could remark that choosing a smooth compact supported function ψ on $\mathbb{R}^+$ such that $\int \psi = 1$, for every $v \in X_s$ the function $h(t) = -\psi(t)X_A(t)\cdot v$ still works. However h is at most as regular as X_A.

2. Fredholm properties of F_A

We show that, as in the Hilbert setting, that the Fredholmness of F_A depend on the Fredholmness of the pair of subspaces (W^s_A, W^u_A).

Lemma 2.1. (cf. [**3**], Proposition 5.2). We have the following characterizations of $\ker F_A$ and $\operatorname{Range} F_A$:

$$\ker F_A = \left\{ u \in C^1_0 \mid u(0) \in W^s_A \cap W^u_A \right\} \tag{98}$$

$$\operatorname{Range} F_A = \left\{ h \in C_0 \mid r^+_{A,P_s} h - r^-_{A,P_u} h \in W^s_A + W^u_A \right\} \tag{99}$$

$$\overline{\operatorname{Range} F_A} = \left\{ h \in C_0 \mid r^+_{A,P_s} h - r^-_{A,P_u} h \in \overline{W^s_A + W^u_A} \right\} \tag{100}$$

Proof. We omit the proof of (98) that comes straightforwardly from the definition of stable and unstable subspaces. Let $h \in \operatorname{Range} F_A$ and $u \in C^1_0$ such that $F_A u = h$. By Proposition 1.2 we have a decomposition $C^1_0(\mathbb{R}^+, E) = \ker F^+_A \oplus \operatorname{Range} R^+_{A,P_s}$. Thus

$$\begin{aligned} u^+ &= X_A(t)u_0 + R^+_{A,P_s} h^+ \\ u^- &= X_A(t)v_0 + R^-_{A,P_u} h^- \end{aligned} \tag{101}$$

where u^+ and u^- are the restrictions of u to the positive (respectively negative) real line. Evaluating in 0 and taking the difference of the two equations we obtain

$$W^s_A + W^u_A \ni u_0 - v_0 = r^-_{A,P_u} h - r^+_{A,P_s} h.$$

To prove the converse let $h \in C_0$ such that $r^+h - r^-h \in W^s_A + W^u_A$. By Propositions 1.2 and 1.3 we have u^+ and u^- such that

$$F^+_A u^+ = h^+, \quad F^-_A u^- = h^-. \tag{102}$$

In order to exhibit an element of C^1_0 such that $F_A u = h$ we want to find suitable u^+ and u^- such that $u^- \# u^+$ is a continuous function and continuously differentiable. Hence it's enough to choose u_0 and v_0 in (101) such that

$$u^+(0) = u^-(0) \tag{103}$$

$$u^{+\prime}(0) = u^{-\prime}(0), \tag{104}$$

as before evaluate (101) in 0 and set (103) in the left sides. If we choose u_0 and v_0 such that $u_0 - v_0 = r^+h - r^-h = w$ the joint function $u_- \# u_+$ is continuous. Differentiating the (101)

$$\begin{aligned} u^{+\prime}(t) &= A(t)X_A(t)u_0 + A(t)R^+_{A,P_s} h^+(t) + h^+(t) \\ u^{-\prime}(t) &= A(t)X_A(t)v_0 + A(t)R^-_{A,P_u} h^-(t) + h^-(t) \end{aligned}$$

we get $A(0)(u_0 - v_0 - w) = 0$, hence any choice in $W^s_A \times W^u_A$ that makes $u^- \# u^+$ continuous it also makes it C^1.

The proof of the left inclusion of (100) is completely similar to the above step. Conversely suppose that h belongs to the right set of the (100). Let $\varepsilon > 0$ and $\delta = 1/(\|I - P_s\| \cdot \|U^{-1}\|)$ where U is the operator defined in (1). Set $w = r^+_{A,P_s} h - r^-_{A,P_u} h$. There exists $x \in W^s_A + W^u_A$ such that $|w - x| < \delta$. By Proposition 1.5

$$r^+ h_\delta = (I - P_s)(w - x), \quad h_\delta = -\varphi U^{-1}(w - x)$$

and $\|h_\delta\| < \varepsilon$. Since h_δ has compact support in $(0, +\infty)$ it can be extended on $\mathbb{R}^-$ with the constant value 0. Thus

$$r^+(h - h_\delta) - r^-(h - h_\delta) = w - r^+ h_\delta = x + P_s(w - x)$$

is an element of $W^s_A + W^u_A$ hence, by (99), $h - h_\delta$ is in the image of F_A. $\square$

We conclude the chapter with the relationship between the Fredholm properties of F_A and the Fredholm properties of the pair (W^s_A, W^u_A).

Theorem 2.2. (cf. [**3**], Theorem 5.1). If A is an asymptotically hyperbolic path the following facts hold:

(i) F_A has closed range if and only if $W^s_A + W^u_A$ is closed,
(ii) F_A is onto if and only if $W^s_A + W^u_A = E$,
(iii) F_A is semi-Fredholm if and only (W^s_A, W^u_A) is a semi-Fredholm pair; in this case we also have $\operatorname{ind} F_A = \operatorname{ind}(W^s_A, W^u_A)$.

Proof. If $W^s_A + W^u_A$ is closed the two sets on the right of (98) and (99) are equal, hence $\operatorname{Range} F_A$ coincides with its closure. Conversely, suppose $\operatorname{Range} F_A$ is closed and let w be an element of $\overline{W^s_A + W^u_A}$. By Proposition 1.5, there exists h smooth with compact support such that

$$w = P_s w + (I - P_s)w = P_s w + r^+ h - r^- h$$

hence $r^+h - r^-h$ is in the closure of $W^s_A + W^u_A$. Then, by hypothesis $r^+h - r^-h \in W^s_A + W^u_A$, hence $w \in W^s_A + W^u_A$ and i) is proved. Suppose F_A is onto, that is the range of F_A is closed. By i) $W^s_A + W^u_A$ is also closed and there is an isomorphism of Banach spaces

$$C_0/\operatorname{Range} F_A \to E/W^s_A + W^u_A, \quad h + \operatorname{Range} F_A \mapsto r^+h - r^-h. \tag{105}$$

It is injective by (99). Given $x \in E$ the element $h + \operatorname{Range} F_A$ such that $r^+h - r^-h = (I - P^s)x$ is in the counter-image of $x + W^s_A + W^u_A$, therefore is surjective. The continuity follows straightforwardly from the definition of the norm for a quotient space. In fact, for every $u \in C^1_0$, we have

$$\begin{aligned}\operatorname{dist}(r^+h - r^-h, W^s_A + W^u_A) &\le \operatorname{dist}(r^+h - r^-h, r^+F_A u - r^-F_A u)\\ &\le (\|r^+\| + \|r^-\|)|h - F_A u|.\end{aligned}$$

Taking the infimum over C^1_0 we prove that the application is bounded. We conclude with the open mapping theorem. If F_A is onto the quotient spaces $C_0/\operatorname{Range} F_A$ is the null space, then, by (105) $W^s_A + W^u_A = E$ and the converse is similar, hence ii) is proved. If F_A is semi-Fredholm $\operatorname{Range} F_A$ is closed, hence $W^s_A + W^u_A$ is also closed. By (98) and (105) the index of F_A and the one of the pair (W^s_A, W^u_A) coincide, this proves iii). $\square$

CHAPTER 5

Spectral flow

Given a continuous path of essentially hyperbolic operators, we can define an integer called *spectral flow*. The definition we provide in this chapter generalizes the one given by J.Phillips for paths of Fredholm and self-adjoint operators and coincides with the one given by C. Zhu and Y. Long in [**54**]. We wish to make our notation coherent with the latter, thus we use the notation $[P-Q]$ for relative dimension $\dim(P,Q)$ when $P-Q$ is a compact operator. We show that the definition of spectral flow depends only on the class of fixed-endpoints homotopy of a path. Moreover, the spectral flow of the catenation of two paths is the sum of the spectral flows of the paths, hence we have a group homomorphism

$$\mathrm{sf}_{A_0} : \pi_1(e\mathcal{H}(E), A_0) \to \mathbb{Z}.$$

In chapter 2 we established a homotopy equivalence between the space of essentially hyperbolic operators $e\mathcal{H}(E)$ and the space of idempotents $\mathcal{P}(\mathcal{C})$ of the Calkin algebra, we denoted it by Ψ and defined it as

$$\Psi(A) = P^+(p(A))$$

where $P^+(p(A))$ is the eigenprojector relative to the positive complex half-plane. In Theorem 2.3 we prove that there is a strict relation between the spectral flow and the homomorphism φ defined through the exact sequence of the bundle $(\mathcal{P}(E), \mathcal{P}(\mathcal{C}), p)$. Precisely,

$$\mathrm{sf}_{A_0} = -\varphi_{P^+(A_0)} \circ \Psi$$

Thus the spectral flow inherits all the properties of the index φ. The equality holds for every Banach space and gives a characterization of the paths whose spectral flow is zero and necessary and sufficient conditions in order to have nontrivial spectral flow.

In the last section we extend the definition of spectral flow to asymptotically hyperbolic and essentially hyperbolic paths. We prove that if A is also an *essentially splitting path* the differential operator F_A is Fredholm and

$$\operatorname{ind} F_A = -\mathrm{sf}(A) = \dim(E^-(A(+\infty)), E^-(A(-\infty))).$$

In general, none of the these equalities holds. Counterexamples are known even in the Hilbert spaces.

1. Essentially hyperbolic operators

We recall that an operator A is said *essentially hyperbolic* if $A + \mathcal{L}_c$ is a hyperbolic element of the Calkin algebra $\mathcal{C}$. We denote by $e\mathcal{H}(E)$ the set of the essentially hyperbolic operators.

Lemma 1.1 (Structure of the spectrum)**.** *Let A be a bounded operator, D the set of isolated points of $\sigma(A)$. Then $\partial\sigma(A) \setminus D \subset \sigma_e(A)$.*

Proof. We argue by contradiction: let $\lambda_0 \in \sigma(A) \setminus D$. If $\lambda_0 \notin \sigma_e(A)$ $A - \lambda_0$ is Fredholm of index k. There exists $r > 0$ such that for every $\lambda \in B(\lambda_0, r) \setminus \{\lambda_0\}$ the operator $A - \lambda$ is Fredholm of the same index and $\dim\ker(A-\lambda)$ and $\dim\operatorname{coker}(A-\lambda)$ have constant dimension, by Theorem B.6. Since λ_0 is a boundary point there are $z, w \in B(\lambda_0, r) \setminus \{\lambda_0\}$ such that $z \in \sigma(A)$ and $w \in \rho(A)$. But $A - w \in GL(E)$ implies that $B(\lambda_0, r) \setminus \{\lambda_0\} \subset \rho(A)$, hence $z \in \rho(A)$ and we get a contradiction. □

Theorem 1.2. *An operator B is essentially hyperbolic if and only if $B = A + K$, $K \in \mathcal{L}_c(E)$, A hyperbolic.*

Proof. Let A be a hyperbolic operator. We want to prove that $A + K$ is essentially hyperbolic, in fact, by Proposition B.2 we have $\sigma_e(A + K) = \sigma_e(A)$. Since A is hyperbolic its spectrum does not meet the imaginary axis. Suppose B is essentially hyperbolic. We show that $F = \sigma(B) \cap i\mathbb{R}$ is an isolated set in $\sigma(B)$ and therefore finite (since is compact). We argue by contradiction. Suppose λ is not isolated. By hypothesis $B - \lambda$ is Fredholm. Let C be the connected component of λ in $\sigma(B) \cap i\mathbb{R}$. It is a closed interval of the imaginary axis. Let

$$J = -i(C \cap i\mathbb{R}), \ a = \max J.$$

By Proposition B.5 $B - a$ is Fredholm with the same index as $B - \lambda$. By Theorem B.6 there exists $r > 0$ such that, for every $w \in B(ia, r)$ the operator $B - w$ is Fredholm and

$$\dim \ker(B - w), \quad \dim \operatorname{coker}(B - w)$$

are constants, for every $w \in B(ia, r) \setminus \{ia\}$. Since a connected component is maximal respect to the inclusion ia is not an internal point of $\sigma(B) \cap i\mathbb{R}$, hence there exists $0 < t < r$ such that $i(a + t)$ is not in the spectrum of B, hence $B - i(a + t)$ is invertible and its kernel and co-kernel are the null space, hence $B - i(a - t)$ is also invertible, thus the connected component of λ consists of $\{\lambda\}$. This proves that λ is not an internal point of $\sigma(B)$; it is not isolated neither, by hypothesis. Therefore Lemma 1.1 allows us to conclude that $\lambda \in \sigma_e(B)$ which contradicts the hypothesis.

Now we can write the spectrum as $\sigma(B) = \sigma^+ \cup \sigma^- \cup \{\lambda_1, \ldots, \lambda_n\}$ and choose a family of paths that surrounds $\sigma(B)$ in $\mathbb{C}$, say $\Gamma = \{\gamma^+, \gamma^-, \gamma_1, \ldots, \gamma_n\}$. We have projectors $\{P^+, P^-, P_i\}$. Since all the points of $\sigma(B) \cap i\mathbb{R}$ are isolated *eigenvalues* of B, each $B - \lambda_i$ is a Fredholm operator of index 0. By Theorem 5.28, Ch. IV, §5.4 of [**30**], each eigenprojector P_i has finite rank. Thus

$$B = \left(B(P^+ + P^-) + \sum_{i=1}^{n} P_i \right) - (I - B) \sum_{i=1}^{n} P_i. \tag{106}$$

□

The space $e\mathcal{H}(E)$ is an open subset of $\mathcal{L}(E)$ hence is locally arcwise connected. Theorem 1.2 and Proposition 1.1 allow us to connect the operator B to the square root of unit

$$P^+(B) - P^-(B) + \sum_{i=1}^{n} P_i.$$

Moreover, if there exists a path that connects $2P - I$ and $2Q - I$ in $e\mathcal{H}(E)$, by Theorem 4.2, there exists T invertible such that $TPT^{-1} - Q$ is a compact operator. For instance, if P is a finite rank projector and E is an infinite dimensional space we always have at least the components: the one that contains $2P - I$ and the one of $2(I - P) - I$. We denote them by $e\mathcal{H}_+(E)$ and $e\mathcal{H}_-(E)$ respectively. By Theorem 1.2 we have

$$e\mathcal{H}_+(E) = \{ A \in \mathcal{H}(E) \mid \operatorname{Re} z > 0 \ \forall z \in \sigma_e(A) \}$$
$$e\mathcal{H}_-(E) = \{ A \in \mathcal{H}(E) \mid \operatorname{Re} z < 0 \ \forall z \in \sigma_e(A) \}.$$

These are star-shaped to I and $-I$ respectively, hence contractible. There are infinite dimensional Banach spaces (see Corollary 19 of [**26**]) where the only complemented subspaces are the finite dimensional and the closed infinite dimensional. For such spaces $e\mathcal{H}_+(E)$ and $e\mathcal{H}_-(E)$ are the only connected components of $e\mathcal{H}(E)$.

2. The spectral flow in Banach spaces

A definition of spectral flow for Banach spaces and essentially hyperbolic operators has been given in [**54**]. For sake of completeness we restate it and show that, using the homotopy lifting properties of the locally trivial bundle $(\mathcal{P}(E), \mathcal{P}(\mathcal{C}), p)$, the spectral flow can be computed more easily and some properties, like homotopy invariance can be proved without considering partitions of the unit interval.

Let A be a continuous path on $[0,1]$ of essentially hyperbolic operators. By composition, we have a continuous path

$$a(t) = p(P^+(A(t))) \in \mathcal{P}(\mathcal{C}(E)).$$

By Theorem 4.2, there exists a continuous path of projectors, P such that $p(P) = a$. We have an integer associated to it

$$\mathrm{sf}(A; P) = \big[P(0) - P^+(A(0))\big] - \big[P(1) - P^+(A(1))\big]. \tag{107}$$

Moreover, given Q such that $p(Q) = a$, by Theorem 3.3, we have

$$\begin{aligned}\mathrm{sf}(A; Q) = \big[Q(0) - P^+(A(0))\big] - \big[Q(1) - P^+(A(1))\big] &= [Q(0) - P(0)] + \big[P(0) - P^+(A(0))\big] \\ - [Q(1) - P(1)] - \big[P(1) - P^+(A(1))\big] &= \mathrm{sf}(A; P).\end{aligned}$$

Definition 2.1. Given A as above, we define the *spectral flow* as the integer $\mathrm{sf}(A; P)$ where P is any of the paths of projectors such that $p(P(t)) = P^+(p(A(t)))$. We denote it by $\mathrm{sf}(A)$.

Proposition 2.2. *The spectral flow satisfies the following properties:*

(i) *It is well behaved with respect to the catenation of paths; thus, on the fundamental group, the spectral flow induces a $\mathbb{Z}$-valued group homomorphism;*
(ii) *the spectral flow of a constant path or a path in $\mathcal{H}(E)$ is zero;*
(iii) *it is invariant for free-endpoints homotopies in $\mathcal{H}(E)$ and for fixed-endpoints homotopies in $e\mathcal{H}(E)$.*

Proof. i). Let A, B be two paths such that $A(1) = B(0)$. We can choose paths of projectors P and Q such that $p(P) = \Psi(A)$ and $p(Q) = \Psi(B)$, with $Q(0) = P(1)$. Denote by C and R the catenation of A, B and P, Q respectively. Then,

$$\begin{aligned}\mathrm{sf}(A * B) &= [R_0 - P^+(C_0)] - [R_1 - P^+(C_1)] \\ &= [P_0 - P^+(A_0)] - [Q_1 - P^+(B_1)] = [P_0 - P^+(A_1)] \\ &- [P_1 - P^+(A_1)] + [Q_0 - P^+(B_0)] - [Q_1 - P^+(B_1)] \\ &= \mathrm{sf}(A) + \mathrm{sf}(B).\end{aligned}$$

ii). If A is constant, the path P can be chosen to be constant. Hence the spectral flow is zero. If A is hyperbolic, $P^+(A(t))$ is continuous and can be chosen as lifting path of $\Psi(A)$. Hence,

$$\mathrm{sf}(A) = [P^+(A(0)) - P^+(A(0))] - [P^+(A(1)) - P^+(A(1))] = 0.$$

iii). Let $H\colon I \times I \to e\mathcal{H}(E)$ be a continuous map. There exists $P\colon I \times I \to \mathcal{P}(E)$ such that

$$P(t,s) - P^+(H(t,s)) \in \mathcal{L}_c(E), \text{ for every } t, s.$$

Let $H(\cdot, 0) = A$ and $H(\cdot, 1) = B$. We have

$$\mathrm{sf}(A) = [P(0,0) - P^+(H(0,0))] - [P(1,0) - P^+(H(1,0))].$$

For $i = 0,1$ and every s, the operator $P(i,s) - P^+(H(i,s))$ is compact. The right summand is constant or continuous, whether the homotopy has fixed endpoints in $e\mathcal{H}(E)$ or laying in $\mathcal{H}(E)$. In both cases

$$[P(i,s) - P^+(H(i,s))] = k_i \text{ for all } s, i = 0,1.$$

Thus, $\mathrm{sf}(A) = k_0 - k_1 = \mathrm{sf}(B)$. ∎

Given a projector P of E, we consider the connected component of $e\mathcal{H}(E)$ of the hyperbolic element $2P-I$. On its fundamental group, we have defined the spectral flow. We have the following

Theorem 2.3. *For every projector P,* $\mathrm{sf}_{2P-I} = -\varphi_P \circ \Psi_*$.

PROOF. Given a loop A in $e\mathcal{H}(E)$, there exists a path of projectors P such that $P - P^+(A_t)$ is compact. By definition of φ_P,

$$\varphi_P(\Psi_*(A)) = [P_1 - P_0] = [P_1 - P^+(A_1)] - [P_0 - P^+(A_1)] = -\mathrm{sf}_{2P-I}(A).$$

□

The theorem says, in particular, that the homomorphisms have the same kernel. Hence we have a characterization of the kernel of the spectral flow.

Proposition 2.4. *A path loop A has spectral flow equal to zero if and only if there exists a continuous loop β in $\mathcal{P}(E)$ such that*

$$\beta(t) - P(A(t); \mathbb{H}^+)$$

is compact for every $t \in [0,1]$.

The theorem states also that they have the same images. Thus we have a characterization of the image of the spectral flow also.

Proposition 2.5. *Given a Banach space E and a projector P there exists a loop of essentially hyperbolic operators based on $2P-I$ with spectral flow k if and only if the projector P is connected to a projector Q such that $P-Q$ is compact and $[P-Q]=k$.*

In general all the facts proved for the index φ are true for the spectral flow: if $P \in \mathcal{P}(E)$ and the hypotheses h1), with $m=1$ and h2) hold, the spectral flow is an isomorphism on $\pi_1(e\mathcal{H}(E), 2P-I)$ with $\mathbb{Z}$. If E satisfies the hypotheses of Proposition 8.4 it is surjective.

s-sections of spectral projectors. Essentially hyperbolic operators coincides are the *admissible* operator defined in [**54**], for which C. Zhu and Y. Long define the spectral flow. In order to compute the spectral flow, we use a continuous path of projectors P such that $P - P^+(A(t))$ is compact for every t. According to their DEFINITION 2.1, P is a *s-section* for $P^+(A(t))$ on $[0,1]$. In DEFINITION 2.6 of [**54**], in order to compute the spectral flow, they divide the unit interval in sub-intervals where a *s-section* of *spectral projectors* exists. In fact, globally defined s-sections of spectral projectors do not exist in general. Consider, for instance

$$A(t) = (2P-1) + (2t-1)P_m$$

where P, P_m are projectors such that $PP_m = P_m P = 0$ and P_m has finite rank m and P has infinite-dimensional kernel and image. In conclusion, if we do not put restrictions on the choice of an s-section, we always have *globally defined* s-sections.

3. The Fredholm index and the spectral flow

Given an asymptotically hyperbolic path A in $e\mathcal{H}(E)$ the spectral flow can be defined as follows: since $\mathcal{H}(E)$ is an open subset of $\mathcal{L}(E)$ there exists $\delta > 0$ such that $A((-\infty,-\delta] \cup [\delta,+\infty)) \subset \mathcal{H}(E)$. Then define

$$\mathrm{sf}(A) = \mathrm{sf}(A, [-\delta, \delta]). \tag{108}$$

That the definition does not depend on the choice of δ follows from ii) of Proposition 2.2.

Definition 3.1. A splitting $E = E_1 \oplus E_2$ is called *essential* for an operator T if there exists a compact perturbation T_0 of T such that $T_0(E_i) \subset E_i$.

In fact it is easy to check that the above splitting is essential for an operator T if and only if $[T, P(E_1, E_2)]$ is a compact operator. Given an asymptotically hyperbolic path A we denote by $E^+(+\infty)$ and $E^-(+\infty)$ the images of the spectral projectors of $A(+\infty)$. Similarly we define $E^+(-\infty)$ and $E^-(-\infty)$.

Definition 3.2. An asymptotically hyperbolic path is called *essentially splitting* if and only if the following conditions hold:

(i) the splittings $E = E^+(+\infty) \oplus E^-(+\infty)$ and $E = E^+(-\infty) \oplus E^-(-\infty)$ are essential for $A(t)$, $t > 0$ and $t \leq 0$ respectively;

(ii) $E^-(-\infty)$ is compact perturbation of $E^-(+\infty)$.

We can prove the following

Theorem 3.3. (cf. Theorem 6.3, [**3**]). If A is asymptotically hyperbolic and essentially splitting, the operator F_A is Fredholm and $\operatorname{ind} F_A = \dim(E^-(A(+\infty)), E^-(A(-\infty)))$.

Proof. Denote by $P^\pm(+\infty)$ and $P^\pm(-\infty)$ the spectral projectors of $A(\pm\infty)$. The following paths

$$A_+(t) = A(t) - [A(t), P^-(+\infty)] \quad \text{if } t > 0$$
$$A_-(t) = A(t) - [A(t), P^+(-\infty)] \quad \text{if } t \leq 0$$

are compact perturbations of A and leave respectively $E^\pm(+\infty)$ and $E^\pm(-\infty)$ invariant. Since $A_+(+\infty) = A(+\infty)$ by Lemma 4.2 we have

$$W^s_{A_+} = E^-(+\infty), \; W^u_{A_-} = E^+(-\infty).$$

By Theorem 5.4 W^s_A and W^u_A are compact perturbation of $E^-(+\infty)$ and $E^+(-\infty)$. respectively. By hypothesis $(E^-(+\infty), E^+(-\infty))$ is a Fredholm pair. By Proposition 5.13, the pair (W^s_A, W^u_A) is Fredholm, hence, by Theorem 2.2, F_A is Fredholm and

$$\begin{aligned}\operatorname{ind} F_A = \dim(W^s_A, W^u_A) &= \dim(W^s_A, E^-(+\infty)) + \operatorname{ind}(E^-(+\infty), E^+(-\infty)) \\ &+ \dim(E^+(-\infty), W^u_A) = \dim(E^-(+\infty), E^-(-\infty)).\end{aligned}$$

□

For essential splitting path we are able to compute the spectral flow. First we need the following

Lemma 3.4. *Let A be an asymptotically hyperbolic and essentially hyperbolic path. It is essentially splitting also if and only if the set $\{ P^+(A(t)) \mid t \in \mathbb{R}\}$ is contained in the same class of compact perturbation.*

Proof. Suppose A is essentially splitting and consider the restriction on half line $\mathbb{R}^+$; hence, using the decomposition $E = E^+ \oplus E^-$, we can write

$$A(t) = \begin{pmatrix} A_+ & K_\pm \\ K_\mp & A_- \end{pmatrix}$$

where $K_\pm$ and $K_\mp$ are compact operators because A is essentially splitting. Since $A_+(+\infty)$ is hyperbolic there exists $t_+ > 0$ such that $A_+([t_+, +\infty)) \subset \mathcal{H}(E^+)$ and

$$\|P^+(A_+(t)) - P^+(A_+(+\infty))\| < 1.$$

But $A_+(+\infty)$ has positive spectrum, hence $P^+(A_+(+\infty)) = I$. Since at distance smaller than 1 from the identity there are not projectors other than the identity, $P^+(A_+(t))$ is the identity too on E^+ if $t \in [t_+, +\infty)$. Since A is essentially hyperbolic on E, A_+ is also essentially hyperbolic on E^+ and we have a path in $[0, t^+]$

$$A_+ \colon [0, t_+] \to e\mathcal{H}(E^+), \; A_+(t_+) \in e\mathcal{H}_+(E^+);$$

since $e\mathcal{H}_+(E^+)$ is a connected component $A_+([0, t^+])$ is contained in $e\mathcal{H}_+(E^+)$. Thus the positive eigenspaces have finite co-dimension for every $t > 0$. It is easy to check that two projectors

$P^+(A_+(s))$ and $P^+(A_+(s'))$ with ranges of finite co-dimension have compact difference: the operator

$$P^+(A_+(s)) - P^+(A(s')) = (P^+(A_+(s)) - I) + I - P^+(A(s'))$$

is sum of finite rank operators. Similarly $P^+(A_-(+\infty)) = 0$ and there exists $t_- < 0$ such that the positive projector of $A_-(t)$ is zero for $t \leq t_-$. Thus $A_-(t)$ for $0 \geq t \geq t_-$ is a path of continuous essentially hyperbolic operators that intersect a connected component, that is $e\mathcal{H}_-(E^-)$; by continuity of A the whole path lies $e\mathcal{H}_-(E^-)$. If $t_0 \geq \max\{t_+, -t_-\}$ we can write for every $t \geq 0$

$$\begin{aligned} P^+(A(t)) &\sim_c P^+(A_+(t)) + P^+(A_-(t)) \sim_c P^+(A_+(t_0)) + P^+(A_-(t_0)) \\ &= I_{E^+} \oplus 0_{E^-} = P^+(+\infty) \end{aligned}$$

where $\sim_c$ denotes the relation of compact perturbation. Similarly, we can prove that $P^+(A(t))$ is compact perturbation of $P^+(-\infty)$ for every $t \leq 0$. By hypothesis, $P^+(+\infty) - P^+(-\infty)$ is compact, hence all the positive projectors (and thus the negative) are compact perturbation one of each other. Conversely, if $\{P^+(A(t) \mid t \in \mathbb{R}\}$ is in the same class of compact perturbation, we have

$$[A(t), P^-(+\infty)] = [A(t), P^-(A(t))] - [A(t), P^-(A(t)) - P^-(+\infty)]$$

for $t > 0$. The first term of the second member is 0, the last is compact by hypothesis. The proof for $t \leq 0$ is similar. □

We conclude the chapter with the proof that for an asymptotically hyperbolic path which is essentially splitting and essentially hyperbolic there holds $\mathrm{sf}(A) = -\operatorname{ind} F_A$.

Theorem 3.5. *Let A be an asymptotically hyperbolic path and essentially hyperbolic such that $\{P^+(A(t)) \mid t \in \mathbb{R}\}$ are compact perturbation of each other. Then*

$$\mathrm{sf}(A) = -\dim(E^-(A(+(\infty))), E^-(A(-\infty))) \tag{109}$$

Proof. Let $\delta > 0$ such that $A((-\infty, -\delta] \cup [\delta, +\infty)) \subset \mathcal{H}(E)$. Since all the projectors are compact perturbation of each other, the spectral flow can be computed by using $P \equiv P^+(A(\delta))$. Hence

$$\mathrm{sf}(A) = [P^+(A(\delta)) - P^+(A(-\delta))].$$

Since A is hyperbolic in $(-\infty - \delta] \cup [\delta, +\infty)$ the path $P^+(A(t))$ is continuous on this subset. By Theorem 3.3,

$$\dim(E^-(A(+\infty)), E^-(A(-\infty))) = -[P^+(A(\delta)) - P^+(A(-\delta))] = -\mathrm{sf}(A).$$

□

Thus, Theorems 3.5 and 3.4 give for essentially splitting paths in $e\mathcal{H}(E)$ the equality $\operatorname{ind} F_A = -\mathrm{sf}(A)$. If A is not essentially splitting counterexamples are known even in a Hilbert space; here we describe the Example 7 of [**3**], Ch. 7.

Example 3.6. In Proposition 4.3 we showed how to patch a discontinuity of a path A without changing the stable space of A_+ and the unstable space of A_-. Here we describe another method; let X and Y be closed isomorphic subspaces that admit isomorphic topological complements X' and Y'. Define $P = P(X, X')$ and $Q = P(Y, Y')$. We have a piecewise continuous path

$$A(t) = \begin{cases} 2P - I & t \geq 1 \\ 2Q - I & t \leq -1 \end{cases};$$

call A^s and A^u the restrictions of A to the positive and negative half-line; by Proposition 4.3, we know that $W^s_{A^s} = X$, $W^u_{A^u} = Y$. There exists an invertible operator T such that $TQT^{-1} = P$

which means, in particular, that $TY = X$. If $GL(E)$ is connected, there also exists a path U that $U(-1) = I$ and $U(1) = T$. Define

$$A_U(t) = \begin{cases} 2P - I & t \geq 1 \\ U(t)(2P - I)U(t)^{-1} & -1 \leq t \leq 1 \\ 2Q - I & t \leq -1 \end{cases} ;$$

the path A_U is continuous and hyperbolic, hence, by ii) of Proposition 2.2, $\mathrm{sf}(A_U)$ is zero. By iii) of Theorem 2.2, the operator F_A is Fredholm if and only if the pair (X, Y) is Fredholm. Thus,

$$\mathrm{sf}(A_U) \neq -\operatorname{ind} F_{A_U}$$

if (X, Y) is a Fredholm pair of index $k \neq 0$.

The result of Theorem 3.5 is meaningful for Hilbert spaces too. It is interesting detecting a class of paths of essentially hyperbolic operators such that (109) holds and the spectral flow does not depend on the endpoints alone, but also on the homotopy class of the path.

APPENDIX A

The Cauchy problem

Let E be a Banach space and let f be a function defined on a open subset $\Omega \subseteq \mathbb{R} \times E$ with values in E. We denote by $\Omega_t = \{u \in E \mid (t,u) \in \Omega\}$. We require f to have these properties:

(i) f is continuous
(ii) for any $t \in \mathbb{R}$ such that $\Omega_t \neq \emptyset$ there exists an open subset $\mathbb{R} \supseteq U_t \ni t$ and a constant M such that $f(t', \cdot)$ is a Lipschitz function with constant M for every t' in U_t.

Theorem A.1 (Cauchy)**.** *Let f and Ω be as above. Then for every $(t_0, u_0) \in \Omega$ there exists an open ball $B(t_0, r)$ and $u \in C^1(B(t_0,r), E)$ such that $(t, u(t)) \in \Omega$ for every $t \in B(t_0, r)$ and*

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases} ;$$

moreover, if there exists an open interval $J \ni t_0$ and $v \in C^1(J, E)$ satisfying the same conditions as $(u, B(t_0, r))$ u and v coincide in the intersection $B(t_0, r) \cap J$.

Proof. Set $z_0 = (t_0, u_0)$. There exists an open neighbourhood of z_0, $D(t_0, a) \times B(u_0, b') \subseteq \Omega$. By compactness of $D(t_0, a)$ we can find a open ball $B(u_0, b)$ such that $f(D(t_0, a) \times B(u_0, b))$ is bounded, call m its bound. For any $r \leq a$ let E_r be the space $C(J_r, B(u_0, b))$ endowed with the supremum topology. If $v \in E_r$ $(t, v(t)) \in J_r \times B(u_0, b) \subseteq \Omega$, thus we can define

$$\Phi_f(v) = u_0 + \int_{t_0}^{t} f(s, v(s)) ds.$$

Since $\left| \int_{t_0}^{t} f(s, v(s)) ds \right| \leq rM$ for every $t \in J_r$ we have

$$\Phi_f(v)(t) \in B(u_0, mr).$$

Still by compactness of $D(t_0, a)$, by property iii), there exists $k \in \mathbb{R}^+$ such that for every $t \in D(t_0, a)$ the function $f(t, \cdot)$ is Lipschitz with constant k in Ω_t. Let $v, w \in E_r$. Hence

$$\|\Phi_f(v) - \Phi_f(w)\| \leq kr\|v - w\|.$$

If we choose $rm < b$ and $kr < 1$ we make Φ_f a contraction of E_r into itself. Hence Φ_f has a unique fixed point u. Then $(u, B(t_0, r))$ fulfills the requirements. $\square$

Proposition A.2. *Suppose f and Ω as in the theorem. If u and v are two solutions defined on a connected open interval J and coincide in $t_0 \in J$ then u and v coincide in J.*

Proof. Let $A = \{t \in J \mid u(t) = v(t)\}$. Since u and v are continuous A is a closed subset of J. By hypothesis we know that is nonempty. We prove that A is also open (hence $A = J$). Let $t' \in A$, $u_0 = u(t') = v(t')$. By Theorem A.1 there exists a solution $w \in C^1(B(t', r_0), E)$ such that $w(t') = u_0$. By uniqueness of local solutions $B(t', r_0) \subseteq A$. $\square$

Definition A.3. Let (u, J) be a solution. Then (v, J') is a *prolongation* of (u, J) if $J \supseteq I$ and $v(t) = u(t)$ for every $t \in J$.

Using Zorn's Lemma it is easy to prove that for a solution (u, J) there exists a unique maximal prolongation (v, J'). There many criterions to establish when a solution (u, J) can be extended to a bigger interval J'. Here's an example:

Lemma A.4. *Let $(u, B(t_0, r))$ be a solution of (f, Ω) and suppose that the set $\{f(t, u(t))\}$ is bounded in E and I_{t_0+r} and I_{t_0-r} are nonempty. Then there is a prolongation $(w, B(t_0, r'))$, $r' > r$.*

The Lemma can be used to prove the existence of global maximal solution in some particular case. First we need the

Lemma A.5 (Gronwall)**.** *Let w, ϕ, ψ be continuous real valued functions on the compact interval $[a, b]$ such that the estimate*

$$w(t) \leq \phi(t) + \int_a^t w(s)\psi(s)ds;$$

for every $a \leq t \leq b$. Then for every t in the interval the estimate

$$w(t) \leq \phi(t) + \int_a^t \phi(s) \left(\exp \int_s^t \psi(\xi)d\xi \right) ds$$

also holds.

Using Gronwall's Lemma we can prove the following statement.

Proposition A.6. *Suppose Ω is the product $J \times E$ where J an open connected interval of $\mathbb{R}$. If for every $t_0 \in J$ there exists a function $k \in C(J, E)$ such that*

$$|f(t, u) - f(t, v)| \leq k(|t - t_0|)|u - v|, \quad t_0 \in J;$$

then every solution admits a prolongation to the whole interval J.

It is easy to check that the pair (f, Ω) satisfies the three conditions of the Theorem A.1. Thus, given (t_0, u_0), there exists a maximal solution $(u, B(t_0, r))$. Since the domain Ω is a product the sets I_{t_0+r} and I_{t_0-r} are nonempty. Moreover, for every $t \in B(t_0, r)$ we have the estimate

$$|u(t)| \leq |u_0| + \int_{t_0}^t k(|s - t_0|)|u(s) - u_0|ds;$$

applying the Gronwall's Lemma we can conclude that u is bounded, hence admits a prolongation by Lemma A.4.

The Proposition A.6 applies to the particular case: let $\Omega = J \times E$ be the domain of f and $A \in C(J, \mathcal{L}(E))$, $b \in C(J, E)$ be two continuous functions. The Cauchy problem

$$f(t, u) = A(t)u + b(t), \quad \Omega = J \times E$$

admits unique global solutions defined on J. We conclude by remarking that the theorems of existence, prolongation and the related results can be restated in a more general setting: by *step function* we mean a finite sum of characteristic functions. Let $\mathscr{C}(J, E)$ be the vector space of step function. As a subset of $L^\infty(J, E)$ we can consider the closure $\overline{\mathscr{C}}$.

Definition A.7. An element of $\overline{\mathscr{C}}$ is called *regulated function*.

Here are the hypotheses of the Theorem A.1 for regulated functions: we f and Ω to solve the conditions

(i) for every $w \in C(J, E)$ such that $\{(t, w(t))\} \subseteq \Omega$ $f(t, w(t))$ is regulated,
(ii) for any point $(t, u) \in \Omega$ there are an open neighbourhood $B(t, r) \times B(u, b)$ and $M \in \mathbb{R}^+$ such that f is bounded $B(t, r) \times B(u, b)$, and $f(s, \cdot)$ is Lipschitz with constant M.

For the proofs and more details see [**18**].

APPENDIX B

Fredholm operators

Given an operator $T\colon E\to F$ we can consider the spaces $\ker T$ and $E/\mathrm{Range}T$. The latter is called co-kernel and is denoted by $\mathrm{coker}T$.

Definition B.1. An operator $T\in\mathcal{L}(E,F)$ is called semi-Fredholm if $\ker T$ and $\mathrm{Range}T$ are closed and at least one of $\ker T$ and $\mathrm{coker}T$ has finite dimension. It is said Fredholm if both have finite dimension.

The *Fredholm index* of a (semi)Fredholm operator is $\mathrm{ind}T=\dim\ker T-\dim\mathrm{coker}T$. We denote by $\mathcal{F}(E,F)$ the set of Fredholm operators.

Proposition B.2. *If $T\colon E\to F$ is a Fredholm operator and K a compact operator then $T+K$ is Fredholm operator and $\mathrm{ind}(T+K)=\mathrm{ind}T$.*

Proposition B.3. *An operator $T\in\mathcal{L}(E,F)$ is Fredholm if and only if is essentially invertible, that is, there exists $S\in\mathcal{L}(F,E)$ such that*

$$\begin{aligned}ST&=I+K\\ TS&=I+H\end{aligned}$$

where K and H are compact operators on E and F respectively.

Proof. Since $\ker T$ and $\mathrm{Range}T$ are complemented subspaces of E and F respectively there are $X\subset E$ and $Y\subset F$ such that $E=\ker T\oplus X$ and $F=Y\oplus\mathrm{Range}T$. The restriction of T to X maps isomorphically X onto $\mathrm{Range}T$, let σ be its inverse. Hence, given a pair (y,r) in F we have

$$T\circ(0\oplus\sigma)(y,r)=r;$$

hence

$$T\circ(0\oplus\sigma)=P(\mathrm{Range}T,Y)=I-P(Y,\mathrm{Range}T)$$

where the last term denotes the projector onto Y along $\mathrm{Range}T$. Since Y has finite dimension it is a perturbation of the identity by a finite-rank operator, hence compact. Similarly

$$(0\oplus\sigma)\circ T=P(X,\ker T)=I-P(\ker T,X)$$

is a compact perturbation of the identity. Hence we can choose $S=0\oplus\sigma$. In order to prove the converse observe that if S is an essential inverse of T we have the inclusions

$$\begin{aligned}\ker T&\subset\ker S\circ T=\ker(I+K),\\ \mathrm{Range}T&\supset\mathrm{Range}T\circ S=\mathrm{Range}(I+H)\end{aligned}$$

where the right members have finite dimension and finite co-dimension because by Proposition B.2 a compact perturbation of the identity is Fredholm. $\square$

Proposition B.4. *Let $A\in\mathcal{L}(E,F)$ and $B\in\mathcal{L}(F,G)$ be two Fredholm operators. Then BA is Fredholm and its index is $\mathrm{ind}B+\mathrm{ind}A$.*

PROOF. For the sake of simplicity we denote by k and c the dimension of the kernel and the co-kernel respectively. Set $T = BA$. Since A is Fredholm there exists a finite-dimensional subspace $X \subset E$ such that

$$\ker T = \ker A \oplus X;$$

the restriction of A to X is an isomorphism with $\ker B \cap \operatorname{Range} A$. Thus

$$k(T) = k(A) + \dim \ker B \cap \operatorname{Range} A. \tag{110}$$

The image of T is $B(\operatorname{Range} A)$. Consider the inclusion of subspaces

$$B(\operatorname{Range} A) \subset \operatorname{Range} B \subset G;$$

the co-dimension of $B(\operatorname{Range} A)$ in $\operatorname{Range} B$ can be computed as the co-dimension of $\operatorname{Range} A + \ker B$ in F, hence

$$\begin{aligned} c(T) &= c(B) + \operatorname{codim}(\operatorname{Range} A + \ker B) \\ &= c(B) + \operatorname{codim} \operatorname{Range} A - (k(B) + \dim \operatorname{Range} A \cap \ker B). \end{aligned} \tag{111}$$

Thus adding the results of (110) and (111) we obtain

$$\begin{aligned} \operatorname{ind} T = k(T) - c(T) = k(A) + \dim \ker B \cap \operatorname{Range} A - c(B) - c(A) \\ - k(B) - \dim \operatorname{Range} A \cap \ker B = \operatorname{ind} A + \operatorname{ind} B. \end{aligned}$$

□

Proposition B.5. *The subset $\mathcal{F}(E, F) \subset \mathcal{L}(E, F)$ is open and the Fredholm index is a locally constant function with values in $\mathbb{Z}$.*

PROOF. We use the Proposition B.3. Let T be a Fredholm operator and S be an essential inverse, that is $TS - I$ is a compact operator. For every operator H such that $\|H\| < \|S\|^{-1}$ we have

$$(T + H)S = TS + HS = I + K + HS = (I + HS) + K$$

where K is a compact operator; since $I + HS$ is invertible we can multiply both terms by its inverse and obtain

$$(T + H)S(I + HS)^{-1} = I + K(I + HS)^{-1}$$

hence $S(I_F + HS)^{-1}$ is an essential right inverse for $T + H$. Similarly we can write $S(T + H) = I + SH + K'$ where K' is compact. Since $I + SH$ is invertible we obtain

$$(I + SH)^{-1}S(T + H) = I + (I + SH)^{-1}K'$$

and prove that $T + H$ has an essential left inverse also. Hence $B(T, \|S\|^{-1}) \subset \mathcal{F}(E, F)$. We compute the index of $T + H$ using the Propositions B.4 and B.2

$$\begin{aligned} \operatorname{ind}(T + H) = -\operatorname{ind} S(I + HS)^{-1} = -\operatorname{ind} S - \operatorname{ind}(I + HS)^{-1} \\ = -\operatorname{ind} S = \operatorname{ind} T. \end{aligned}$$

□

The preceding statement and the Proposition B.2 say that the index of a Fredholm operator is stable under small or compact perturbations. Here we state a more specific result regarding the dimension of the kernel and the co-kernel

Theorem B.6. (cf. THEOREM 5.31, ch. IV §5.5 of [**30**].) *Let T be a semi-Fredholm operator from E to F and A bounded. There exists $\delta > 0$ such that, for every $0 < |\lambda| < \delta$ the quantities*

$$\dim \ker(T + \lambda A), \quad \dim \operatorname{coker}(T + \lambda A)$$

are constants.

In order to prove the theorem we need the following lemma.

Lemma B.7. *Let T be an operator with finite-dimensional kernel from E to F and $X \subset E$ a closed subspace. Then $T(X) \subset F$ is closed.*

Proof. We use the fact that an open linear operator maps closed subspaces containing the kernel in closed subspaces. The purpose is to show that there exists $Y \subset E$ closed such that $T(Y) = T(X)$ and $Y \supset \ker T$. Such space can be taken as $Y = \ker T + X$ which is closed because the kernel has finite dimension. □

We are now able to prove the theorem. First we show that the theorem cannot be extended to a neighbourhood of zero. Let P be a projector of finite co-dimension non surjective, hence it is a Fredholm operator and let $A = I - P$. Let $x \in \ker(P + \lambda A)$ with $\lambda \neq 0$. We can write

$$Px = -\lambda(I - P)x$$

hence both $-\lambda(I - P)x$ and Px are zero. Since $\lambda \neq 0$ we also have $(I - P)x = 0$ thus $x = Px + (I - P)x = 0$. We have proved that $P + \lambda A$ is injective, but P is not injective.

Suppose first that $\ker T$ has finite dimension. Using induction we can build two decreasing sequences of closed subspaces $\{E_n\}$, $\{F_n\}$ of E and F respectively as follows

$$\begin{cases} E_0 = E \\ E_{n+1} = A^{-1}(TE_n) \end{cases} \qquad \begin{cases} F_0 = F \\ F_{n+1} = TE_n \end{cases}$$

these are all closed spaces by the previous lemma. We have $AE_n \subset F_n$ and $TE_n = F_{n+1}$ for any $n \in \mathbb{N}$. Let

$$E_\omega = \bigcap_{n \geq 0} E_n$$

$$F_\omega = \bigcap_{n \geq 0} F_n;$$

If $x \in \ker(T + \lambda I)$ and $\lambda \neq 0$ using induction on the equality $\lambda^{-1}Tx = -Ax$ it is easy to check that $x \in E_\omega$. It is clear that $T(E_\omega) \subset F_\omega$; we prove now that $T(E_\omega) = F_\omega$. Given $y \in E_\omega$

$$T^{-1}(\{y\}) \cap E_\omega = T^{-1}(\{y\}) \cap \big(\bigcap_{n \geq 1} E_n\big) = \bigcap_{n \geq 1} \big(T^{-1}(\{y\}) \cap E_n\big);$$

since $F_{n+1} = T(E_n)$ for $n \geq 1$ the last member is a decreasing intersection of finite-dimensional, since $\ker T$ has finite dimension, of affine subspaces. Hence the intersection is nonempty. Call T_ω the restriction of T to E_ω. We proved that T_ω is surjective and Fredholm. By Proposition B.5 there exists $\delta > 0$ such that the operator $T_\omega + \lambda A_\omega$ is Fredholm, of constant index, and surjective. If $|\lambda| < \delta$ and $\lambda \neq 0$

$$\operatorname{ind}(T_\omega + \lambda A_\omega) = \dim \ker(T + \lambda A).$$

and is still constant as long as $\lambda \neq 0$. If $\operatorname{coker} T$ has finite dimension the same steps can be repeated for T^*.

APPENDIX C

Spectral decomposition

We recall some basic definitions and results on spectral theory. Given a Banach algebra $\mathcal{B}$ with unit 1, the spectrum of an element $x \in \mathcal{B}$ is the set

$$\{ \lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin G(\mathcal{B}) \}$$

where $G(\mathcal{B})$ is set of invertible elements of the algebra; since this is an open subset of the algebra, the spectrum is a closed subset of the complex plane. It is usually denoted by $\sigma(x)$ or $\sigma_{\mathcal{B}}(x)$. Moreover, the following properties hold:

(i) $\sigma(x)$ is compact;
(ii) $\sigma(tx) = t\sigma(x)$ and $\sigma(x+t) = \sigma(x) + t$;
(iii) given p and q idempotents elements of $\mathcal{B}$ such that $p + q = 1$, we define two subalgebras

$$\mathcal{B}_p = \{pyp : y \in \mathcal{B}\}, \quad \mathcal{B}_q = \{qyq : y \in \mathcal{B}\}.$$

Each of the elements pxp and qxq has a spectrum in the respective algebra it belongs and

$$\sigma(x) = \sigma_{\mathcal{B}_p}(pxp) \cup \sigma_{\mathcal{B}_q}(qxq).$$

Definition C.1. Let $\Omega \subset \mathbb{C}$ be an open subset of the complex plane and $K \subset \Omega$, compact. Let Γ be a collection of continuous curves $\gamma_i : [a,b] \to \mathbb{C}$ such that $\gamma_i \cap K = \emptyset$. We say that Γ *surrounds* K in Ω if

$$\mathrm{Ind}_\Gamma(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{d\lambda}{\lambda - \zeta} = \begin{cases} 1 & \text{if } \zeta \in K \\ 0 & \text{if } \zeta \notin \Omega \end{cases}$$

where $\mathrm{Ind}_\Gamma(\zeta)$ is the sum of $\mathrm{ind}_{\gamma_i}(\zeta)$.

Lemma C.2. *Suppose $\mathcal{B}$ is a Banach algebra, $x \in \mathcal{B}$, $\alpha \in \mathbb{C}$, $\alpha \notin \sigma(x)$ and Γ surrounds $\sigma(x)$ in Ω. Then*

$$\frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n (\lambda - x)^{-1} d\lambda = (\alpha - x)^n.$$

for every $n \in \mathbb{Z}$.

The proof is made by induction on n. The case $n = 0$ is provided by the Neumann series (see [**46**], Lemma 10.24).

Let $x \in \mathcal{B}$ and σ_+ and σ_- closed subsets of $\sigma(x)$ such that $\sigma(x) = \sigma_- \cup \sigma_+$ and $\sigma_- \cap \sigma_+$. There is a pair of open subsets

$$\sigma_+ \subset \Omega_+, \ \sigma_- \subset \Omega_-$$
$$\partial\Omega_+ = \gamma_+, \ \partial\Omega_- = \gamma_-$$

where $\gamma_\pm$ are continuous curves and $\gamma_\pm$ surrounds $\sigma_\pm$ in $\Omega_\pm$.

Theorem C.3. *Let x and $\gamma_\pm$ as above. Then, the integrations*

$$p^+(x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma^+} (\lambda - x)^{-1} d\lambda$$
$$p^-(x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma^-} (\mu - x)^{-1} d\mu.$$

are projectors of $\mathcal{B}$, called spectral projectors. *In the Banach algebras $p^+\mathcal{B}p^+$ and $p^-\mathcal{B}p^-$ the elements xp^+ and xp^- have spectrum σ_+ and σ_- respectively.*

Using the *Fubini-Tonelli* theorem it can be checked that $p^+p^- = p^-p^+ = 0$. Applying the previous lemma with $n = 0$ we also have $p^+(x) + p^-(x) = 1$. Hence

$$p^{+^2} = p^+, \quad p^{-^2} = p^-.$$

Theorem C.4. Let $\Omega \subset \mathbb{C}$ and $x \in \mathcal{B}$ such that $\sigma(x) \subset \Omega$. Let f be a holomorphic function on Ω and Γ surrounding $\sigma(x)$ in Ω. Thus, the integration

$$\widehat{f}(x) = \frac{1}{2\pi i}\int_\Gamma f(z)(x-z)^{-1}dz$$

defines an element of $\mathcal{B}$. The following properties hold:

(i) $\widehat{fg}(x) = \widehat{f}(x)\widehat{g}(x)$;
(ii) $\widehat{g \circ f}(x) = \widehat{g}(\widehat{f}(x))$;
(iii) $\sigma(\widehat{f}(x)) = f(\sigma(x))$;
(iv) on the subset $\{x : \sigma(x) \subset \Omega\}$, $\widehat{f}$ is continuous.

Example C.5. Let A be a bounded operator such that $\|A\| < 1$. There exists R such that $R^2 = I + A$. We consider the power series expansion in a neighbourhood of the origin of $f(z) = \sqrt{1+z}$. Thus, $R = \widehat{f}(A)$ is a solution of the equation. Moreover, the path

$$t \mapsto \widehat{f}(tA)$$

is continuous and connects the operator R to the identity.

APPENDIX D

Continuous sections of linear maps

We recall some classical theorem that regards continuous selection We begin with the result of Bartle and Graves. Let X and Y be Banach spaces and let $L\colon E \to F$ be a linear surjective application. We do not require L to be bounded. Define

$$I(L) = \sup_{|y|=1} \inf_{Lx=y} |x|.$$

It is easy to check that if L is injective also and L^{-1} is bounded $I(L) = \|L^{-1}\|$. Let T be a paracompact Hausdörff space. The conditions of the theorem are the following: for every $t \in T$ we are given a bounded surjective operator $S(t) \in \mathcal{L}(X, Y)$ which is *strongly continuous*. Define

$$M_0(S) = \sup_{t \in T} \|S(t)\|, \; N_0(S) = \sup_{t\in T} I(S(t))$$

the map $s\colon C(T, X) \to C(T, Y), \; x \mapsto sx(t) = S(t)x(t)$ is well defined. Structures of Banach space on $C(T, X)$ and $C(T, Y)$ are not required.

Theorem D.1. *Suppose both M_0 and N_0 are finite. Fix $N > N_0$ and $\varepsilon > 0$. For every $y \in C(T, Y)$ there exists $x \in C(T, X)$ such that $sx = y$ and*

$$|x(t)| \leq N|y(t)| + \varepsilon. \tag{112}$$

for every $t \in T$.

For the proof see [**7**], Theorem 4. As application of this results consider the situation of two Banach spaces E, F. Let T be a topological space and $y \in C(T, F)$ and $x \in C(T, E)$ such that $x(t) \neq 0$ for every $t \in T$. Let $\hat{x} = x/|x|$

Corollary D.2. For every $\delta, \varepsilon > 0$ there exists $U_\delta^\varepsilon \in C(T, \mathcal{L}(E, F))$ such that $U_\delta^\varepsilon(t)x(t) = y(t)$ and

$$\|U_\delta^\varepsilon(t)\| \leq (1+\delta)\frac{y(t)}{|x(t)|} + \varepsilon$$

Proof. We briefly check that the conditions of the theorem are fulfilled. As Banach spaces we choose $X = \mathcal{L}(E, F)$ and $Y = F$. Since $x(t) \neq 0$ for every $t \in T$ we have a map

$$S\colon C(T, \mathcal{L}(E, F)) \to C(T, F), \; U \mapsto U \cdot (x/|x|).$$

Strong continuity is trivial. Let $t \in T$ and $y \in F$. By Hahn-Banach there exists $\xi \in E^*$ such that $\langle \xi, \hat{x}(t)\rangle = 1$, $|\xi| = 1$. Then the operator

$$U \cdot z = \langle \xi, z\rangle y$$

maps $\hat{x}(t)$ in y and $\|U\| = |y|$. On the other side there can be no operator U such that $U\hat{x}(t) = y$ and $\|U\| < |y|$. This proves that $s(t)$ is surjective and $I(s(t)) = 1$. Thus $N_0(S) = 1$ and clearly $M_0(S) = 1$. Fix $\delta, \varepsilon > 0$. Let $y \in C(T, F)$ be a continuous function. Since $1 + \delta > N_0$ there exists $U \in C(T, \mathcal{L}(E, F))$ such that

$$U(t)\hat{x}(t) = y(t)/|x(t)|, \; \|U(t)\| \leq (1+\delta)\frac{|y(t)|}{|x(t)|} + \varepsilon.$$

Thus $U(t)x(t) = y(t)$ for every $t \in T$. $\square$

Proposition D.3. *Let E, F Banach spaces and $f \in \mathcal{L}(E, F)$ a bounded surjective operator. There exists a continuous map $s \in C(F, E)$ such that $f \circ s = id$.*

Proof. The Theorem D.1 can be applied as follows: since F is metric is a paracompact space. For every $x \in F$ we define

$$L(x)\colon C(F, E) \to C(F, F),\ s \mapsto f \circ s.$$

Since L is constant on F is clearly strongly continuous, in fact is bounded. Then there exists $s \in C(F, E)$ such that $Ls = id$, thus $f \circ s = id$. □

Proposition D.4. *Let $\mathcal{A}$ and $\mathcal{B}$ Banach algebras, $\varphi\colon \mathcal{A} \to \mathcal{B}$ a surjective homomorphism. There are local section of $\varphi\colon G(\mathcal{A}) \to \varphi(G(\mathcal{A}))$.*

Proof. First let s be a continuous right inverse of $\varphi\colon \mathcal{A} \to \mathcal{B}$. Such a section exists by Proposition D.3. Let y_0 in $\varphi(G(\mathcal{A}))$ and $x_0 \in G(\mathcal{A})$ such that $\varphi(x_0) = y_0$. We can define another right inverse of φ such that

$$S(y) = s(y) + x_0 - s(y_0),\ S(y_0) = x_0.$$

Since $G(\mathcal{A}) \subset \mathcal{A}$ is open, there exists $\delta > 0$ such that $B(x_0, \delta) \subset G(\mathcal{A})$. Thus $S^{-1}(B(x_0, \delta)) \subset \varphi(G(\mathcal{A}))$ and the restriction of S to $S^{-1}(B(x_0, \delta))$ is a local section on a neighbourhood of y_0. □

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