Università di Pisa

Dipartimento di Matematica
Corso di Laurea in Matematica

Lecture notes of

Geometric Measure Theory

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Contents

Chapter 1

Recap of basic Measure Theory

The aim of this chapter is to recall briefly the basic notions of Measure Theory which can be found in any book of Measure Theory (see, for instance, [4], [5], [6]).

1.1 Different notion of measures

Positive measures

Let be given a set $\mathbb{X}$ and a σ-algebra $\mathcal{M}$ in $\mathbb{X}$.

Definition 1.1.1 (Positive measure)**.** A positive measure μ over $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, +\infty]$ such that

- $\mu(\emptyset) = 0$,
- μ is σ-additive, that is, given $(E_n)_n$ a countable pairwise disjoint collection of sets in $\mathcal{M}$, then

$$\sum_n \mu(E_n) = \mu\left(\bigcup_n E_n\right).$$

Definition 1.1.2 (Borel measure)**.** Given a topological space $\mathbb{X}$, we denote as $\mathcal{B}(\mathbb{X})$ the σ-algebra of Borel sets; a positive measure μ over $\mathcal{B}(\mathbb{X})$ is said to be Borel.

Definition 1.1.3 (Support of a measure)**.** Let μ be a Borel measure on a topological space $\mathbb{X}$. The support of μ is the closure of the set defined by the point $x \in \mathbb{X}$ s.t. x has a fundamental system of neighbourhoods $V(x)$ s.t. $\mu(U) > 0$ for all $U \in V(x)$. It is usually denoted as $\mathrm{supp}(\mu)$.

Remark 1.1.4. By definition 1.1.3, the support of a measure μ is always a closed subset of $\mathbb{X}$. However, if $(\mathbb{X}, d)$ is a metric space, for all $x \in \mathrm{supp}(\mu)$ for all $r > 0$ there exists a ball $B(x, \delta)$ s.t. $0 < \delta < r$ and $\mu(B(x, r)) > 0$.

Definition 1.1.5 (Locally finite measure)**.** We say that a Borel measure μ on a topological space $\mathbb{X}$ is locally finite if each point $x \in \mathbb{X}$ has an open neighbourhood $U(x)$ s.t. $\mu(U(x)) < +\infty$.

Remark 1.1.6. A locally finite Borel measure μ on a topological space $\mathbb{X}$ turns out to be finite on compact sets.

Definition 1.1.7 (Radon measure)**.** Given a topological space $\mathbb{X}$ and a Borel measure μ on $\mathbb{X}$, we say that μ is a Radon measure if it is locally finite and

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\} \quad \forall E \in \mathcal{B}(\mathbb{X}).$$

Radon-Nikodym decomposition

Definition 1.1.8 (Absolute continuity)**.** Given μ, λ positive measures over $\mathcal{M}$, we say that λ is absolutely continuous with respect to μ if $\mu(A) = 0$ implies that $\lambda(A) = 0$ for all $A \in \mathcal{M}$. We write $\lambda << \mu$.

Definition 1.1.9 (Mutual singularity)**.** Given μ, λ positive measures over $\mathcal{M}$, we say that λ and μ are mutually singular if there exist $A, B \in \mathcal{M}$ such that

- $A \cap B = \emptyset$, $A \cup B = \mathbb{X}$,
- $\mu(A) = 0$, $\lambda(B) = 0$.

We write $\mu \perp \lambda$. We say that μ is concentrated in B and λ is concentrated in A.

Definition 1.1.10 (Mass of a measure)**.** Given a positive measure μ on $\mathcal{M}$, we denote as

$$\mathbb{M}(\mu) = \|\mu\|_1 \coloneqq \mu(\mathbb{X})$$

the mass of the measure μ.

Definition 1.1.11 (Restriction of a measure)**.** Given a positive measure μ on $\mathcal{M}$ and a set $F \in \mathcal{M}$, we denote as $\mu \llcorner F$ the positive measure defined by

$$\mu \llcorner F(E) \coloneqq \mu(F \cap E) \quad \forall E \in \mathcal{M}.$$

Definition 1.1.12 (Measure defined by density)**.** Given a measure μ over $\mathcal{M}$ and a measurable function $f : \mathbb{X} \to [0, +\infty]$, we denote as

$$f \cdot \mu(E) \coloneqq \int_E f \, d\mu.$$

We say that f is the density of $f \cdot \mu$ with respect to μ.

Remark 1.1.13. In the setting of 1.1.12, $f \cdot \mu$ is a positive measure on $\mathcal{M}$. Moreover, $f \cdot \mu$ is absolutely continuous with respect to μ. We also have that

$$\|f \cdot \mu\|_1 = \int_{\mathbb{X}} f \, d\mu = \|f\|_{L^1(\mu)}.$$

Definition 1.1.14 (σ-finite measure)**.** Given a measure μ over $\mathcal{M}$, we say that μ is σ-finite if there exists a countable family $(E_n)_n \subseteq \mathcal{M}$ such that

- $\mu(E_n) < +\infty$,
- $\mathbb{X} = \bigcup_n E_n$.

Hence, the following fundamental result holds true.

Theorem 1.1.15 (Radon-Nikodym)**.** *Let μ, λ be σ-finite positive measures over $\mathcal{M}$. Then, there exist positive measures λ_a, λ_s over $\mathcal{M}$ such that*

- *$\lambda = \lambda_a + \lambda_s$;*
- *$\lambda_a << \mu$ and $\lambda_s \perp \mu$.*

The decomposition is unique; hence, λ_a is the absolutely continuous part of λ with respect to μ and λ_s is the singular part of λ with respect to μ. Moreover $\lambda_a = f \cdot \mu$ for a suitable measurable function $f : \mathbb{X} \to [0, +\infty]$. f is unique up to μ-null sets and it is called Radon-Nikodym derivative of λ with respect to μ.

Remark 1.1.16. In the following, we will prove further properties of the Radon-Nikodym derivative.

Approximation results Under very mild assumptions on the ambient space $\mathbb{X}$, Borel measures have strong approximation properties.

Theorem 1.1.17. *Let $\mathbb{X}$ be a separable and locally compact metric space; let μ be a positive Borel measure on $\mathbb{X}$. Assume that μ is locally finite; then, μ is regular, that is for all $E \in \mathcal{B}(\mathbb{X})$ there holds*

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\} \\ &= \inf\{\mu(A) \mid E \subseteq A, A \text{ open}\}.\end{aligned}$$

Vector-valued measures

Let $\mathbb{X}$ be a set, $\mathcal{M}$ be a σ-algebra on $\mathbb{X}$ and F be a Banach space.

Definition 1.1.18 (Vector-valued measure)**.** A measure μ over $\mathcal{M}$ with values in F is a function $\mu : \mathcal{M} \to F$ such that

- $\mu(\emptyset) = 0$,
- for every countable and pairwise disjoint family $(E_n)_n \subseteq \mathcal{M}$ there holds

  $$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n),$$

  where the sum makes sense in the Banach space F and the series is absolutely convergent (i.e. it does not depend on the order).

Assume that $\mathbb{X}$ is a locally compact and separable metric space and F is a finite-dimensional normed space. The following fundamental result hold true.

Theorem 1.1.19. *Let λ be a Borel measure in $\mathbb{X}$ with values in F. Then, λ can be canonically written as $\lambda = f \cdot \mu$, where*

- *μ is a Borel locally finite measure on $\mathbb{X}$ (indeed, μ is the total variation measure of λ),*
- *$f : \mathbb{X} \to F$ is a function in $L^1(\mu)$, that is*

  $$\|f\|_{L^1(\mu)} := \int_{\mathbb{X}} |f| \ d\mu < +\infty,$$

- *for all $E \in \mathcal{B}(\mathbb{X})$ we have that*

  $$\lambda(E) = f \cdot \mu(E) = \int_E f \ d\mu,$$

  where the integral makes sense component-wise.

Moreover, the decomposition is unique if we require that $\|f(x)\| = 1$ for μ-a.e. $x \in \mathbb{X}$.

Definition 1.1.20. We define $\mathcal{M}(\mathbb{X}, F)$ to be the set of the Borel F-valued measures on $\mathbb{X}$.

Definition 1.1.21. Given $\lambda \in \mathcal{M}(\mathbb{X}, F)$, we denote

$$\mathbb{M}(\lambda) = \|\lambda\|_1 \coloneqq \|f\|_{L^1(\mu)},$$

where μ and f are given by theorem 1.1.19. $\|\lambda\|_1$ is called the norm of total variation of λ.

Remark 1.1.22. Definition 1.1.21 agrees with 1.1.10, which makes sense for positive measures.

Theorem 1.1.23. *The space $\mathcal{M}(\mathbb{X}, F)$ endowed with the norm of the total variation (see 1.1.21) is a Banach space.*

Remark 1.1.24. The case where F is an infinite dimensional Banach space in the theorem 1.1.19 is relevant but quite delicate.

Riesz's representation theorem Let $\mathbb{X}$ be a locally compact separable metric space; let F be a finite-dimensional normed space; let g be a function in $\in C_0(\mathbb{X}; F^*)$, that is $g : \mathbb{X} \to F^*$ is continuous and for every $\varepsilon > 0$ there exists a compact set $K \in \mathbb{X}$ such that $\|g(x)\| \leq \varepsilon$ for all $x \in \mathbb{X} \setminus K$. We denote as $< \cdot, \cdot >$ the duality pairing of F, F^*.

Definition 1.1.25. Given a measure $\lambda = f \cdot \mu \in \mathcal{M}(\mathbb{X}, F)$ (where f, μ are given by theorem 1.1.19), we define

$$T_\lambda(g) = \int_{\mathbb{X}} < f(x), g(x) > \ d\mu(x) \coloneqq \int_{\mathbb{X}} g \ d\lambda \quad \forall g \in C_0(\mathbb{X}, F^*).$$

Proposition 1.1.26. *The functional T_λ (see 1.1.25) is well defined, linear and bounded; hence $T_\lambda \in (C_0(\mathbb{X}, F^*))^*$.*

Thus, the following fundamental theorem holds true.

Theorem 1.1.27 (Riesz)**.** *The map $T : \mathcal{M}(\mathbb{X}; F) \to (C_0(\mathbb{X}, F^*))^*$ defined in 1.1.25 is a surjective isometry of Banach spaces.*

Remark 1.1.28. The proof of theorem 1.1.27 is quite hard; indeed, the surjectivity is the non-trivial part.

The identification of $\mathcal{M}(\mathbb{X}, F)$ and $(C_0(\mathbb{X}, F^*))^*$ induces a weak-* topology on $\mathcal{M}(\mathbb{X}, F)$. However, this is the only relevant topology on measures (that induced by the metric structure is too strong and basically useless). The weak-* topology in $\mathcal{M}(\mathbb{X}, F)$ is metrizable on bounded subsets of $\mathcal{M}(\mathbb{X}, F)$.

Definition 1.1.29 (Convergence of measures)**.** Given $(\lambda_n)_n \subseteq \mathcal{M}(\mathbb{X}, F)$ and $\lambda \in \mathcal{M}(\mathbb{X}, F)$, we say that $(\lambda_n)_n$ converges to λ in the sense of measures if $(T_{\lambda_n})_n$ converges to T_λ as functionals in $(C_0(\mathbb{X}, F^*))^*$, that is

$$\lim_{n \to +\infty} \int_{\mathbb{X}} g \ d\lambda_n = \int_{\mathbb{X}} g \ d\lambda \quad \forall g \in C_0(\mathbb{X}, F^*).$$

Lemma 1.1.30 (Lower semicontinuity)**.** *If $(\lambda_n)_n$ is a sequence of measures in $\mathcal{M}(\mathbb{X}, F)$ that converges to a measure λ in the sense of measures, then it holds that*

$$\liminf_{n \to +\infty} \|\lambda_n\|_1 \geq \|\lambda\|_1 .$$

Proof. It suffices to recall that norm is weakly-* lower semicontinuous. ☐

Theorem 1.1.31 (Compactness of measures)**.** *Given a bounded sequence $(\lambda_n)_n \subseteq \mathcal{M}(\mathbb{X}, F)$, up to subsequences, it converges in the sense of measures (see 1.1.29) to a measure $\lambda \in \mathcal{M}(\mathbb{X}, F)$.*

Proof. It is an immediate consequence of Riesz's theorem (see 1.1.27) and Banach-Alaouglu theorem (that is bounded subset in the dual of a Banach space are relatively compact with respect to the weak-* topology; in deed, they are sequentially compact). ☐

In the following, we will only use the sequential weak-* topology dealing with positive-valued measures and in the sequential version.

Proposition 1.1.32. *Suppose $\mathbb{X}$ to be compact. Let $(\mu_n)_n$ be a sequence of Borel, positive finite measure. Assume $(\mu_n)_n$ converges to a Borel positive finite measure μ. Then, it holds that*

$$\liminf_{n\to+\infty} \int_{\mathbb{X}} g \; d\mu_n \geq \int_{\mathbb{X}} g \; d\mu \quad \forall g : \mathbb{X} \to [0, +\infty] \text{ lower semicontinuous.}$$

In particular, we have that

- $\liminf_{n\to+\infty} \mu(A) \geq \mu(A) \quad \forall A \subseteq \mathbb{X}$ *open*;
- $\limsup_{n\to+\infty} \mu_n(C) \leq \mu(C) \quad \forall C \subseteq \mathbb{X}$ *closed*;
- $\lim_{n\to+\infty} \mu_n(E) = \mu(E) \quad \forall E \in \mathcal{B}(\mathbb{X})$ *s.t.* $\mu(\partial E) = 0$.

Proof. The first statement is an immediate consequence of the fact that, given g a lower-semicontinuous nonnegative function on $\mathbb{X}$, there exists a monotone sequence of continuous (in deed Lipschitz) and nonnegative functions $(g_n)_n$ such that

$$g(x) = \sup_n g_n(x) = \lim_{n\to+\infty} g_n(x) \quad \forall x \in \mathbb{X}.$$

Recall that, since $\mathbb{X}$ is compact, then continuous functions are admissible test functions in the weak-* convergence.

In particular, if A is an open set, then $\mathbb{1}_A$ is a nonnegative lower semicontinuous function; hence, the previous argument applies.

For closed sets, apply the statement for open sets to the complementary of C and notice that

$$\lim_{n\to+\infty} \mu_n(\mathbb{X}) = \mu(\mathbb{X}),$$

since $\mathbb{1}_{\mathbb{X}}$ is an admissible test function (in deed, it is continuous and $\mathbb{X}$ is compact).

Thus, the last statement follows immediately from the previous ones and the approximation result stated in 1.1.17. ☐

We state the following result as a simple corollary of the previous statements. In deed, it is a particular version of the more general Prokhorov's theorem.

Corollary 1.1.33. *Let $(\mathbb{X}, d)$ be a compact metric space and $(\mu_n)_n$ be a sequence of Borel probability measures in $\mathbb{X}$. Up to subsequences, there exists a Borel probability measure μ on $\mathbb{X}$ s.t. $(\mu_n)_n$ converges to μ in the sense of measures.*

Proof. Apply theorem 1.1.31 to get that $(\mu_n)_n$ converges (up to subsequences) to a measure μ in the sense of measure. Deduce that $\|\mu\|_1 \leq 1$. Test the weak convergence with the constant function $\mathbb{1}_{\mathbb{X}}$ to obtain that $\mu(\mathbb{X}) = 1$. Then, we have that $\|\mu\|_1 = \mu(\mathbb{X}) = 1$. Conclude that μ is a nonnegative measure. ☐

Outer measures

Let $\mathbb{X}$ be a set.

Definition 1.1.34 (Outer measure)**.** An outer measure on $\mathbb{X}$ is a map $\mu : \mathcal{P}(\mathbb{X}) \to [0, +\infty]$ with the following properties:

- $\mu(\emptyset) = 0$;
- μ is monotone, that is $E \subseteq E'$ implies that $\mu(E) \leq \mu(E')$;
- μ is countably subadditive, that is for any countable family $(E_n)_n$ of subsets in $\mathbb{X}$ (not necessarily disjoint), then

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n).$$

Definition 1.1.35 (μ-measurability)**.** Given an outer measure μ on $\mathbb{X}$, a set $E \subseteq \mathbb{X}$ is μ-measurable (in the sense of Carathéodory) if

$$\mu(F) = \mu(E \cap F) + \mu(E^c \cap F) \quad \forall F \subseteq \mathbb{X}.$$

We denote as $\mathcal{M}_\mu$ the collection of the μ-measurable subsets of $\mathbb{X}$.

Remark 1.1.36. Let μ be an outer measure on $\mathbb{X}$. With the notation introduced in 1.1.11, $E \subseteq \mathbb{X}$ is μ-measurable if and only if

$$\mu = \mu \llcorner E + \mu \llcorner E^c.$$

Notice that the inequality

$$\mu \leq \mu \llcorner E + \mu \llcorner E^c$$

holds for free by the subadditivity of outer measures.

Proposition 1.1.37. *Let μ be an outer measure on $\mathbb{X}$. The class $\mathcal{M}_\mu$ of the μ-measurable sets (see 1.1.35) is a σ-algebra and μ is a σ-additive measure on $\mathcal{M}_\mu$.*

Remark 1.1.38. The proof of proposition 1.1.37 is a long exercise.

Theorem 1.1.39 (Carathéodory)**.** *Let $\mathbb{X}$ be a metric space; let μ be an outer measure on $\mathbb{X}$ additive on distant sets, that is*

$$\mu(E \cup E') = \mu(E) + \mu(E') \quad \forall E, E' \subseteq \mathbb{X} \ s.t. \ \operatorname{dist}(E, E') = \inf_{x \in E, x' \in E'} d(x, x') > 0.$$

Then, $\mathcal{M}_\mu$ contains Borel sets; in particular, μ restricts to a Borel measure on $\mathbb{X}$.

Remark 1.1.40. The proof of theorem 1.1.39 is non-trivial.

Example 1.1.41. Consider the case where $\mathbb{X}$ is any set and μ is the outer measure on $\mathbb{X}$ that counts points; it is immediate to check that $\mathcal{M}_\mu = \mathcal{P}(\mathbb{X})$. On the other hand, if μ is the outer measure such that

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

then $\mathcal{M}_\mu = \{0, \mathbb{X}\}$.

Carathéodory construction Given a metric space $\mathbb{X}$, we want to construct meaningful outer measures on $\mathbb{X}$ which are additive on distant sets. So, we present the Carathéodory construction of outer measures (at least in a simplified version that fits our needs). Let $\mathbb{X}$ be a metric space; let $\mathcal{F}$ be a family of subsets of $\mathbb{X}$ such that $\emptyset \in \mathcal{F}$.

Definition 1.1.42 (Gauge function)**.** We say that $\rho : \mathcal{F} \to [0, +\infty]$ is a gauge function if $\rho(\emptyset) = 0$.

Definition 1.1.43 (δ-covering)**.** Given $\delta > 0$, we say that $(E_i)_i \subseteq \mathcal{P}(\mathbb{X})$ is a δ-covering of $E \subseteq \mathbb{X}$ if $(E_i)_i$ covers E and $\operatorname{diam}(E_i) \leq \delta$ for all i.

Definition 1.1.44. Let ρ be a gauge function on $\mathcal{F}$. Fix $\delta \in (0, +\infty]$. For every $E \subseteq \mathbb{X}$, we define

$$\psi_\delta(E) \coloneqq \inf \left\{ \sum_i \rho(E_i) \;\middle|\; (E_i)_i \subseteq \mathcal{F} \text{ is a } \delta\text{-covering of E} \right\},$$

with the assumption that $\inf(\emptyset) = +\infty$. We define

$$\psi(E) \coloneqq \sup_{\delta > 0} \psi_\delta(E).$$

Remark 1.1.45. In 1.1.44, notice that if δ decreases then $\psi_\delta(E)$ increases; hence

$$\psi(E) = \sup_{\delta > 0} \psi_\delta(E) = \lim_{\delta \to 0} \psi_\delta(E).$$

Proposition 1.1.46. *Given ψ_δ and ψ as in 1.1.44, for all $\delta > 0$ there holds that ψ_δ is an outer measure which is additive on distant sets, that is*

$$\psi_\delta(E \cup E') = \psi_\delta(E) + \psi_\delta(E') \quad \forall E, E' \subseteq \mathbb{X} \text{ s.t. } \operatorname{dist}(E, E') > 0.$$

Hence, the same holds true for ψ, that is ψ is an outer measure which is additive on distant sets.

Remark 1.1.47. The proof of 1.1.46 follows straightforward from the definitions given.

Lebesgue measure in $\mathbb{R}^n$ The Carathéodory construction described above applies in the particular case of the Lebesgue measure. Let $\mathbb{X}$ be $\mathbb{R}^n$ and $\mathcal{F}$ be the collection of n-dimensional rectangles on $\mathbb{R}^n$, that is

$$\mathcal{F} \coloneqq \{R = I_1 \times \cdots \times I_n \mid I_i \subseteq \mathbb{R} \text{ is a bounded interval}\}.$$

We define the gauge function

$$\rho^n(I_1 \times \cdots \times I_n) \coloneqq \prod_{i=1}^{n} (\sup I_i - \inf I_i).$$

Then, we define ψ_δ and ψ as in 1.1.44. ψ is the Lebesgue measure in $\mathbb{R}^n$ and it is usually denoted as $\mathscr{L}^n$. In this specific case, it is easy to check that ψ_δ is independent from δ. In deed, given $\delta > 0$ and a bounded rectangle R, it is easy to decompose R in a finite number of disjoint rectangles (R_i) s.t. $\operatorname{diam}(R_i) \leq \delta$ for all i and

$$\rho^n(R) = \sum_i \rho^n(R_i).$$

Hence, for the Lebesgue measure it holds that

$$\psi_\delta = \psi_\infty = \psi \quad \forall \delta > 0.$$

Lebesgue measure is additive on distant subsets; in particular, it restrict to a Borel measure in $\mathbb{R}^n$. It is easy to check that $\mathscr{L}^n$ is locally finite; in particular, theorem 1.1.17 applies: for any Borel set E there holds that

$$\begin{aligned} \mathscr{L}^n(E) &= \sup\{\mathscr{L}^n(K) \mid K \subseteq E, K \text{ compact}\} \\ &= \inf\{\mathscr{L}^n(A) \mid E \subseteq A, A \text{ open}\}, \end{aligned}$$

or equivalently

$$\forall \varepsilon > 0 \ \exists C \text{ closed }, A \text{ open s.t. } C \subseteq E \subseteq A \text{ and } \mathscr{L}^n(A \setminus C) < \varepsilon.$$

It is possible to show that this approximation property holds for all the sets in $\mathcal{M}_{\mathscr{L}^n}$ (see 1.1.35). However, the collection of the Lebesgue-measurable sets in $\mathbb{R}^n$ is traditionally defined by the property above; that is, we say that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if it holds that

$$\forall \varepsilon > 0 \ \exists C \text{ closed }, A \text{ open s.t. } C \subseteq E \subseteq A \text{ and } \lambda(A \setminus C) < \varepsilon,$$

where λ is the function defined on open set as

$$\lambda(A) := \sup\left\{ \sum_i \rho^n(R_i) \;\middle|\; (R_i)_i \text{ at most countable family of rectangles covering } A \right\}.$$

In deed, one can prove that $E \in \mathcal{M}_\mu$ if and only if E has the property that

$$\forall \varepsilon > 0 \ \exists C \text{ closed }, A \text{ open s.t. } C \subseteq E \subseteq A \text{ and } \lambda(A \setminus C) < \varepsilon.$$

1.2 Hausdorff measure

The Carathéodory construction gives rise to the Hausdorff measure. Let $\mathbb{X}$ be a metric space; fix $d \in [0, +\infty)$. We define a gauge function ρ^d on $\mathcal{F} = \mathcal{P}(\mathbb{X})$. If $d \neq 0$, for all $E \in \mathcal{P}(\mathbb{X})$, we define

$$\rho^d(E) := \operatorname{diam}(E)^d;$$

if $d = 0$, we define

$$\rho^d(E) := \begin{cases} 1 & \text{if } E \neq \emptyset, \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Definition 1.2.1 (Hausdorff measure)**.** With the notation introduced in 1.1.44, we set

$$\mathcal{H}^d_\delta(E) = c_d \psi_\delta(E), \quad \mathcal{H}^d(E) = c_d \psi(E),$$

where $c_d > 0$ is a normalization constant defined as

$$c_d := \begin{cases} \frac{\alpha_d}{2^d} & \text{if } d \in \mathbb{N}, \\ 1 & \text{if } d \notin \mathbb{N} \end{cases}$$

and $\alpha_d = \mathscr{L}^d(B_1)$, where B_1 is the unit ball in $\mathbb{R}^d$. $\mathcal{H}^d$ is called d-dimensional Hausdorff measure in $\mathbb{X}$.

Remark 1.2.2. The class $\mathcal{F}$ in the construction of $\mathcal{H}^d$ can be refined. In fact, $\mathcal{H}^d$ does not change if we replace $\mathcal{F} = \mathcal{P}(\mathbb{X})$ with the following subclasses:

- $\mathcal{F} = \{\text{closed sets}\}$, because $\operatorname{diam}(E) = \operatorname{diam}(\overline{E})$ for all $E \subseteq \mathbb{X}$;
- $\mathcal{F} = \{\text{open sets}\}$, because for all $E \subseteq \mathbb{X}$ for all $\varepsilon > 0$ there exists an open set A s.t. $E \subseteq A$ and $\operatorname{diam}(A) \leq \operatorname{diam}(E) + \varepsilon$;
- if $\mathbb{X}$ is a normed space, we can consider $\mathcal{F} = \{\text{convex sets}\}$, because $\operatorname{diam}(E) = \operatorname{diam}(\operatorname{Conv}(E))$ for all $E \subseteq \mathbb{X}$ ($\operatorname{Conv}(E)$ is the convex hull of E); moreover, we can restrict $\mathcal{F}$ to be the collection of convex closed sets or convex open sets.

However, it is very important to notice that the value of $\mathcal{H}^d$ changes if we replace $\mathcal{F} = \mathcal{P}(\mathbb{X})$ with the collection of the balls in $\mathbb{X}$. In this case, we obtain the so called "spherical Hausdorff measure", denoted as $\mathcal{H}^d_S$. In fact, we remark that in general a set E is not contained in a ball with the same diameter (consider the case of a triangle in $\mathbb{R}^2$). But, $\mathcal{H}^d_S$ agrees with $\mathcal{H}^d$ on d-dimensional surfaces of class C^1 in $\mathbb{R}^m$.

Remark 1.2.3. It follows a list of useful and easy remarks on Hausdorff measure.

- $\mathcal{H}^0$ is the measure that counts points; indeed, we have
$$\mathcal{H}^0_\infty(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ 1 & \text{if } E \neq \emptyset. \end{cases}$$
- Given $d > 0$, $\mathcal{H}^d$ is called "d-dimensional measure" because it has the following fundamental scaling property in the case $\mathbb{X} = \mathbb{R}^n$:
$$\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E) \quad \forall \lambda > 0,\ \forall E \in \mathcal{B}(\mathbb{R}^n).$$
This property is an immediate consequence of the fact that
$$(\operatorname{diam}(\lambda E))^d = \lambda^d (\operatorname{diam}(E))^d \quad \forall \lambda > 0\ \forall E \subseteq \mathbb{R}^n.$$
- If $\mathbb{X}, \mathbb{Y}$ are metric spaces, $d > 0$ and $f : \mathbb{X} \to \mathbb{Y}$ is a Lipschitz map, then
$$\mathcal{H}^d(f(E)) \leq \operatorname{Lip}(f)^d \mathcal{H}^d(E) \quad \forall E \in \mathcal{B}(\mathbb{X}).$$
This follows immediately from the fact that
$$\operatorname{diam}(f(E)) \leq \operatorname{Lip}(f) \operatorname{diam}(E) \quad \forall E \subseteq \mathbb{X}.$$
In particular, if f is an isometry, then the Hausdorff measure is preserved by f, that is
$$\mathcal{H}^d(f(E)) = \mathcal{H}^d(E) \quad \forall E \in \mathcal{B}(\mathbb{X}).$$
This is an immediate consequence of the property stated above, applied to $f : \mathbb{X} \to f(\mathbb{X})$ and $f^{-1} : f(\mathbb{X}) \to \mathbb{X}$: both maps have Lipschitz constant 1.

Remark 1.2.4. We will show that in the case $\mathbb{X} = \mathbb{R}^d$, then $\mathcal{H}^d = \mathcal{H}^d_\delta = \mathscr{L}^d$ for all $\delta \in (0, +\infty]$. In deed, this statement is delicate and requires covering theorems. We point out that the choice of the normalization constant plays a fundamental role here (and only here, to be honest). Notice also that both $\mathcal{H}^d$ and $\mathscr{L}^d$ are translation invariant; hence, a general statement on Haar measures (that will be given later on) implies that $\mathcal{H}^d = c \mathscr{L}^d$. However, the delicate part is to find the right constant c.

Hausdorff dimension

Proposition 1.2.5. *Take $0 \le d < d' < +\infty$ and $E \subseteq \mathbb{X}$. Then:*

- $\mathcal{H}^d(E) < +\infty$ *implies that* $\mathcal{H}^{d'}(E) = 0$*;*
- $\mathcal{H}^{d'}(E) > 0$ *implies that* $\mathcal{H}^d(E) = +\infty$*.*

Proof. It immediately follows from the definition of Hausdorff measure. □

Definition 1.2.6 (Hausdorff dimension)**.** Given $E \subseteq \mathbb{X}$, we define

$$\dim_{\mathcal{H}}(E) := \sup\{d \mid \mathcal{H}^d(E) = 0\} = \inf\{d \mid \mathcal{H}^d(E) = +\infty\}.$$

The number $\dim_{\mathcal{H}}(E)$ is the so-called Hausdorff dimension of E.

Remark 1.2.7. Notice that the definition 1.2.6 is well-posed due to 1.2.5. Moreover, if $d \in [0, +\infty)$ is such that $\mathcal{H}^d(E) \in (0, +\infty)$, then $d = \dim_{\mathcal{H}}(E)$. We remark that if $d = \dim_{\mathcal{H}}(E)$, then $\mathcal{H}^d(E)$ can be 0 or $+\infty$.

The Cantor set The following example is famous and significant; it will be useful in the following.

Example 1.2.8. We define the Cantor set as

$$C := \bigcap_{n=0}^{+\infty} C_n, \quad C_n := \bigcup_{i=1}^{2^n} I_{n,i},$$

where $I_{n,i}$ are closed intervals with length $\frac{1}{3^n}$; in particular, C is a compact set. More explicitly, we define

$$C_0 = [0,1],$$
$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$
$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right],$$

and so on... To be precise, at each level, we split each interval in three equal parts and we remove the central ones (see 1.1).

We want to find the Hausdorff dimension $\overline{d}$ of C and compute $\mathcal{H}^{\overline{d}}(C)$. It is very remarkable that $\overline{d}$ is easy to guess. In deed, we have that

$$C = \left(C \cup \left[0, \frac{1}{3}\right]\right) \cup \left(C \cup \left[\frac{2}{3}, 1\right]\right) = \frac{1}{3}C \cup \left(\frac{2}{3} + \frac{1}{3}C\right),$$

where the union is disjoint. Hence, we deduce that

$$\mathcal{H}^d(C) = 2\mathcal{H}^d\left(\frac{1}{3}C\right) = \frac{2}{3^d}\mathcal{H}^d(C) \quad \forall d \in (0, +\infty).$$

Assuming that $0 < \mathcal{H}^{\overline{d}}(C) < +\infty$, we have that

$$1 = \frac{2}{3^{\overline{d}}},$$

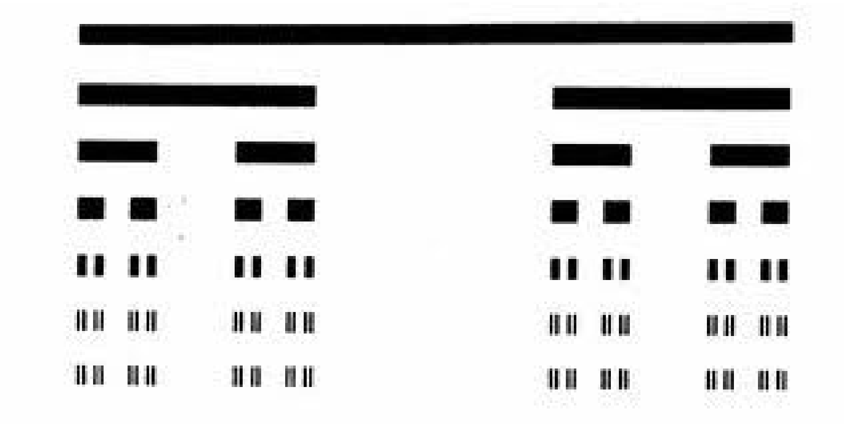

Figure 1.1: The first iterations of the construction of the Cantor set.

that is

$$\overline{d} = \frac{\log 2}{\log 3}.$$

This argument could be made rigorous, if we prove that

$$0 < \mathcal{H}^{\overline{d}}(C) < +\infty;$$

this assumption is very strong and it is the core of the matter: in deed, it implies straightforward that $\dim_{\mathcal{H}}(C) = \overline{d}$. However, we define

$$\overline{d} := \frac{\log 2}{\log 3}.$$

Providing an upper bound for $\mathcal{H}^{\overline{d}}(C)$ is easy, since it suffices to exhibit a covering of C. Fix $n \in \mathbb{N}$ and consider the most natural covering, i.e.

$$\{I_{n,i} \mid i = 1, \dots, 2^n\}.$$

We notice that this is a 3^{-n}-covering of C. Hence, for $\delta \geq 3^{-n}$, there holds that

$$\mathcal{H}^{\overline{d}}_{\delta}(C) \leq \sum_{i=1}^{2^n} (\operatorname{diam}(I_{n,i}))^{\overline{d}} = \frac{2^n}{3^{n\overline{d}}} = 1.$$

Then, we find that

$$\mathcal{H}^{\overline{d}}(C) \leq 1.$$

Providing a lower bound for $\mathcal{H}^{\overline{d}}(C)$ is much more difficult because we have to deal with an arbitrary covering of $(E_i)_i$ of C. Denote $\mathcal{T} := \{I_{n,i} \mid n \in \mathbb{N},\ i = 1, \dots, 2^n\}$; we want to reduce to the case where $(E_i)_i$ is finite and made of elements of $\mathcal{T}$. As we said in 1.2.2, we can assume that $(E_i)_i$ is made of convex open sets, i.e. E_i is an interval. Since C is compact and $\mathcal{H}^{\overline{d}}_{\delta}(C)$ is an infimum, we can assume that $(E_i)_i$ is finite. Since $\operatorname{diam}(E_i) = \operatorname{diam}(\overline{E_i})$, we can freely assume that each E_i is a closed interval. Since we have to estimate an infimum, we can replace E_i with

$$\tilde{E}_i := [\inf(E_i \cap C), \sup(E_i \cap C)].$$

Consider the largest intervals $I_{n_i,j_i}, I_{l_i,p_i} \in \mathcal{T}$ (eventually the same interval) s. t.

$$\inf(E_I \cap C) \in I_{n_i,j_i} \subseteq \tilde{E}_i \cap C, \quad \sup(E_I \cap C) \in I_{l_i,p_i} \subseteq \tilde{E}_i \cap C.$$

Notice that, if $I_{n_i,j_i} = I_{p_i,l_i}$, then $\tilde{E}_i$ coincide with I_{n_i,j_i}; hence $\tilde{E}_i$ already belongs to $\mathcal{T}$. So, we can assume that $I_{n_i,j_i} \neq I_{p_i,l_i}$. Denote

$$G_i := \tilde{E}_i \setminus (I_{n_i,j_i} \cup I_{l_i,p_i});$$

by maximality, it is easy to see that $\tilde{E}_i$ is made of the intervals $I_{n_i,j_i}, G_i, I_{p_i,l_i}$. Moreover, by the construction of the Cantor set, it follows that

$$\operatorname{diam}(G_i) \geq \max\{\operatorname{diam}(I_{n_i,j_i}), \operatorname{diam}(I_{l_i,p_i})\}.$$

The first key step consists of the following inequalities:

$$\begin{aligned}
(\operatorname{diam}(\tilde{E}_i))^{\overline{d}} &= (\operatorname{diam}(I_{n_i,j_i}) + \operatorname{diam}(G_i) + \operatorname{diam}(I_{l_i,p_i}))^{\overline{d}} \\
&\geq \left(\frac{3}{2}(\operatorname{diam}(I_{n_i,j_i}) + \operatorname{diam}(I_{l_i,p_i}))\right)^{\overline{d}} \\
&= 2\left(\frac{\operatorname{diam}(I_{n_i,j_i})}{2} + \frac{\operatorname{diam}(I_{l_i,p_i})}{2}\right)^{\overline{d}} \\
&\geq 2\left[\frac{(\operatorname{diam}(I_{n_i,j_i}))^{\overline{d}}}{2} + \frac{(\operatorname{diam}(I_{l_i,p_i}))^{\overline{d}}}{2}\right] \\
&= (\operatorname{diam}(I_{n_i,j_i}))^{\overline{d}} + \operatorname{diam}(I_{l_i,p_i}))^{\overline{d}};
\end{aligned}$$

we have used that $3^{\overline{d}} = 2$ and that the function $\varphi(s) = s^{\overline{d}}$ is concave in $(0, +\infty)$ (because $0 < \overline{d} < 1$). This argument show that it is convenient to replace $\tilde{E}_i$ with I_{n_i,j_i} and I_{p_i,l_i}.

Having said that, we can assume that $(E_i)_i$ is a finite covering of C made of elements of $\mathcal{T}$, that is $E_i = I_{N_i,k_i}$ for some N_i, k_i. Denote

$$\overline{N} := \max_i N_i.$$

The second key step is the following: replacing I_{N_i,k_i} with the two intervals in which it splits at the next level is "convenient" to compute $\mathcal{H}^{\overline{d}}_\delta$. I_{N_i,k_i} belongs to the level i; when passing to the level $N_i + 1$, it splits into two intervals $I_{N_i+1,p_i}, I_{N_i+1,p_i+1}$, for some p. By the property of $\overline{d}$, it follows that

$$(\operatorname{diam}(I_{N_i,k}))^{\overline{d}} = (\operatorname{diam}(I_{N_i+1,p_i}))^{\overline{d}} + (\operatorname{diam}(I_{N_i+1,p_i+1}))^{\overline{d}}.$$

Thus, we can split the intervals $(I_{N_i,k_i})_i$ until we arrive at the level $\overline{N}$. In other words, we can assume that $(E_i)_i$ is a covering of C made by intervals at the level $\overline{N}$; hence, we deduce that each interval at the level $\overline{N}$ must be taken (maybe more than once). In conclusion, we have that

$$\sum_i (\operatorname{diam}(E_i))^{\overline{d}} \geq \sum_{k=1}^{2^{\overline{N}}} (\operatorname{diam}(I_{\overline{N},k}))^{\overline{d}} = 2^{\overline{N}} \frac{1}{3^{\overline{N}\overline{d}}} = 1,$$

by the property of $\overline{d}$.

Chapter 2

Covering theorems

Covering theorems are fundamental tools in Geometric Measure Theory. For further references, see [5] and [4]. The purpose of this chapter is the following: given a family $\mathcal{F}$ of balls that covers a set E, we want to extract a "better" subcovering $\mathcal{F}'$. The meaning of "better" varies; it could be one of the followings:

- $\mathcal{F}'$ is disjoint;
- the balls in $\mathcal{F}'$ do not overlap to much;
- given some measure μ and $\varepsilon > 0$, then

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon.$$

There are two families of results:

- Vitali-type covering theorems, that hold in every metric space $\mathbb{X}$, but require something about the measure, e.g. doubling properties;
- Besicovitch-type covering theorems, that holds only in $\mathbb{R}^n$ but for every measure μ.

2.1 Vitali-type covering theorems

In the following, we will always assume $\mathbb{X}$ to be a locally compact, separable metric space; every measure μ on $\mathbb{X}$ is implicitly assumed to be Borel and locally finite.

Definition 2.1.1 (Doubling properties)**.** Let μ be a measure on $\mathbb{X}$.

- μ has the doubling property (D.P.) if there exists $M > 0$ s. t.

$$\mu(B(x, 2r)) \leq M\mu(B(x, r)) \ \ \forall r > 0, \ \forall x \in \operatorname{supp}(\mu).$$

- μ has the asymptotic doubling property (A.D.P.) if there holds that

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < +\infty \quad \forall x \in \operatorname{supp}(\mu).$$

Remark 2.1.2. The definition above can be equivalently stated with closed balls; the number 2 can be replaced with any number $\lambda > 1$ (fixed). Moreover, if μ has the doubling property, then every ball in $\mathbb{X}$ has finite measure: in deed, since μ is locally finite, there exists a sufficiently small ball $B(x, r)$ s.t. $\mu(B(x, r)) < +\infty$. The doubling properties guarantees that $\mu(B(x, R)) < +\infty$ for all $R > 0$; in particular, every ball has finite measure. Moreover, if there exists a ball with zero measure, than the same holds for every ball, that is $\mu \equiv 0$.

The following lemma holds in every metric space; indeed, it has nothing to do with measures.

Lemma 2.1.3 (5r-Vitali's lemma). *Let $\mathcal{F}$ be a family of closed balls $B_i = \overline{B}(x_i, r_i)$ s. t.*

$$R \coloneqq \sup_i r_i < +\infty.$$

There exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with the following properties:

- *$\mathcal{G}$ is disjoint and, then, at most countable;*
- *define $\hat{\mathcal{G}} \coloneqq \{\hat{B} \mid B \in \mathcal{G}\}$, where*

$$\hat{\overline{B}}(x_i, r_i) \coloneqq \overline{B}(x_i, 5r_i);$$

  then, it holds that

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} \hat{B}.$$

In particular, if $\mathcal{F}$ covers E, then $\hat{\mathcal{G}}$ covers E and $\mathcal{G}$ is disjoint.

Proof. Divide $\mathcal{F}$ as follows:

$$\mathcal{F} \coloneqq \bigcup_{n=0}^{\infty} \mathcal{F}_n, \quad \mathcal{F}_n \coloneqq \left\{ B = \overline{B}(x_i, r_i) \in \mathcal{F} \;\middle|\; \frac{R}{2^{n+1}} < r_i \leq \frac{R}{2^n} \right\}.$$

Step 1: Take $\mathcal{G}_0$ a maximal disjoint subfamily of $\mathcal{F}_0$ (it clearly exists by the Zorn's lemma). We claim that

$$\bigcup_{B \in \mathcal{F}_0} B \subseteq \bigcup_{B \in \mathcal{G}_0} \hat{B}.$$

In deed, if $B \in \mathcal{F}_0 \setminus \mathcal{G}_0$, the maximality of $\mathcal{G}_0$ guarantees that there exists a ball $B' \in \mathcal{G}_0$ s. t. $B \cap B' \neq \emptyset$. We check that $B \subseteq \hat{B}'$; denote

$$B = \overline{B}(x, r), \quad B' = \overline{B}(x', r'), \quad \hat{B}' = \overline{B}(x', 5r').$$

Pick $y \in B \cap B'$; for all $z \in B$ we have

$$d(x', z) \leq d(x', y) + d(z, y) \leq r' + 2r \leq 5r',$$

since $r, r' \in (R/2, R]$, thus $r \leq 2r'$.

Step 2: Take $\mathcal{G}_1$ a maximal subfamily of $\mathcal{F}_1$ which is disjoint and disjoint from $\mathcal{G}_0$ (it exists by the Zorn's lemma). Then, $\mathcal{G}_0 \cup \mathcal{G}_1$ is clearly disjoint. Refreshing the arguments given in the previous step, it is immediate to prove that

$$\bigcup_{B \in \mathcal{F}_1} B \subseteq \bigcup_{B \in \mathcal{G}_0 \cup \mathcal{G}_1} B.$$

Step 3: Assume that we have defined $\mathcal{G}_0, \dots, \mathcal{G}_n$ s. t. $\mathcal{G}_0 \cup \dots \cup \mathcal{G}_n$ is disjoint and

$$\bigcup_{B \in \mathcal{F}_0 \cup \dots \cup \mathcal{F}_n} B \subseteq \bigcup_{B \in \mathcal{G}_0 \cup \dots \cup \mathcal{G}_n} \hat{B};$$

inductively, we define $\mathcal{G}_{n+1}$ as a maximal subfamily of $\mathcal{F}_{n+1}$ which is disjoint and disjoint from $\mathcal{G}_0 \cup \dots \cup \mathcal{G}_n$. As above, one can check that $\mathcal{G}_0 \cup \dots \cup \mathcal{G}_{n+1}$ is disjoint and that

$$\bigcup_{B \in \mathcal{F}_0 \cup \dots \cup \mathcal{F}_{n+1}} B \subseteq \bigcup_{B \in \mathcal{G}_0 \cup \dots \cup \mathcal{G}_{n+1}} \hat{B}.$$

In conclusion, the subfamily

$$\mathcal{G} := \bigcup_{n=0}^{\infty} \mathcal{G}_n$$

has the desired properties. □

Example 2.1.4. The case where $\mathcal{F} = \{\overline{B}(0, r) \mid r \in (0, +\infty)\}$ in $\mathbb{R}^d$ shows that the assumption on the uniform boundedness of the radii of the balls in the lemma above (see 2.1.3) cannot be dropped.

First Vitali's covering theorem The definition below will play a fundamental role in the following of the section.

Definition 2.1.5 (Fine covering). Let $E \subseteq \mathbb{X}$ and $\mathcal{F}$ be a family of closed balls with the property that for all $x \in E$ for all $\delta > 0$ there exists a ball $B \in \mathcal{F}$ with radius at most δ and s.t. $x \in B$. We say that $\mathcal{F}$ is a fine covering of E.

Lemma 2.1.6. *Let μ be a locally finite, Borel measure on $\mathbb{X}$ with the doubling property ($M > 0$ is the doubling constant in 2.1.1). Let $E \subseteq \mathbb{X}$ be a Borel set s.t. $\mu(E) < +\infty$; let $\mathcal{F}$ be a fine covering of E (see 2.1.5). Then, there exists $\mathcal{G} \subseteq \mathcal{F}$ disjoint and s.t.*

$$\mu\left(E \cap \bigcup_{B \in \mathcal{G}} B\right) \geq \frac{1}{2M^3} \mu(E).$$

Proof. Without loss of generality, we can assume that $E \subseteq \operatorname{supp}(\mu)$. Since $\mu(E) < +\infty$, the approximation result on measures (see 1.1.17) guarantees that there exists an open set A s.t. $E \subseteq A$ and

$$\mu(A \setminus E) \leq \frac{\mu(E)}{2M^3}.$$

Denote

$$\mathcal{F}' := \{B \in \mathcal{F} \mid B \subseteq A,\ B \text{ has radius at most } 1\}.$$

We claim that $\mathcal{F}'$ is a covering of E (in deed, we check that $\mathcal{F}'$ is fine). Take $x \in E$; since A is open, there exists $0 < \delta < 1$ s.t. $x \in \overline{B}(x, r) \subseteq A$. Since $\mathcal{F}$ is a fine covering of E, there exists a ball $\overline{B}(y, r) \in \mathcal{F}$ s.t. $x \in \overline{B}(x, r)$ and $r \leq \delta/10$; in particular, we have that

$$x \in \overline{B}(y, r) \subseteq \overline{B}(x, \delta) \subseteq A;$$

thus $\overline{B}(y, r) \in \mathcal{F}'$.

We apply the lemma 2.1.3 to $\mathcal{F}'$ and we obtain $\mathcal{G}$ disjoint and such that $\hat{\mathcal{G}}$ covers E. Then, we have that

$$\mu(E) \leq \mu\left(\bigcup_{B\in\mathcal{G}} \hat{B}\right) \leq \sum_{B\in\mathcal{G}} \mu(\hat{B}) \leq M^3 \sum_{B\in\mathcal{G}} \mu(B).$$

Thus, we obtain

$$\sum_{B\in\mathcal{G}} \mu(B) \geq \frac{\mu(E)}{M^3}.$$

In conclusion, we have that

$$\mu\left(E \cap \bigcup_{B\in\mathcal{G}} B\right) \geq \mu\left(\bigcup_{B\in\mathcal{G}} B\right) - \mu(A \setminus E) \geq \frac{\mu(E)}{M^3} - \frac{\mu(E)}{2M^3} = \frac{\mu(E)}{2M^3}.$$

□

Theorem 2.1.7 (Vitali's covering theorem - 1)**.** *Let μ be a locally finite, Borel measure on $\mathbb{X}$ with the doubling property ($M > 0$ is the doubling constant in 2.1.1). Let $E \subseteq \mathbb{X}$ be a Borel set s.t. $\mu(E) < +\infty$; let $\mathcal{F}$ be a fine covering of E (see 2.1.5). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ s.t.*

- *$\mathcal{F}'$ is disjoint and countable (at most);*
- *$\mathcal{F}'$ covers μ-a.a. of E;*
- *$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.*

Proof. Without loss of generality, we can assume that $E \subseteq \operatorname{supp}(\mu)$. Fix $\varepsilon > 0$ and an open set A_0 s.t. $E \subseteq A_0$ and $\mu(A_0) \leq \mu(E) + \varepsilon$ (such A_0 exists thank to the approximation result stated in 1.1.17).

Step 1: Define

$$\mathcal{F}_0 := \{B \in \mathcal{F} \mid B \subseteq A_0\}.$$

Since $\mathcal{F}$ is a fine covering of E, then $\mathcal{F}_0$ is still a fine covering of E (this can be checked as shown in the proof of lemma 2.1.6). Lemma 2.1.6 guarantees that there exists $\mathcal{G}_0 \subseteq \mathcal{F}_0$ disjoint and s.t.

$$\mu\left(E \cap \bigcup_{B\in\mathcal{G}_0} B\right) \geq \frac{1}{2M^3}\mu(E).$$

We can choose $\mathcal{G}_0' \subseteq \mathcal{G}_0$ finite and s.t.

$$\mu\left(E \cap \bigcup_{B\in\mathcal{G}_0'} B\right) \geq \frac{1}{3M^3}\mu(E).$$

Thus, we obtain

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{G}_0'} B\right) \leq \left(1 - \frac{1}{3M^3}\right)\mu(E).$$

Step 2: Define

$$E_1 \coloneqq E \setminus \bigcup_{B \in \mathcal{G}_0'} B,$$

$$\mathcal{F}_1 \coloneqq \left\{ B \in \mathcal{F} \;\middle|\; B \subseteq A_0, B \cap \left(\bigcup_{\tilde{B} \in \mathcal{G}_0'} \tilde{B} \right) = \emptyset \right\}.$$

We claim that $\mathcal{F}_1$ is a fine covering of E_1. Since $\mathcal{G}_0'$ is finite and the balls are closed, then $\bigcup_{\tilde{B} \in \mathcal{G}_0'} \tilde{B}$ is closed. Take $x \in E_1$ and $\delta > 0$; there exists $0 < r < \delta$ s.t.

$$\overline{B}(x,r) \cap \left(\bigcup_{\tilde{B} \in \mathcal{G}_0'} \tilde{B} \right) = \emptyset, \quad \overline{B}(x,r) \subseteq A_0.$$

Since $\mathcal{F}$ is a fine covering of E, there exists $B' \in \mathcal{F}$ s.t.

$$x \in B' \subseteq \overline{B}(x,r);$$

thus, $B' \in \mathcal{F}_1$.

We can apply lemma 2.1.6 to $\mathcal{F}_1$ and E_1; arguing as in the previous step, we obtain $\mathcal{G}_0'$ finite and disjoint s.t.

$$\mu\left(E_1 \setminus \bigcup_{B \in \mathcal{G}_1'} B \right) \leq \left(1 - \frac{1}{3M^3} \right) \mu(E_1) \leq \left(1 - \frac{1}{3M^3} \right)^2 \mu(E).$$

By construction, there holds that $\mathcal{G}_1'$ is also disjoint from $\mathcal{G}_0'$.

Step 3: We inductively define

$$E_n \coloneqq E_{n-1} \setminus \bigcup_{B \in \mathcal{G}_0' \cup \cdots \cup \mathcal{G}_{n-1}'} B;$$

arguing as in the previous step, we find $\mathcal{G}_n'$ finite, disjoint and disjoint from $\mathcal{G}_0' \cup \cdots \cup \mathcal{G}_{n-1}'$, s.t. $\bigcup_{B \in \mathcal{G}_n'} B \subseteq A_0$ and

$$\mu\left(E_n \setminus \bigcup_{B \in \mathcal{G}_n'} B \right) \leq \left(1 - \frac{1}{3M^3} \right)^{n+1} \mu(E).$$

Step 4: We define

$$\mathcal{F}' \coloneqq \bigcup_{n=0}^{\infty} \mathcal{G}_n';$$

notice that $\mathcal{F}'$ is disjoint, countable and each ball in $\mathcal{F}'$ is contained in A_0. Thus

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu\left(\bigcup_{B \in \mathcal{F}'} B \right) \leq \mu(A_0) \leq \mu(E) + \varepsilon.$$

In conclusion, for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}\mu\left(E \setminus \bigcup_{B\in\mathcal{F}'} B\right) &\leq \mu\left(E \setminus \bigcup_{B\in\mathcal{G}'_0\cup\cdots\cup\mathcal{G}'_n} B\right)\\ &= \mu\left(E_1 \setminus \bigcup_{B\in\mathcal{G}'_1\cup\cdots\cup\mathcal{G}'_n} B\right)\\ &= \cdots = \mu\left(E_n \setminus \bigcup_{B\in\mathcal{G}'_n} B\right)\\ &\leq \left(1-\frac{1}{3M^3}\right)^{n+1} \mu(E).\end{aligned}$$

Taking the limit as $n \to +\infty$, we find that

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{F}'} B\right) = 0.$$

□

Second Vitali's covering theorem

Lemma 2.1.8. *Let μ be a locally finite, Borel measure on $\mathbb{X}$ with the doubling property ($M > 0$ is the doubling constant in 2.1.1). Let $E_0 \subseteq \mathbb{X}$ be a Borel set s.t. $\mu(E_0) = 0$; let $\mathcal{F}$ be a fine covering of E (see 2.1.5). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ ($\mathcal{F}'$ is not necessarily disjoint) s.t.*

- *$\mathcal{F}'$ is at most countable,*
- *$\mathcal{F}'$ covers E_0,*
- *$\sum_{B\in\mathcal{F}'} \mu(B) < \varepsilon$.*

Proof. We can reduce to the case where $E_0 \subseteq \operatorname{supp}(\mu)$. In deed, $\mathbb{X} \setminus \operatorname{supp}(\mu)$ is a open set and, since $\mathcal{F}$ is a fine covering, then $E_0 \setminus \operatorname{supp}(\mu)$ can be covered with at most countably many balls in $\mathcal{F}$ (recall that $\mathbb{X}$ is separable, hence every covering admits a countable or finite subcovering) contained in $\mathbb{X}\setminus\operatorname{supp}(\mu)$. Those balls are not necessarily disjoint and they have measure zero.

Hence, suppose that $E_0 \subseteq \operatorname{supp}(\mu)$. Given $\varepsilon > 0$, take an open set A s.t. $E_0 \subseteq A$ and $\mu(A) \leq \varepsilon/M^3$, where $M > 0$ is the doubling constant defined in 2.1.1 (such A exists because of the approximation result stated in 1.1.17). Since $\mathcal{F}$ is a fine covering of E_0, then

$$\mathcal{F}' := \{B \in \mathcal{F} \mid B \subseteq A,\ B \text{ has radius at most } 1\}$$

is still a covering of E_0 (it can be checked as in the proof of lemma 2.1.6). Denote

$$\check{\mathcal{F}}' := \{\check{B} \mid B \in \mathcal{F}'\},$$

where $\check{\overline{B}}(x,r) = \overline{B}(x, r/5)$. We apply lemma 2.1.3 to $\check{\mathcal{F}}'$ and we obtain $\mathcal{G} \subseteq \check{\mathcal{F}}'$ disjoint s.t.

$$E_0 \subseteq \bigcup_{B\in\mathcal{G}} \hat{B},$$

since $\mathcal{F}'$ covers E_0. It is immediate to see that $\hat{\mathcal{G}} \subseteq \mathcal{F}' \subseteq \mathcal{F}$ and that

$$E_0 \subseteq \bigcup_{B \in \hat{G}} B.$$

Since $\mathcal{G}$ is disjoint and $\mathbb{X}$ is a separable metric space, $\mathcal{G}$ is countable. As for the measure, we have:

$$\sum_{B \in \mathcal{G}} \mu(B) = \mu\left(\bigcup_{B \in \mathcal{G}} B\right) \leq \mu(A) \leq \frac{\varepsilon}{M^3};$$

then

$$\sum_{B \in \hat{G}} \mu(B) = \sum_{B \in \mathcal{G}} \mu(\hat{B}) \leq M^3 \sum_{B \in \mathcal{G}} \mu(B) \leq \varepsilon.$$

□

Theorem 2.1.9 (Vitali's covering theorem - 2)**.** *Let μ be a locally finite, Borel measure on $\mathbb{X}$ with the doubling property ($M > 0$ is the doubling constant in 2.1.1). Let $E \subseteq \mathbb{X}$ be a Borel set s.t. $\mu(E) < +\infty$; let $\mathcal{F}$ be a fine covering of E (see 2.1.5). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ s.t.*

- *$\mathcal{F}'$ is at most countable (not necessarily disjoint);*
- *$\mathcal{F}'$ covers E;*
- *$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.*

Proof. Apply first Vitali's covering theorem (see 2.1.7) to cover μ-a.a. of E; use lemma 2.1.8 to cover the remaining part. □

We conclude with some remarks.

Remark 2.1.10.
- For first Vitali's covering theorem (see 2.1.7) to hold it is enough that μ has the asymptotically doubling property (see 2.1.1).
- The previous statement can be adapted in the case where $\mathcal{F}$ is a family of open balls. In particular, first Vitali's theorem (see 2.1.7) holds if $\mathcal{F}$ is a family of open balls s.t. $\mu(\partial B) = 0$ for all $B \in \mathcal{F}$ (it suffices to apply the theorem as stated in 2.1.7 to $\overline{F} = \{\overline{B} \mid B \in \mathcal{F}\}$). Also second Vitali's theorem (see 2.1.9) works for such $\mathcal{F}$. Recall that, if μ is a Borel measure which is finite on balls, for all $x \in \mathbb{X}$ there holds that $\mu(\partial B(x, r)) = 0$ for all $r > 0$ except at most countably many (depending from x).
- These theorems work for more general sets than just balls.

The Vitali's covering theorems works for a larger class of measure. We state (without proofs) a more general results, that we will invoke in the following chapters.

Theorem 2.1.11. *Let μ be a locally finite, Borel measure on $\mathbb{X}$ with the doubling property. Let F be a Borel set in $\mathbb{X}$. Denote $\lambda := \mu \llcorner F$. Let $E \subseteq \mathbb{X}$ be a Borel set s.t. $\lambda(E) < +\infty$; let $\mathcal{F}$ be a fine covering of E (see 2.1.5). Then, the followings hold true:*

- *for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ s.t.*
    1. *$\mathcal{F}'$ is disjoint and countable (at most);*

2. *$\mathcal{F}'$ covers λ-a.a. of E;*
3. $\sum_{B\in\mathcal{F}'}\mu(B) \le \lambda(E)+\varepsilon$.

- *for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ s.t.*

1. *$\mathcal{F}'$ is at most countable (not necessarily disjoint);*
2. *$\mathcal{F}'$ covers E;*
3. $\sum_{B\in\mathcal{F}'}\mu(B) \le \lambda(E)+\varepsilon$.

2.2 Besicovitch-type covering theorems

In this section, we work in $\mathbb{R}^d$. We start with an elementary lemma.

Lemma 2.2.1. *Let $B_1 = B(x_1, r_1), \dots, B_n = B(x_n, r_n)$ be balls in $\mathbb{R}^d$ s.t.*

1. *$B_1 \cap B_n \neq \emptyset$ for all $i < n$,*
2. *$r_i \ge r_n/2$ for all $i < n$,*
3. *$x_i \notin B_j$ for all $i \neq j < n$.*

Then, there exists $N(d) \in \mathbb{N}$ depending exclusively from d s.t. $n \le N(d)$.

Remark 2.2.2. The proof of this lemma relies on "elementary" geometry; since it is very technical, we will not give it. However, we just remark that the statement is very reasonable. The first condition says that each ball touches B_n; the second condition says that the balls $B_1, \dots B_{n-1}$ are not too small with respect to B_n; the third condition says that the balls do not overlap too much. The situation is resumed in figure 2.1.

First Besicovitch's covering theorem

Lemma 2.2.3 (Besicovitch)**.** *Let $\mathcal{F}$ be a family of balls in $\mathbb{R}^d$ with bounded radii. Let E be the set of the centers of the balls in $\mathcal{F}$. Then, there exist $\mathcal{G}_1, \dots, \mathcal{G}_N \subseteq \mathcal{F}$ s.t.*

- *each $\mathcal{G}_i$ is disjoint,*
- *$\mathcal{G} := \bigcup_{i=1}^N \mathcal{G}_i$ covers E,*
- *N depends only on d (in deed, N is the integer given by lemma 2.2.1).*

Proof. **Step 1**: First we consider the case where the balls in $\mathcal{F}$ have comparable radii, that is for all $B(x_i, r_i), B(x_j, r_j) \in \mathcal{F}$ we have that $r_i \le 2r_j$. Given $N = N(d)$ as in lemma 2.2.1, take $(\mathcal{G}_1, \dots, \mathcal{G}_N)$ with the following properties:

- each $\mathcal{G}_i$ is a disjoint subfamily of $\mathcal{F}$;
- for all $B = B(x, r) \in \mathcal{G}_i$, then $x \notin B'$, where B' is any ball of $\mathcal{G}_j$, for all $j \neq i$;
- $(\mathcal{G}_1, \dots, \mathcal{G}_N)$ is maximal with respect to the partial order

$$(\mathcal{G}'_1, \dots, \mathcal{G}'_N) \le (\mathcal{G}_1, \dots, \mathcal{G}_N)$$

defined by

$$\mathcal{G}'_i \subseteq \mathcal{G}_i \quad \forall i.$$

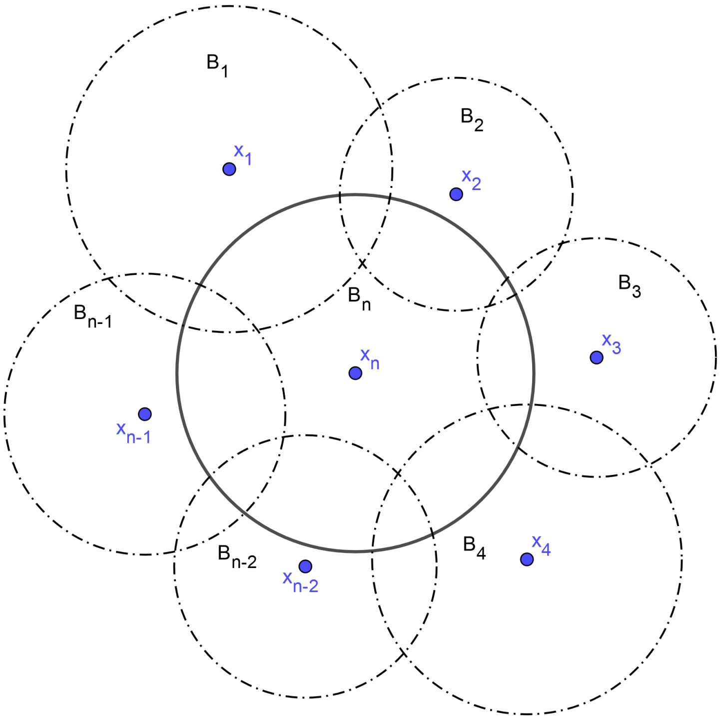

Figure 2.1: The balls in Lemma 2.2.1.

The existence of such $(\mathcal{G}_1, \dots, \mathcal{G}_N)$ follows immediately by the Zorn's lemma. Notice that we allow some of the $\mathcal{G}_i$ to be empty. Now, take $\tilde{B} = B(x, r) \in \mathcal{F}$; we claim that

$$x \in \bigcup_{i=1}^{N} \left(\bigcup_{B \in \mathcal{G}_i} B \right).$$

Suppose that $\mathcal{G}_N = \emptyset$. Assume by contradiction that

$$x \notin \bigcup_{i=1}^{N} \left(\bigcup_{B \in \mathcal{G}_i} B \right);$$

thus, it holds that

$$(\mathcal{G}_1, \dots, \mathcal{G}_{N-1}, \mathcal{G}_N) \lneq (\mathcal{G}_1, \dots, \mathcal{G}_{N-1}, \{\tilde{B}\});$$

however, $(\mathcal{G}_1, \dots, \mathcal{G}_{N-1}, \mathcal{G}_N)$ is maximal and this is a contradiction. Hence, we are left with the case where any $\mathcal{G}_i$ is not empty. Still, assume by contradiction that

$$x \notin \bigcup_{i=1}^{N} \left(\bigcup_{B \in \mathcal{G}_i} B \right);$$

then, by maximality, for all i there exists $B_i \in \mathcal{G}_i$ s.t. $B \cap B_i \neq \emptyset$. Then, the family $\{B_1, \dots, B_N, B\}$ contradicts lemma 2.2.1.

Step 2: We consider the general case, where $\mathcal{F} = \{B(x_i, r_i)\}_i$ and

$$R \coloneqq \sup_i r_i < +\infty.$$

We consider the decomposition

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n, \quad \mathcal{F}_n := \left\{ B(x_i, r_i) \in \mathcal{F} \;\middle|\; \frac{R}{2^{n+1}} < r \leq \frac{R}{2^n} \right\}.$$

Now, we extract $\mathcal{G}_{0,1}, \ldots, \mathcal{G}_{0,N}$ from $\mathcal{F}_0$ as in the previous step. Thus, $\bigcup_{i=1}^N \mathcal{G}_{0,i}$ covers the set of the centers of the balls in $\mathcal{F}_0$.

Define

$$\mathcal{F}_1' := \left\{ B(x_i, r_i) \in \mathcal{F}_1 \;\middle|\; x_i \notin \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_{0,i}} B \right\}.$$

Extract $\mathcal{G}_{1,1}, \ldots \mathcal{G}_{1,N}$ from $\mathcal{F}_1'$ as in the previous step, with the additional assumption that $\mathcal{G}_{1,i}$ is disjoint from $\mathcal{G}_{0,i}$ for all $i = 1, \ldots, N$. We claim that $\bigcup_{i=1}^N \mathcal{G}_{0,i} \cup \mathcal{G}_{1,i}$ covers the set of the centers of the balls in $\mathcal{F}_0 \cup \mathcal{F}_1'$. Notice that if $B \in \mathcal{F}_0$, then the center of B is covered by $\bigcup_{i=1}^N \mathcal{G}_{0,i}$. Let $B \in \mathcal{F}_1'$; assume by contradiction that

$$x \notin \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_{0,i} \cup \mathcal{G}_{1,i}} B.$$

In particular, it follows that

$$x \notin \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_{1,i}} B;$$

with the same arguments explained in the previous step, we can easily reduce to the case in which each of the $\mathcal{G}_{1,i}$ is not empty. Thus, for all $i = 1, \ldots, N$ there exists $B_i \in \mathcal{G}_{1,i}$ s.t. $B \cap B_i \neq \emptyset$. Hence, the family $(B_1, \ldots, B_N, B)$ contradicts lemma 2.2.1. At this point, it is immediate to see that $\bigcup_{i=1}^N \mathcal{G}_{0,i} \cup \mathcal{G}_{1,i}$ covers the set of the centers of the balls in $\mathcal{F}_0 \cup \mathcal{F}_1$. Also note that $\mathcal{G}_{0,i} \cup \mathcal{G}_{1,i}$ is a disjoint family of balls (by construction).

Step 3: By induction, we define $\mathcal{G}_{n,1}, \ldots, \mathcal{G}_{n,N}$ s.t.

- $\mathcal{G}_{0,i} \cup \cdots \cup \mathcal{G}_{n,i}$ is a disjoint family of balls in $\mathcal{F}$,
- $\bigcup_{i=1}^N \mathcal{G}_{0,i} \cup \cdots \cup \mathcal{G}_{n,i}$ covers the set of the centers of the balls in $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$.

Finally, we can define

$$\mathcal{G}_i := \bigcup_{n=0}^{\infty} \mathcal{G}_{n,i}.$$

Notice that the balls in $\mathcal{G}_i$ are pairwise disjoint and $\bigcup_{i=1}^N \mathcal{G}_i$ covers the set of the centers of the balls in $\bigcup_{n=0}^\infty \mathcal{F}_n = \mathcal{F}$. □

Remark 2.2.4. Lemma 2.2.5 says that, given $\mathcal{F}$ a family of balls with bounded radii, we can cover the set of the centers of these balls with at most N subfamilies $\mathcal{G}_i$, s.t. each $\mathcal{G}_i$ is disjoint.

Lemma 2.2.5. *Let μ be a Borel, locally finite measure on $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be a Borel set s.t. $\mu(E) < +\infty$. Let $\mathcal{F}$ be a family of balls with bounded radii whose centers cover E. Then, there exists $\mathcal{G} \subseteq \mathcal{F}$ disjoint s.t.*

$$\mu\left(E \cap \left(\bigcup_{B \in \mathcal{G}} B\right)\right) \geq \frac{1}{N}\mu(E),$$

where $N = N(d) > 0$ is the integer given by lemma 2.2.1 (it only depends on the dimension of the space d).

Proof. Apply lemma 2.2.3 to $\mathcal{F}$ and obtain $\mathcal{G}_1, \dots \mathcal{G}_N \subseteq \mathcal{F}$ s.t. each $\mathcal{G}_i$ is disjoint and

$$E \subseteq \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_i} B.$$

Then, we have

$$\mu(E) = \mu\left(\bigcup_{i=1}^{N}\left(E \cap \bigcup_{B\in\mathcal{G}_i} B\right)\right) \le \sum_{i=1}^{N} \mu\left(E \cap \bigcup_{B\in\mathcal{G}_i} B\right).$$

Thus, there exists i_0 s.t.

$$\mu\left(E \cap \left(\bigcup_{B\in\mathcal{G}_{i_0}} B\right)\right) \ge \frac{1}{N}\mu(E).$$

To conclude, simply denote $\mathcal{G} \coloneqq \mathcal{G}_{i_0}$. $\square$

The definition below will play a fundamental role in the following results.

Definition 2.2.6 (Besicovitch's covering)**.** Let $E \subseteq \mathbb{R}^d$; let $\mathcal{F}$ be a family of balls in $\mathbb{R}^d$ s.t.

$$\inf\{r \mid B(x,r) \in \mathcal{F}\} = 0 \quad \forall x \in E.$$

We say that $\mathcal{F}$ is a Besicovitch's covering of E.

Remark 2.2.7. Note that every Besicovitch's covering of E (see 2.2.6) is a fine covering of E (see 2.1.5).

Theorem 2.2.8 (Besicovitch's covering theorem - 1)**.** *Let μ be a Borel, locally finite measure on $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be a Borel set s.t. $\mu(E) < +\infty$. Let $\mathcal{F}$ be a family of closed balls which is a Besicovitch's covering of E (see 2.2.6). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ s.t.*

- *$\mathcal{F}'$ is disjoint and at most countable,*
- *$\mathcal{F}'$ covers μ-a.a. of E,*
- *$\sum_{B\in\mathcal{F}'} \mu(B) \le \mu(E) + \varepsilon$.*

Proof. Fix $\varepsilon > 0$ and an open set A s.t. $E \subseteq A$ and $\mu(A) \le \mu(E) + \varepsilon$ (the existence of such A is provided by the approximation result stated in 1.1.17).

Step 1: Define

$$\mathcal{F}_0 \coloneqq \{B = \overline{B}(x,r) \in \mathcal{F} \mid B \subseteq A,\ r \le 1\}.$$

Since $\mathcal{F}$ is a Besicovitch's covering of E, then the same holds for $\mathcal{F}_0$. Given $N = N(d)$ as in lemma 2.2.1, choose $\mathcal{G}_0 \subseteq \mathcal{F}_0$ as in lemma 2.2.5: more precisely, $\mathcal{G}_0$ is disjoint and s.t.

$$\mu\left(E \cap \bigcup_{B\in\mathcal{G}_0} B\right) \ge \frac{\mu(E)}{N}.$$

We can take $\mathcal{G}_0' \subseteq \mathcal{G}_0$ finite s.t.

$$\mu\left(E \cap \bigcup_{B\in\mathcal{G}_0'} B\right) \geq \frac{\mu(E)}{2N}.$$

Thus, we have that

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{G}_0'} B\right) \leq \left(1 - \frac{1}{2N}\right)\mu(E).$$

Step 2: Set

$$E_1 \coloneqq E \setminus \bigcup_{B\in\mathcal{G}_0'} B$$

and

$$\mathcal{F}_1 \coloneqq \left\{B \in \mathcal{F}_0 \,\middle|\, B \cap \bigcup_{B'\in\mathcal{G}_0'} B' = \emptyset\right\}.$$

Since $\mathcal{G}_0'$ is finite, then $\bigcup_{B'\in\mathcal{G}_0'} B'$ is closed; hence, it is immediate to show that $\mathcal{F}_1$ is a Besicovitch's covering of E_1. As in the previous step, take $\mathcal{G}_1' \subseteq \mathcal{F}_1$ finite and disjoint s.t.

$$\mu\left(E_1 \setminus \bigcup_{B\in\mathcal{G}_1'} B\right) \leq \left(1 - \frac{1}{2N}\right)\mu(E_1) \leq \left(1 - \frac{1}{2N}\right)^2\mu(E).$$

Since

$$E_1 \setminus \bigcup_{B\in\mathcal{G}_1'} = E \setminus \bigcup_{B\in\mathcal{G}_0'\cup\mathcal{G}_1'} B,$$

we have that

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{G}_0'\cup\mathcal{G}_1'} B\right) \leq \left(1 - \frac{1}{2N}\right)^2\mu(E).$$

Notice that $\mathcal{G}_0' \cup \mathcal{G}_1'$ is disjoint.

Step 3: Inductively, we define $\mathcal{G}_0', \ldots, \mathcal{G}_n'$ s.t.

1. each $\mathcal{G}_i'$ is finite and disjoint,
2. $\bigcup_{i=0}^n \mathcal{G}_i'$ is finite and disjoint,
3. there holds that

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{G}_0'\cup\cdots\cup\mathcal{G}_n'} B\right) \leq \left(1 - \frac{1}{2N}\right)^{n+1}\mu(E).$$

Thus, if we define $\mathcal{F}' \coloneqq \bigcup_{n=0}^\infty \mathcal{G}_n'$, the conclusion follows immediately. □

Second Besicovitch's covering theorem

Lemma 2.2.9. *Let μ be a Borel, locally finite measure on $\mathbb{R}^d$. Let $E_0 \subseteq \mathbb{R}^d$ be a Borel set s.t. $\mu(E_0) = 0$. Let $\mathcal{F}$ be a family of closed balls which is a Besicovitch's covering of E_0 (see 2.2.6). Then, for all $\varepsilon > 0$ there exists $\mathcal{G} \subseteq \mathcal{F}$ (not necessarily disjoint) s.t.*

- *$\mathcal{G}$ is at most countable,*
- *$\mathcal{G}$ covers E_0,*
- *$\sum_{B\in\mathcal{G}} \mu(B) \leq \varepsilon$.*

Proof. Fix $\varepsilon > 0$ and choose an open set A s.t. $E_0 \subseteq A$ and $\mu(A) \leq \varepsilon/N$, where N is the integer given by lemma 2.2.1 (the existence of such A is provided by the approximation result stated in 1.1.17). Define

$$\mathcal{F}' \coloneqq \{B = \overline{B}(x,r) \in \mathcal{F} \mid B \subseteq A,\ r \leq 1\}.$$

Notice that $\mathcal{F}'$ is still a Besicovitch's covering of E_0. We can use lemma 2.2.3 to find $\mathcal{G}_1, \dots, \mathcal{G}_N$ disjoint s.t. $\mathcal{G} \coloneqq \bigcup_{i=1}^N \mathcal{G}_i$ covers E_0. Then, we have that

$$\sum_{B\in\mathcal{G}_i} \mu(B) = \mu\left(\bigcup_{B\in\mathcal{G}_i} B\right) \leq \mu(A) \leq \frac{\varepsilon}{N};$$

thus

$$\sum_{B\in\mathcal{G}} \mu(B) \leq N\frac{\varepsilon}{N} = \varepsilon.$$

□

Theorem 2.2.10 (Besicovitch's covering theorem - 2)**.** *Let μ be a Borel, locally finite measure on $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be a Borel set s.t. $\mu(E) < +\infty$. Let $\mathcal{F}$ be a family of closed balls which is a Besicovitch's covering of E (see 2.2.6). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ (not necessarily disjoint) s.t.*

- *$\mathcal{F}'$ is at most countable;*
- *$\mathcal{F}'$ covers E;*
- *$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.*

Proof. Cover μ-a.a. of E using the first Besicovitch's covering theorem (see 2.2.8); then, use the lemma above (see 2.2.9) to cover the remaining part (which is μ-null). □

2.3 $\mathscr{L}^d$ *vs* $\mathcal{H}^d$ in $\mathbb{R}^d$

Covering theorems are very powerful tools in Geometric Measure Theory that have a huge variety of applications. Here, as an example, we show that the Lebesgue measure and the d-dimensional Hausdorff measure agree in $\mathbb{R}^d$. In particular, we prove that $\mathcal{H}^d = \mathcal{H}^d_\delta = \mathscr{L}^d$ in $\mathbb{R}^d$ for all $\delta > 0$. The inequality $\mathscr{L}^d \geq \mathcal{H}^d_\delta$ is a straightforward application of covering theorems.

Proposition 2.3.1. *Given $\delta > 0$ and $E \subseteq \mathbb{R}^d$ a Borel set, then $\mathcal{H}^d_\delta(E) \leq \mathscr{L}^d(E)$. In particular, it holds that $\mathcal{H}^d(E) \leq \mathscr{L}^d(E)$.*

Proof. If $\mathscr{L}^d(E) = +\infty$, there is nothing to prove. So, we can assume that $\mathscr{L}^d(E)$ is finite. Given $\varepsilon > 0$, we can use second Besicovitch's covering theorem (see 2.2.10) to find a countable family $\mathcal{F} = (B_i)_i$ of balls with radii bounded from above by δ s.t.

- $E \subseteq \bigcup_i B_i$,
- $\sum_i \mathscr{L}^d(B_i) \leq \mathscr{L}^d(E) + \varepsilon$.

Denote α_d the volume of the unit ball in $\mathbb{R}^d$ and recall the definition of Hausdorff measure (see 1.2.1). Thus

$$\varepsilon + \mathscr{L}^d(E) \geq \sum_i \mathscr{L}^d(B_i) \geq \sum_i \frac{\alpha_d}{2^d}(\mathrm{diam}(B_i))^d \geq c_d \sum_i (\mathrm{diam}(B_i))^d \geq \mathcal{H}^d_\delta(E).$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\mathscr{L}^d(E) \geq \mathcal{H}^d_\delta(E)$. Taking the supremum with respect to $\delta > 0$, we conclude the proof. $\square$

Remark 2.3.2. For the reverse inequality, it is enough to show that $\mathcal{H}^d_\infty \geq \mathscr{L}^d$. If we prove this claim, we conclude that

$$\mathscr{L}^d \geq \mathcal{H}^d \geq \mathcal{H}^d_\delta \geq \mathcal{H}^d_\infty \geq \mathscr{L}^d \quad \forall \delta \in (0, +\infty],$$

that is $\mathcal{H}^d = \mathscr{L}^d$.

The proof of the claim above is based on the isodiametric inequality, which relies on the Steiner symmetrization. We briefly estabilish this usefull tools.

Definition 2.3.3 (Steiner symmetrization)**.** Given $E \subseteq \mathbb{R}^d$ Borel and an hyperplane $V \subseteq \mathbb{R}^d$, the Steiner symmetrization of E with respect to V is the set E' defined by

$$E' \coloneqq \{(x', x'') \in V \times V^\perp \mid |x''| \leq h(x')\},$$

where

$$h(x') \coloneqq \frac{1}{2}\mathscr{L}^1(E_{x'}), \quad E_{x'} \coloneqq (x' + V^\perp) \cap E.$$

Remark 2.3.4. In other words, the Steiner symmetrization of E with respect to the hyperplane V (see 2.3.3) is the set E' where all the sections of E made by segments parallel to $V^\perp$ have been centered with respect to V (see, for instance, figure 2.2).

The following properties follows straightforward from definition 2.3.3.

Lemma 2.3.5. *Given $E \subseteq \mathbb{R}^d$ Borel and an hyperplane $V \subseteq \mathbb{R}^d$, denote E' the Steiner symmetrization of E with respect to V (see 2.3.3). Then, the followings hold true:*

1. *E' is Lebesgue measurable and $\mathscr{L}^d(E) = \mathscr{L}^d(E')$;*
2. *diam$(E') \leq$ diam(E).*

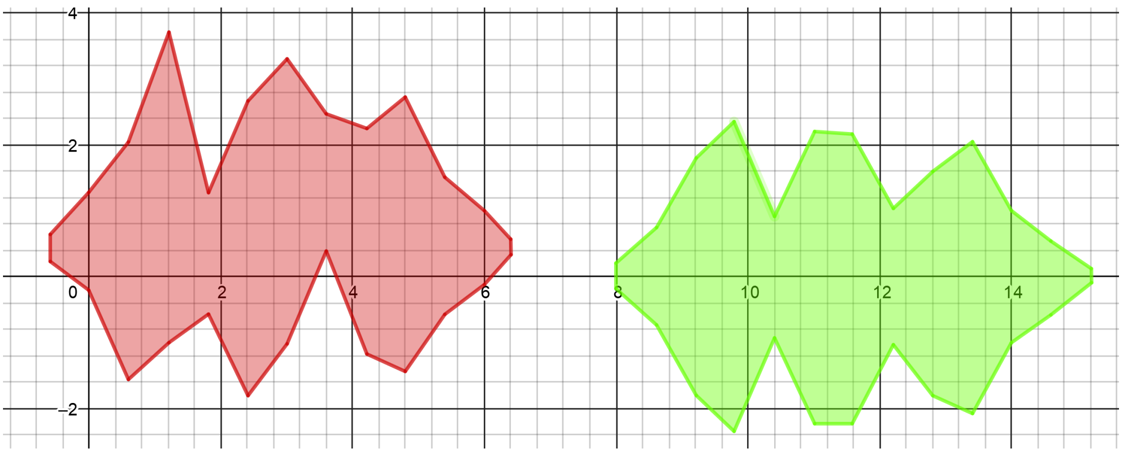

Figure 2.2: In red the set E and in green the set E', which is the Steiner symmetric of E with respect to the axis $y = 0$ in $\mathbb{R}^2$.

Proof. The first statement is an immediate consequence of Fubini's theorem.

As for the second statement, we can assume that E is bounded (otherwise, $\operatorname{diam}(E) = +\infty$ and there is nothing to prove). Under this assumption, clearly E' is bounded too. Denote by $P_V : \mathbb{R}^d \to V$ the orthogonal projection to V. Given $\varepsilon > 0$, there exist $x_1, x_2 \in E'$ s.t.

$$(\operatorname{diam}(E'))^2 \leq |x_1 - x_2|^2 + \varepsilon.$$

Without loss of generality, we can assume that x_1, x_2 are "extremal" points in E', in the sense that $x_1 = (x_1', h(x_1')), x_2 = (x_2', -h(x_2')) \in P_V(E) \times V^\perp$. Thus

$$(\operatorname{diam}(E'))^2 \leq |x_1 - x_2|^2 + \varepsilon = |x_1' - x_2'|^2 + |h(x_1') + h(x_2')|^2 + \varepsilon.$$

Note that

$$\begin{aligned} h(x_1') + h(x_2') &\leq \frac{1}{2}\left(\sup E_{x_1'} - \inf E_{x_1'}\right) + \frac{1}{2}\left(\sup E_{x_2'} - \inf E_{x_2'}\right) \\ &= \frac{1}{2}\left(\sup E_{x_1'} - \inf E_{x_2'}\right) + \frac{1}{2}\left(\sup E_{x_2'} - \inf E_{x_1'}\right) \\ &\leq \max\left\{\sup E_{x_1'} - \inf E_{x_2'}; \sup E_{x_2'} - \inf E_{x_1'}\right\}. \end{aligned}$$

Choose $x_1'' \in E_{x_1'}, x_2'' \in E_{x_2'}$ s.t.

$$\left(\max\left\{\sup E_{x_1'} - \inf E_{x_2'}; \sup E_{x_2'} - \inf E_{x_1'}\right\}\right)^2 \leq |x_1'' - x_2''|^2 + \varepsilon.$$

Denote $\tilde{x}_1 = (x_1', x_1'')$ and $\tilde{x}_2 = (x_2', x_2'')$; thus, we obtain that

$$\begin{aligned} (\operatorname{diam}(E'))^2 &\leq |x_1' - x_2'|^2 + \left(\max\left\{\sup E_{x_1'} - \inf E_{x_2'}; \sup E_{x_2'} - \inf E_{x_1'}\right\}\right)^2 \\ &\leq |x_1' - x_2'|^2 + |x_1'' - x_2''|^2 + 2\varepsilon \\ &= |\tilde{x}_1 - \tilde{x}_2|^2 + 2\varepsilon \\ &\leq (\operatorname{diam}(E))^2 + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows immediately. □

Remark 2.3.6. Pick V_1, V_2 two orthogonal hyperplanes in $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be Borel. Denote by E' the Steiner symmetrization of E with respect to V_1 and E'' the Steiner symmetrization of E' with respect to V_2. It is immediate to see that the two Steiner symmetrizations commute (it is crucial that V_1, V_2 are orthogonal hyperplanes). Moreover, since Steiner symmetric with respect to V implies symmetric with respect to V, then E'' is symmetric with respect to V_1 and V_2. So, if we pick $V_1, \dots, V_d$ orthogonal hyperplanes in $\mathbb{R}^d$, the set $E^{(d)}$ given by d Steiner symmetrizations of E (with respect to those hyperplanes) will be symmetric with respect to $V_1, \dots, V_d$. In particular, $E^{(d)}$ will contain the origin and it will be symmetric with respect to the origin. Moreover, lemma 2.3.5 guarantees that $\operatorname{diam}(E^{(d)}) \leq \operatorname{diam}(E)$ and $\mathscr{L}^d(E^{(d)}) = \mathscr{L}^d(E)$.

Now, we are ready to state and prove the isodiametric inequality.

Theorem 2.3.7 (Isodiametric inequality)**.** *For every $E \subseteq \mathbb{R}^d$ Borel there holds that*

$$\mathscr{L}^d(E) \leq \frac{\alpha_d}{2^d}(diam(E))^d.$$

In other words, the ball has the largest volume among the Borel sets with prescribed diameter.

Proof. Let $E^{(d)}$ be the Steiner symmetrization of E with respect to the coordinate hyperplanes. As explained in 2.3.6, $E^{(d)}$ contains the origin. Thus, we have that

$$E^{(d)} \subseteq \overline{B}\left(0, \frac{\operatorname{diam}(E^{(d)})}{2}\right) = B.$$

Since $\operatorname{diam}(E) \geq \operatorname{diam}(E^{(d)}) = \operatorname{diam}(B)$, we deduce that

$$\alpha^d\left(\frac{\operatorname{diam}(E)}{2}\right)^d \geq \alpha_d\left(\frac{\operatorname{diam}(B)}{2}\right)^d = \mathscr{L}^d(B) \geq \mathscr{L}^d(E^{(d)}) = \mathscr{L}^d(E).$$

□

Finally, we can show the inequality $\mathcal{H}^d_\infty \geq \mathscr{L}^d$.

Theorem 2.3.8. *Given $E \subseteq \mathbb{R}^d$ Borel, then $\mathcal{H}^d_\infty(E) \geq \mathscr{L}^d(E)$.*

Proof. Take $(E_i)_i$ a covering of E; by the isodiametric inequality (see 2.3.7), it follows that

$$\frac{\alpha_d}{2^d}\sum_i(\operatorname{diam}(E_i))^d \geq \sum_i \mathscr{L}^d(E_i) \geq \mathscr{L}^d(E).$$

To conclude, take the infimum with respect to the coverings of E. □

Chapter 3

Densities of sets

In the following chapter, we assume $\mathbb{X}$ to be a locally compact, separable metric space (or $\mathbb{X} = \mathbb{R}^d$). The following results can be found in [5], [4] and [1], [3] (as for the fractals construction).

3.1 Density of a set with respect to a measure

Definition 3.1.1 (Density of a set)**.** Let μ be a Borel locally finite measure on $\mathbb{X}$. Take $x \in \operatorname{supp}(\mu)$. We define the upper density of E at x with respect to μ as

$$\Theta^*_\mu(E, x) := \limsup_{r \to 0} \frac{\mu(E \cap \overline{B}(x, r))}{\mu(\overline{B}(x, r))};$$

we define the lower density of E at x with respect to μ as

$$\Theta_{*\mu}(E, x) := \liminf_{r \to 0} \frac{\mu(E \cap \overline{B}(x, r))}{\mu(\overline{B}(x, r))}.$$

We say that E admits density at x with respect to μ if the lower density and the upper density defined above coincide. In this case, we write

$$\Theta_\mu(E, x) := \Theta_{*\mu}(E, x) = \Theta^*_\mu(E, x).$$

Theorem 3.1.2. *Let $E \subseteq \mathbb{X}$ be a Borel set; let μ be a Borel locally finite measure on $\mathbb{X}$. Assume that one of the followings assumptions hold true:*

1. $\mathbb{X} = \mathbb{R}^d$*;*
2. μ *is doubling.*

Then, $\Theta_\mu(E, x)$ exists at μ-a.e. $x \in \mathbb{X}$ (see 3.1.1) and there holds that

$$\Theta_\mu(E, x) = \lim_{r \to 0} \frac{\mu(E \cap \overline{B}(x, r))}{\mu(\overline{B}(x, r))} = \begin{cases} 1 & \text{for } \mu\text{-a.e } x \in E, \\ 0 & \text{for } \mu\text{-a.e } x \in E^c. \end{cases}$$

Proof. **Step 1**: We show that for μ-a.e. $x \notin E$, then $\Theta_\mu(E, x)$ exists and it is 0. Given $\delta > 0$, define

$$E_\delta := \{x \notin E \mid \Theta^*_\mu(E, x) > \delta\}.$$

We claim that $\mu(E_\delta) = 0$; if we show this property for all $\delta > 0$, we conclude that $\Theta_\mu(E, x)$ exists and it is zero at μ-a.e. $x \in E^c$. Set $\lambda := \mu \llcorner E$ and define

$$\begin{aligned}\mathcal{F} &:= \{\overline{B}(x,r) \mid x \in E_\delta,\ r \le 1,\ \mu(E \cap \overline{B}(x,r)) \ge \delta\mu(\overline{B}(x,r))\} \\ &= \{\overline{B}(x,r) \mid x \in E_\delta,\ r \le 1,\ \lambda(\overline{B}(x,r)) \ge \delta\mu(\overline{B}(x,r))\}\end{aligned}$$

Assume that $\mathbb{X} = \mathbb{R}^d$; then, it is immediate to see that $\mathcal{F}$ is a Besicovitch's covering of E_δ (see 2.2.6) and λ is a Borel locally finite measure on $\mathbb{R}^d$. Since $\lambda(E_\delta) = 0$ (λ is concentrated on E and $E_\delta \subseteq E^c$), given $\varepsilon > 0$, second Besicovitch's covering theorem (see 2.2.10) guarantees that there exists $\mathcal{G} \subseteq \mathcal{F}$ s.t. $\mathcal{G}$ covers E_δ and $\sum_{B\in\mathcal{G}} \lambda(B) \le \varepsilon$. Thus, we have

$$\varepsilon \ge \sum_{B\in\mathcal{G}} \lambda(B) \ge \delta \sum_{B\in\mathcal{G}} \mu(B) \ge \delta\mu(E_\delta).$$

It follows that

$$\mu(E_\delta) \le \frac{\varepsilon}{\delta} \quad \forall \varepsilon > 0;$$

we conclude that $\mu(E_\delta) = 0$.

If $\mathbb{X}$ is an arbitrary locally compact separable metric space and μ is doubling, the proof is absolutely the same, except that we need to use Vitali's covering theorems. In particular, $\mathcal{F}$ is a fine covering of E_δ (see 2.1.5). Thus, apply theorem 2.1.11 (with the measure $\lambda = \mu \llcorner E$, the covering $\mathcal{F}$ and the set E_δ). Then, we conclude as in the case in which $\mathbb{X} = \mathbb{R}^d$.

Step 2: The fact that $\Theta_\mu(E, x)$ exists and it is 1 μ-a.e. in E follows from the previous step applied to E^c and the fact that

$$\Theta^*_\mu(E^c, x) = 1 - \Theta_{*\mu}(E, x)$$

for all $x \in \operatorname{supp}(\mu)$. □

3.2 d-dimensional density

Definition 3.2.1 (d-dimensional density)**.** Given a Borel set $E \subseteq \mathbb{X}$, $d \in (0, +\infty)$ and $x \in \mathbb{X}$, we define the upper d-dimensional density of E at x as

$$\Theta^*_d(E, x) := \limsup_{r\to 0} \frac{\mathcal{H}^d(E \cap \overline{B}(x,r))}{\alpha_d r^d},$$

where $\alpha_d = 2^d$ if d is not integer, otherwise α_d is the volume of the unit ball in $\mathbb{R}^d$, namely it is the same constant in the construction of the Hausdorff measure (see 1.2.1). We define the lower d-dimensional density of E at x as

$$\Theta_{*d}(E, x) := \liminf_{r\to 0} \frac{\mathcal{H}^d(E \cap \overline{B}(x,r))}{\alpha_d r^d}.$$

Remark 3.2.2. In the following, we will provide estimates for the upper d-dimensional density; we will never deal with the lower d-dimensional density. However, we mention that there exists a Borel set in $\mathbb{R}^m$ and $d \in (0, +\infty)$ s.t. $0 < \mathcal{H}^d(E) < +\infty$ and $\Theta_{*d}(E, x) = 0$ for $\mathcal{H}^d$-a.e. $x \in E$.

Theorem 3.2.3. *Define $E \subseteq \mathbb{X}$ be a Borel set. Assume that $\mathcal{H}^d(E) < +\infty$ for some $d \in (0, +\infty)$. Then, the followings hold true:*

- $\Theta^*_d(E,x) = 0$ *for* $\mathcal{H}^d$*-a.e.* $x \notin E$*;*
- $\Theta^*_d(E,x) \geq \frac{1}{2^d}$ *for* $\mathcal{H}^d$*-a.e.* $x \in E$*;*
- *if* $\mathbb{X} = \mathbb{R}^m$*, then* $\Theta^*_d(E,x) \leq 1$ *for* $\mathcal{H}^d$*-a.e.* $x \in E$*;*
- *if* $\mathbb{X}$ *is any locally compact, separable metric space, then* $\Theta^*_d(E,x) \leq 5^d$ *for* $\mathcal{H}^d$*-a.e.* $x \in E$*.*

Proof. Let $\mu \coloneqq \mathcal{H}^d \llcorner E$; since $\mathcal{H}^d(E) < +\infty$, then μ is a finite measure.

Step 1: We prove the first statement. Given $\delta > 0$, we define

$$E_\delta \coloneqq \{x \notin E \mid \Theta^*_d(E,x) > \delta\}.$$

If we show that $\mathcal{H}^d(E_\delta) = 0$ for every $\delta > 0$, then we immediately conclude that $\Theta^*_d(E,x) = 0$ for $\mathcal{H}^d$-a.e. $x \notin E$. Since $E_\delta \cap E = \emptyset$, then $\mu(E_\delta) = 0$. Fix $\varepsilon > 0$ and pick an open set A s.t. $E_\delta \subseteq A$ and $\mu(A) \leq \varepsilon$ (the existence of such an open set is provided by the approximation result of measures stated in 1.1.17). Fix $\rho > 0$ and define

$$\begin{aligned}\mathcal{F} &\coloneqq \{\overline{B}(x,r) \mid x \in E_\delta,\ r \leq \rho,\ \overline{B}(x,r) \subseteq A,\ \mathcal{H}^d(E \cap \overline{B}(x,r)) \geq \delta\alpha_d r^d\}\\ &= \{\overline{B}(x,r) \mid x \in E_\delta,\ r \leq \rho,\ \overline{B}(x,r) \subseteq A,\ \mu(\overline{B}(x,r)) \geq \delta\alpha_d r^d\}.\end{aligned}$$

Clearly, $\mathcal{F}$ is a covering of E_δ. We can apply the Vitali's $5r$-lemma (see 2.1.3); thus, we find $\mathcal{G} \subseteq \mathcal{F}$ countable and disjoint s.t. $\hat{\mathcal{G}}$ covers E_δ. Hence, we have

$$\begin{aligned}\varepsilon \geq \mu(A) &\geq \mu\left(\bigcup_{B \in \mathcal{G}} B\right)\\ &= \sum_{B \in \mathcal{G}} \mu(B) \geq \delta\frac{\alpha_d}{2^d}\sum_{B \in \mathcal{G}}(\mathrm{diam}(B))^d\\ &\geq \delta\frac{\alpha_d}{2^d 5^d}\sum_{B \in \mathcal{G}}(\mathrm{diam}(\hat{B}))^d \geq \delta\frac{1}{5^d}\mathcal{H}^d_{10\rho}(E_\delta),\end{aligned}$$

where the last inequality follows from the fact that $\hat{G}$ is a 10ρ-covering of E_δ. We deduce that

$$\mathcal{H}^d_{10\rho}(E_\delta) \leq \varepsilon\frac{5^d}{\delta}.$$

Since ε is arbitrary, we conclude that $\mathcal{H}^d_{10\rho}(E_\delta) = 0$; taking the supremum with respect to ρ, we obtain that $\mathcal{H}^d(E_\delta) = 0$.

Step 2: We prove the second statement. Given $\lambda < \frac{1}{2^d}$ and $r_0 > 0$, we define

$$\begin{aligned}E_{\lambda,r_0} &\coloneqq \{x \in E \mid \mathcal{H}^d(E \cap \overline{B}(x,r)) \leq \lambda\alpha_d r^d\ \forall r \leq r_0\}\\ &= \{x \in E \mid \mu(\overline{B}(x,r)) \leq \lambda\alpha_d r^d\ \forall r \leq r_0\}\end{aligned}$$

We claim that $\mathcal{H}^d(E_{\lambda,r_0}) = 0$; then, we deduce that $\mathcal{H}^d(E_\lambda) = 0$ for all $\lambda < \frac{1}{2^d}$, where

$$E_\lambda \coloneqq \{x \in E \mid \Theta^*_d(E,x) < \lambda\}.$$

This is enough to conclude that $\Theta^*_d(E,x) \geq \frac{1}{2^d}$ for $\mathcal{H}^d$-a.e. $x \in E$.

Fix $\lambda < \frac{1}{2^d}$ and $r_0 > 0$; denote $E_{\lambda,r_0} = \tilde{E}$. Take $\delta \leq r_0$; by the definition of $\mathcal{H}^d_\delta$ (see 1.2.1), it follows that for all $\varepsilon > 0$ there exists $(E_i)_i$ a δ-covering of $\tilde{E}$ s.t.

$$\frac{\alpha_d}{2^d}\sum_i (\operatorname{diam}(E_i))^d \leq \mathcal{H}^d_\delta(\tilde{E}) + \varepsilon.$$

For all i, define $r_i \coloneqq \operatorname{diam}(E_i) \leq \delta$ and choose $x_i \in E_i \cap \tilde{E}$ (if the intersection is empty, then we can remove E_i). Then, $E_i \subseteq \overline{B}(x_i, r_i)$; hence, we obtain

$$\begin{aligned}
\mathcal{H}^d(\tilde{E}) + \varepsilon \geq \mathcal{H}^d_\delta(\tilde{E}) + \varepsilon &\geq \frac{\alpha_d}{2^d}\sum_i (\operatorname{diam}(E_i))^d \\
&= \frac{\alpha_d}{2^d}\sum_i r_i^d \geq \frac{1}{\lambda 2^d}\sum_i \mu(\overline{B}(x_i, r_i)) \\
&\geq \frac{1}{\lambda 2^d}\mu(\tilde{E}) = \frac{1}{\lambda 2^d}\mathcal{H}^d(\tilde{E}).
\end{aligned}$$

To resume, we have shown that

$$\frac{1}{\lambda 2^d}\mathcal{H}^d(\tilde{E}) \leq \mathcal{H}^d(\tilde{E}) + \varepsilon.$$

Since ε is arbitrary, we obtain that

$$\frac{1}{\lambda 2^d}\mathcal{H}^d(\tilde{E}) \leq \mathcal{H}^d(\tilde{E}).$$

Since $\mathcal{H}^d(\tilde{E}) \leq \mathcal{H}^d(E) < +\infty$ and $\frac{1}{\lambda 2^d} < 1$, we conclude that $\mathcal{H}^d(\tilde{E}) = 0$.

Step 3: We prove the third statement. Assume $\mathbb{X} = \mathbb{R}^m$. Given $m > 1$ we define

$$E_m \coloneqq \{x \in E \mid \Theta^*_d(E, x) > m\}.$$

We claim that $\mathcal{H}^d(E_m) = 0$; hence, we deduce that $\Theta^*_d(E, x) \leq 1$ for $\mathcal{H}^d$-a.e. $x \in E$. Fix $\delta > 0$ and define

$$\begin{aligned}
\mathcal{F} &\coloneqq \{\overline{B}(x, r) \mid x \in E_m,\ r \leq \delta,\ \mathcal{H}^d(E \cap \overline{B}(x, r)) \geq m\alpha_d r^d\} \\
&= \{\overline{B}(x, r) \mid x \in E_m,\ r \leq \delta,\ \mu(\overline{B}(x, r)) \geq m\alpha_d r^d\}.
\end{aligned}$$

Clearly, $\mathcal{F}$ is a Besicovitch covering of E_m (see 2.2.6); second Besicovitch's covering theorem (see 2.2.10) guarantees that for all $\varepsilon > 0$ there exists $\mathcal{G} \subseteq \mathcal{F}$ s.t. $\mathcal{G}$ covers E_m and there holds that

$$\sum_{B \in \mathcal{G}} \mu(B) \leq \mu(E_m) + \varepsilon.$$

Then, we have that

$$\begin{aligned}
\mathcal{H}^d(E_m) + \varepsilon = \mu(E_m) + \varepsilon &\geq \sum_{B \in \mathcal{G}} \mu(B) \\
&\geq \sum_{B \in \mathcal{G}} m\alpha_d r^d = m\frac{\alpha_d}{2^d}\sum_{B \in \mathcal{G}} (\operatorname{diam}(B))^d \\
&\geq m\mathcal{H}^d_{2\delta}(E_m).
\end{aligned}$$

To resume, we have that

$$m\mathcal{H}^d_{2\delta}(E_m) \leq \mathcal{H}^d(E_m) + \varepsilon.$$

Taking the supremum with respect to δ, we obtain that

$$m\mathcal{H}^d(E_m) \le \mathcal{H}^d(E_m) + \varepsilon;$$

since $\varepsilon > 0$ is arbitrary, we have that

$$m\mathcal{H}^d(E_m) \le \mathcal{H}^d(E_m).$$

Since $\mathcal{H}^d(E_m) \le \mathcal{H}^d(E) < +\infty$ and $m > 1$, it follows that $\mathcal{H}^d(E_m) = 0$.

Step 4: We prove the forth statement. Consider the case of any locally compact, separable metric space $\mathbb{X}$. Given $m > 5^d$, we define

$$E_m := \{x \in E \mid \Theta^*_d(E, x) > m\}.$$

We claim that $\mathcal{H}^d(E_m) = 0$; hence, we deduce that $\Theta^*_d(E, x) \le 5^d$ for $\mathcal{H}^d$-a.e. $x \in E$. Fix $\varepsilon > 0$ and pick an open set A s.t. $E_m \subseteq A$ and $\mu(A) \le \mu(E_m) + \varepsilon$ (the existence of such an open set is provided by the approximation result of measures stated in 1.1.17). Fix $\delta > 0$ and define

$$\begin{aligned}\mathcal{F} &:= \{\overline{B}(x, r) \mid x \in E_m,\ r \le \delta,\ \overline{B}(x, r) \subseteq A,\ \mathcal{H}^d(E \cap \overline{B}(x, r)) \ge m\alpha_d r^d\}\\ &= \{\overline{B}(x, r) \mid x \in E_m,\ r \le \delta,\ \overline{B}(x, r) \subseteq A,\ \mu(\overline{B}(x, r)) \ge m\alpha_d r^d\}.\end{aligned}$$

Clearly, $\mathcal{F}$ is a covering of E_m. We can apply the Vitali's $5r$-lemma (see 2.1.3); thus, we find $\mathcal{G} \subseteq \mathcal{F}$ countable and disjoint s.t. $\hat{\mathcal{G}}$ covers E_m. Hence, we have

$$\begin{aligned}\varepsilon + \mathcal{H}^d(E_m) &= \varepsilon + \mu(E_m) \ge \mu(A)\\ &\ge \mu\left(\bigcup_{B\in\mathcal{G}} B\right) = \sum_{B\in\mathcal{G}} \mu(B)\\ &\ge m\frac{\alpha_d}{2^d}\sum_{B\in\mathcal{G}}(\mathrm{diam}(B))^d\\ &\ge m\delta\frac{\alpha_d}{2^d 5^d}\sum_{B\in\mathcal{G}}(\mathrm{diam}(\hat{B}))^d\\ &\ge \frac{m}{5^d}\mathcal{H}^d_{10\delta}(E_m),\end{aligned}$$

where the last inequality follows from the fact that $\hat{G}$ is a 10δ-covering of E_m. To resume, we have that

$$\frac{m}{5^d}\mathcal{H}^d_{10\delta}(E_m) \le \varepsilon + \mathcal{H}^d(E_m).$$

Taking the supremum with respect to δ, we obtain that

$$\frac{m}{5^d}\mathcal{H}^d(E_m) \le \varepsilon + \mathcal{H}^d(E_m).$$

Then, we deduce that

$$\frac{m}{5^d}\mathcal{H}^d(E_m) \le \mathcal{H}^d(E_m).$$

Since $\mathcal{H}^d(E_m) \le \mathcal{H}^d(E) < +\infty$ and $\frac{m}{5^d} > 1$, it follows that $\mathcal{H}^d(E_m) = 0$. $\square$

We can generalize the definition of d-dimensional upper density of a set (see 3.2.1) in the following sense.

Definition 3.2.4 (d-dimensional upper density of a measure)**.** Given a locally finite Borel measure μ, $x \in \mathbb{X}$ and $d \in (0, +\infty)$ we define the d-dimensional upper density of μ at x as follows:

$$\Theta_d^*(\mu, x) := \limsup_{r \to 0} \frac{\mu(\overline{B}(x, r))}{\alpha_d r^d},$$

where α_d is as in definition 3.2.1.

Remark 3.2.5. Clearly, definitions 3.2.1 and 3.2.4 agrees if $\mu = \mathcal{H}^d \llcorner E$, for a Borel set $E \subseteq \mathbb{X}$.

Theorem 3.2.3 can be generalized as follows.

Theorem 3.2.6. *Let $d \in (0, +\infty)$. If $\mu = \rho \cdot \mathcal{H}^d$, with $\rho \in L^1_{loc}(\mathbb{X}, \mathcal{H}^d)$, then there holds that*

$$\frac{1}{2^d}\rho(x) \leq \Theta_d^*(\mu, x) \leq C_d \rho(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{X},$$

where C_d is a constant that depends only on d. Moreover, for $\mathcal{H}^d$-a.e. $x \in \mathbb{X}$ s.t. $\rho(x) > 0$ there holds

$$0 < \Theta_d^*(\mu, x) < +\infty;$$

in particular, this holds for μ-a.e. $x \in \mathbb{X}$.

Theorem above 3.2.6 can be reversed.

Theorem 3.2.7. *Let μ be a locally finite measure Borel measure on $\mathbb{X}$ and $d \in (0, +\infty)$. Assume that $\Theta_d^*(\mu, x) < +\infty$ for μ-a.e. $x \in \mathbb{X}$; then μ is absolutely continuous with respect to $\mathcal{H}^d$.*

Theorem 3.2.8. *Let μ be a locally finite measure Borel measure on $\mathbb{X}$ and $d \in (0, +\infty)$. Assume that $\Theta_d^*(\mu, x) > 0$ for μ-a.e. $x \in \mathbb{X}$. Then, there exists a Borel set E which is σ-finite with respect to $\mathcal{H}^d$ and s.t. μ is supported on E.*

If we put together the theorems above, we obtain the following result.

Theorem 3.2.9. *Let μ be a locally finite Borel measure on $\mathbb{X}$ and $d \in (0, +\infty)$. Assume that $0 < \Theta_d^*(\mu, x) < +\infty$ for $\mathcal{H}^d$-a.e. $x \in \mathbb{X}$. Then, there exists $\rho \in L^1_{loc}(\mathbb{X}, \mathcal{H}^d)$ s.t. $\mu = \rho \cdot \mathcal{H}^d$. Moreover, for $\mathcal{H}^d$-a.e. $x \in \mathbb{X}$ there holds that*

$$\frac{1}{C_d}\Theta_d^*(\mu, x) \leq \rho(x) \leq 2^d \Theta_d^*(\mu, x).$$

Proof. The statement is an immediate corollary of 3.2.8, 3.2.9, Radon-Nykodim theorem (see 1.1.15), which requires σ-finiteness, and 3.2.7. □

3.2.1 On the Radon-Nykodim derivative

The notion of d-dimensional density has a wide number of applications. Here, we provide one of them.

Lemma 3.2.10. *Let μ, λ be finite, Borel measures on $\mathbb{X}$; assume $\mathbb{X} = \mathbb{R}^n$ or μ to be doubling or $\mu = \mu' \llcorner F$, where μ' is doubling and $F \subseteq \mathbb{X}$ is Borel. If $\lambda \perp \mu$, then for μ-a.e. $x \in \mathbb{X}$ there holds that*

$$\frac{d\lambda}{d\mu}(x) := \lim_{r \to 0} \frac{\lambda(\overline{B}(x, r))}{\mu(\overline{B}(x, r))} = 0.$$

Proof. Since $\mu \perp \lambda$, there exists a Borel set S s.t. $\mu(\mathbb{X} \setminus S) = 0$ and $\lambda(S) = 0$. Without loss of generality, we can assume that $S \subseteq \operatorname{supp}(\mu)$. Thus, for μ-a.e. $x \in S$, we can well define the quantity

$$\frac{d\lambda}{d\mu}(x) := \limsup_{r \to 0} \frac{\lambda(\overline{B}(x,r))}{\mu(\overline{B}(x,r))}.$$

Given $m > 0$, define

$$E_m := \left\{ x \in S \;\middle|\; \frac{d\lambda}{d\mu}(x) > m \right\}.$$

If we show that $\mu(E_m) = 0$ for all $m > 0$, we conclude that for μ-a.e. $x \in F$ there holds

$$\frac{d\lambda}{d\mu}(x) = \limsup_{r \to 0} \frac{\lambda(\overline{B}(x,r))}{\mu(\overline{B}(x,r))} = 0;$$

hence, the conclusion follows immediately. Fix $m > 0$; for all $\varepsilon > 0$ there exists an open set A s.t. $E_m \subseteq A$ and $\lambda(A) \leq \varepsilon$, since $E_m \subseteq F$ and $\lambda(E_m) = 0$ (the approximation theorem 1.1.17 provides the existence of such an open set). Consider the family

$$\mathcal{F} := \{\overline{B}(x,r) \mid r \leq 1,\ \overline{B}(x,r) \subseteq A,\ \lambda(\overline{B}(x,r)) \geq m\mu(\overline{B}(x,r))\}.$$

Assume that $\mathbb{X} = \mathbb{R}^n$. Then, $\mathcal{F}$ is clearly a Besicovitch covering of E_m (see 2.2.6). Then, first Besicovitch's covering theorem (see 2.2.10) provides the existence of a disjoint subfamily $\mathcal{G} \subseteq \mathcal{F}$ that covers μ-a.a. of E_m. Thus, we have

$$\mu(E_m) \leq \sum_{B \in \mathcal{G}} \mu(B) \leq \frac{1}{m} \sum_{B \in \mathcal{G}} \lambda(B) = \frac{1}{m} \lambda\left(\bigcup_{B \in \mathcal{G}} B \right) \leq \frac{\lambda(A)}{m} \leq \frac{\varepsilon}{m}.$$

Since ε is arbitrary, we deduce that $\mu(E_m) = 0$.

The case of $\mathbb{X}$ locally compact, separable metric space and μ doubling is completely similar. It suffices to notice that $\mathcal{F}$ is a fine covering of E_m (see 2.1.5) and then use first Vitali's covering theorem (see 2.1.9).

The case of $\mu = \mu' \llcorner F$ (μ' doubling and $F \subseteq \mathbb{X}$ Borel) is completely similar. It suffices to use the Vitali's covering theorem in the more general version stated in 2.1.11. $\square$

Definition 3.2.11 (Approximate L^p-continuity)**.** Let μ be locally finite Borel measure on a locally compact and separable metric space $\mathbb{X}$; let f be a Borel function in $L^p(\mathbb{X}, \mu)$ for some $p \in [1, +\infty)$. We say that $\overline{x} \in \operatorname{supp}(\mu)$ is a point of L^p approximate continuity for f if the following holds true:

$$\lim_{r \to 0} \fint_{\overline{B}(x,r)} |f(x) - f(\overline{x})|^p \, d\mu(x) = 0.$$

Theorem 3.2.12. *Let μ be locally finite Borel measure on a locally compact and separable metric space $\mathbb{X}$; let f be a Borel function in $L^p(\mathbb{X}, \mu)$ for some $p \in [1, +\infty)$. Assume μ doubling of $\mathbb{X} = \mathbb{R}^n$. Then μ-a.e. $x \in \mathbb{X}$ is a point of L^p approximate continuity for f (see 3.2.11).*

Proof. Fix $\varepsilon > 0$; Lusin's theorem provides the existence of a continuous function $\tilde{f}$ and a closed set E s.t. $\mu(E^c) \leq \varepsilon$ and $f = \tilde{f}$ on E. We claim that μ-a.e. $x \in E$ is a point of L^p-approximate continuity for f. Since ε is arbitrary, this is enough to conclude. Denote $\lambda := |f|^p\, \mu \llcorner E^c$, $\tilde{\mu} := \mu \llcorner E$. Fix $\overline{x} \in \operatorname{supp}(\mu) \cap E$; denote $B_r := \overline{B}(\overline{x}, r)$. Thus

$$\begin{aligned}
\fint_{B_r} |f - f(\overline{x})|^p \, d\mu &= \frac{1}{\mu(B_r)} \int_{B_r \cap E} |f - f(\overline{x})|^p \, d\mu + \frac{1}{\mu(B_r)} \int_{B_r \setminus E} |f - f(\overline{x})|^p \, d\mu \\
&= \frac{1}{\mu(B_r)} \int_{B_r \cap E} \left|\tilde{f} - \tilde{f}(\overline{x})\right|^p \, d\mu + \frac{1}{\mu(B_r)} \int_{B_r \setminus E} |f - f(\overline{x})|^p \, d\mu \\
&\le \left(\sup_{B_r} \tilde{f} - \inf_{B_r} \tilde{f}\right)^p + \frac{2^{p-1}}{\mu(B_r)} \int_{B_r \setminus E} \left(|f(\overline{x})|^p + |f|^p\right) \, d\mu \\
&\le \left(\sup_{B_r} \tilde{f} - \inf_{B_r} \tilde{f}\right)^p + 2^{p-1} |f(\overline{x})|^p \frac{\mu(B_r \setminus E)}{\mu(B_r)} + 2^{p-1} \frac{\lambda(B_r)}{\mu(B_r)}.
\end{aligned}$$

At this point, we have three addenda and we can estimate each of them separately.

- Since $\tilde{f}$ is continuous, we have that

$$\lim_{r \to 0} \left(\sup_{B_r} \tilde{f} - \inf_{B_r} \tilde{f}\right)^p = 0.$$

- Assume that $f(\overline{x})$ is a real number (recall that $\overline{f}$ is finite at μ-a.e. $x \in \mathbb{X}$, since $f \in L^p(\mathbb{X}, \mu)$). Theorem 3.1.2 guarantees that

$$\lim_{r \to 0} \frac{\mu(B_r \setminus E)}{\mu(B_r)} = \Theta_\mu(E^c, \overline{x}) = 0$$

  for μ-a.e. $\overline{x} \in \mathbb{X}$. Then, we conclude that

$$\lim_{r \to 0} |f(\overline{x})|^p \frac{\mu(B_r \setminus E)}{\mu(B_r)} = 0 \text{ for } \mu\text{-a.e. } x \in E.$$

- As for the last addendum, we have that

$$\frac{\lambda(B_r)}{\mu(B_r)} = \frac{\lambda(B_r)}{\tilde{\mu}(B_r)} \frac{\mu(B_r \cap E)}{\mu(B_r)} \le \frac{\lambda(B_r)}{\tilde{\mu}(B_r)}.$$

  Since $\lambda \perp \tilde{\mu}$, the lemma 3.2.10 guarantees that

$$\limsup_{r \to 0} \frac{\lambda(B_r)}{\tilde{\mu}(B_r)} = 0 \text{ for } \tilde{\mu}\text{-a.e. } \overline{x} \in E.$$

  This is enough to conclude.

□

Corollary 3.2.13. *Let μ be a locally finite Borel measure on a locally compact and separable metric space $\mathbb{X}$; let f be a Borel function in $L^p(\mathbb{X}, \mu)$ for some $p \in [1, +\infty)$. Assume μ doubling of $\mathbb{X} = \mathbb{R}^n$. Then, the following holds for μ-a.e. $x \in \mathbb{X}$:*

$$\lim_{r \to 0} \fint_{\overline{B}(\overline{x}, r)} f(y) \, d\mu(y) = f(\overline{x}).$$

Proof. It is an immediate consequence of theorem see 3.2.12, the Jensen's inequality and the locally finiteness of μ. □

Corollary 3.2.14. *Let μ, λ be locally finite Borel measures on a locally compact and separable metric space $\mathbb{X}$; assume μ doubling or $\mathbb{X} = \mathbb{R}^n$. Let us consider the decomposition of λ with respect to μ given by theorem 1.1.15 $\lambda = f \cdot \mu + \lambda_s$, where f is a nonnegative function in $L^1_{loc}(\mathbb{X}, \mu)$ and λ_s is the singular part of λ with respect to μ. Then, the following holds true:*

$$f(x) = \lim_{r \to 0} \frac{\lambda(\overline{B}(x,r))}{\mu(\overline{B}(x,r))} = \lim_{r \to 0} \frac{f \cdot \mu(\overline{B}(x,r))}{\mu(\overline{B}(x,r))} \text{ for } \mu\text{-a.e. } x \in \mathbb{X}.$$

Proof. Since $\lambda_s \perp \mu$, lemma 3.2.10 guarantees that

$$\frac{d\lambda_s}{d\mu}(x) := \lim_{r \to 0} \frac{\lambda_s(\overline{B}(x,r))}{\mu(\overline{B}(x,r))} = 0 \text{ for } \mu\text{-a.e. } x \in \mathbb{X}.$$

By corollary 3.2.13, we obtain that

$$\frac{d(f \cdot \mu)}{d\mu}(x) = \lim_{r \to 0} \frac{1}{\mu(\overline{B}(x,r))} \int_{\overline{B}(x,r)} f \; d\mu = f(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{X}.$$

Then, we immediately conclude that

$$\frac{d\lambda}{d\mu}(x) = \frac{d(f \cdot \mu)}{d\mu}(x) = f(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{X}.$$

□

3.3 Self-similar fractals

We conclude this chapter on densities with the beautiful construction of self-similar fractals according to Hutchinson. As we will see, the main theorem will be an elegant application of the results established on the d-dimensional density.

Definition 3.3.1 (Contractive similarity)**.** Take $b \in \mathbb{R}^n$, $\lambda \in (0,1)$ and $R \in O(n)$; we say that the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x) = b + \lambda R(x)$$

is a contractive similarity. λ is known as scaling coefficient.

Definition 3.3.2 (Self-similar fractal)**.** Given a set $E \subseteq \mathbb{R}^n$, we say that E is a self-similar fractal (according to Hutchinson) if there exist $\Phi_1, \dots, \Phi_N$ contractive similarities (see 3.3.1) s.t.

$$E = \bigcup_{i=1}^{N} \Phi_i(E).$$

Example 3.3.3. The Cantor set C defined in 1.2.8 is a self-similar fractal. The contractive similarities are

$$\Phi_1(x) = \frac{1}{3}x, \quad \Phi_2(x) = \frac{2}{3} + \frac{1}{3}x.$$

As in the example of the Cantor set, it is easy to guess the Hausdorff dimension of a self-similar fractal.

Lemma 3.3.4. *Let $E \subseteq \mathbb{R}^n$ be a self-similar fractal and let $\Phi_1, \dots, \Phi_n$ contractive similarities such that*

$$E = \bigcup_{i=1}^{N} \Phi_i(E),$$

with scaling coefficients $\lambda_1, \dots, \lambda_N \in (0,1)$ (see 3.3.1 and 3.3.2). Assume that the $\Phi_i(E)$ are pairwise disjoint and that there exists $d \in (0,+\infty)$ s.t. $0 < \mathcal{H}^d(E) < +\infty$. Then, d is the unique solution of the equation

$$1 = \sum_{i=1}^{N} \lambda_i^d. \tag{3.1}$$

Proof. First, we notice that equation (3.1) has a unique solution. In deed, the function

$$\rho(d) := \sum_{i=1}^{N} \lambda_i^d$$

is well defined, positive and convex in $(0,+\infty)$. Moreover, we have that

$$\lim_{d \to +\infty} \rho(d) = 0, \quad \lim_{d \to 0} \rho(d) = N > 1.$$

Since $\lambda_i \in (0,1)$, the condition $N = 1$ is not compatible with the definition of self-similar fractal (see 3.3.2). Then, equation (3.1) has a unique solution in $(0,+\infty)$.

Let $d \in (0,+\infty)$ be as in the assumption. Since E is disjoint union of the $\Phi_i(E)$, then we have

$$\mathcal{H}^d(E) = \sum_{i=1}^{N} \mathcal{H}^d(\Phi_i(E)) = \sum_{i=1}^{N} \lambda_i^d \mathcal{H}^d(E);$$

since $0 < \mathcal{H}^d(E) < +\infty$, we conclude that d is the unique solution of (3.1). □

The issue of lemma 3.3.4 is that we should know that $0 < \mathcal{H}^d(E) < +\infty$. The following theorem solves this problem; however, we recall some fundamental preliminary notions.

Definition 3.3.5 (Hausdorff distance)**.** Let $(\mathbb{X}, d)$ be a metric space; given C_1, C_2 compact non-empty sets in $\mathbb{X}$, we define the Hausdorff distance between C_1 and C_2 as

$$d_H(C_1, C_2) := \inf\{r > 0 \mid C_1 \subseteq \mathcal{U}_r(C_2)\ C_2 \subseteq \mathcal{U}_r(C_1)\},$$

where $\mathcal{U}_r(C) := \{x \in \mathbb{X} \mid \operatorname{dist}(x, C) < r\}$.

Remark 3.3.6. In other words, the Hausdorff distance between two compact sets (see 3.3.5) says how much it is necessary to enlarge each of them to include the other.

The following classical theorem hold true.

Theorem 3.3.7. *Let $(\mathbb{X}, d)$ be a metric space; let $\tilde{X}$ be the collection of non-empty compact sets in $\mathbb{X}$.*

- *The Hausdorff distance is a distance on $\tilde{X}$.*
- *If $(\mathbb{X}, d)$ is complete, then so is $(\tilde{X}, d_H)$.*

- *If* $(\mathbb{X}, d)$ *is compact, then so is* $(\tilde{X}, d_H)$.

Theorem 3.3.8 (Hutchinson)**.** *Let be given* $\Phi_1, \dots, \Phi_N$ *contractive similarities of* $\mathbb{R}^n$ *(see 3.3.1) with scaling factors* $\lambda_1, \dots, \lambda_n$. *Let* d *be the unique solution of* (3.1). *Then, the followings hold true:*

- *there exists a unique compact set* $C \subseteq \mathbb{R}^n$ *s.t.* $C = \bigcup_{i=1}^N \Phi_i(C)$*;*
- $\mathcal{H}^d(C) < +\infty$*;*
- *assume that the "open set condition" hold true (there exists an open bounded set* $V \subseteq \mathbb{R}^n$ *s.t.* $\Phi_i(V) \subseteq V$ *for all* i *and the* $\Phi_i(V)$ *are pairwise disjoint). Then* $\mathcal{H}^d(C) > 0$.

Proof. We define

$$\lambda_{max} := \max_{i=1,\dots,N} \lambda_i < 1.$$

$$\lambda_{min} := \min_{i=1,\dots,N} \lambda_i > 0.$$

Step 1: Let $\mathbb{X}$ be the space of compact non-empty sets in $\mathbb{R}^n$ endowed with the Hausdorff distance d_H. Let $\Phi : \mathbb{X} \to \mathbb{X}$ be the map given by

$$\Phi(E) := \bigcup_{i=1}^{n} \Phi_i(E).$$

Theorem 3.3.7 guarantees that $(\mathbb{X}, d_H)$ is a complete metric space. We claim that Φ is a contraction of $(\mathbb{X}, d_H)$; then, Φ admits a unique fixed point $C \in \mathbb{X}$. In other words, there exists a unique compact set C s.t. $C = \Phi(C) = \bigcup_{i=1}^N \Phi_i(C)$. Take E, E' compact non-empty sets in $\mathbb{R}^n$. Using only the definition of the Hausdorff distance as an infimum and the fact that $\Phi_i(\cdot) = x_i + \lambda_i R_i(\cdot)$ where $R_i \in O(n)$, we obtain

$$\begin{aligned} d_H(\Phi(E), \Phi(E')) &= d_H\left(\bigcup_{i=1}^N \Phi_i(E), \bigcup_{i=1}^N \Phi_i(E')\right) \\ &\leq \sup_{i=1,\dots,N} d_H(\Phi_i(E), \Phi_i(E')) \\ &= \sup_{i=1,\dots,N} \lambda_i d_H(E, E') \\ &= \lambda_{max} d_H(E, E'). \end{aligned}$$

Since $\lambda_{max} < 1$, we conclude that Φ is a contraction.

Step 2: Providing an estimate from above for the Hausdorff measure is easy, since it suffices to work with a specific covering. Notice that

$$C = \bigcup_{i_1=1}^N \Phi_{i_1}(C) = \bigcup_{i=1}^N \Phi_{i_1}\left(\bigcup_{i_2=1}^N \Phi_{i_2}(C)\right) = \bigcup_{i_1, i_2=1}^N \Phi_{i_1} \circ \Phi_{i_2}(C).$$

We want to iterate this argument; we define $I := \{1, \dots, N\}$ and for all $m \in \mathbb{N}$ we denote $\underline{i} = (i_1, \dots, i_m)$ a multi-index in I^m. We also denote $\Phi_{\underline{i}} := \Phi_{i_1} \circ \cdots \circ \Phi_{i_m}$ and $\lambda_{\underline{i}} := \lambda_{i_1} \cdots \lambda_{i_m}$. It is immediate to see that $\Phi_{\underline{i}}$ is a contractive similarity of scaling factor $\lambda_{\underline{i}}$. For simplicity, we define $C_{\underline{i}} := \Phi_{\underline{i}}(C)$. With this notation, for all $n \in \mathbb{N}$, we have that

$$C = \bigcup_{\underline{i}\in I^m} \Phi_{\underline{i}}(C) = \bigcup_{\underline{i}\in I^m} C_{\underline{i}}.$$

Note that

$$\operatorname{diam}(C_{\underline{i}}) = \lambda_{\underline{i}}\operatorname{diam}(C) \leq \lambda_{max}^m \operatorname{diam}(C).$$

Since $\lambda_{max} < 1$, given $\delta > 0$, there exists $m \in \mathbb{N}$ s.t. $\lambda_{max}^m \operatorname{diam}(C) < \delta$; hence, for all $\underline{i} \in I^m$, there holds that $\operatorname{diam}(C_{\underline{i}}) \leq \delta$. In particular, $(C_{\underline{i}})_{\underline{i}\in I^m}$ is a δ-covering of C. Let d be as in the assumptions; we deduce that

$$\begin{aligned}\mathcal{H}^d_\delta(C) &\leq c_d \sum_{\underline{i}\in I^m} \operatorname{diam}(C_{\underline{i}})^d \leq c_d \left(\sum_{\underline{i}\in I^m} \lambda_{\underline{i}}^d\right) (\operatorname{diam}(C))^d \\ &= c_d \left(\sum_{i=1}^N \lambda_i^d\right)^m (\operatorname{diam}(C))^d = c_d(\operatorname{diam}(C))^d.\end{aligned}$$

Taking the supremum with respect to δ, we obtain that $\mathcal{H}^d(C) \leq c_d(\operatorname{diam}(C))^d < +\infty$.

Step 3: In some sense, providing an estimate from below for $\mathcal{H}^d(C)$ is much harder, since we have to consider any covering of C. However, we follow a different way. We construct a probability measure μ on C s.t. $\Theta^*_d(\mu, x) < +\infty$ for all $x \in C$. As stated in theorem 3.2.7, this implies that $\mu << \mathcal{H}^d$; since $\mu(C) = 1 > 0$, we deduce that $\mathcal{H}^d(C) > 0$. Instead of the "open set condition", we assume the following slightly stronger condition: there exists a bounded open set $V \subseteq \mathbb{R}^n$ s.t. $\Phi_i(V) \subseteq V$ for all i and $\Phi_i(\overline{V})$ are pairwise disjoint. We claim that this condition (the closures of $\Phi_i(V)$ are disjoint) implies that $\Phi_i(C)$ are pairwise disjoint. In deed, it is enough to show that $C \subseteq \overline{V}$ (recall that $\Phi_i(\overline{V})$ are disjoint). Let $\Phi : \mathbb{X} \to \mathbb{X}$ be the contraction defined in the first step, where $\mathbb{X}$ is the collection of non-empty compact sets in $\mathbb{R}^n$. We define $\tilde{\mathbb{X}}$ as the collection of non-empty compact sets in $\overline{V}$. Since $\Phi(V) \subseteq V$, we have that $\Phi(\overline{V}) = \overline{\Phi(V)} \subseteq \overline{V}$. In other words, Φ restricts to a contraction of $\tilde{X}$. Since $\overline{V}$ is compact, theorem 3.3.7 guarantees that $\tilde{X}$ endowed with the Hausdorff distance is a compact metric space. Then, Φ has a fixed point $\tilde{C}$ in $\tilde{X}$. Clearly, $\tilde{C}$ is also a fixed point of Φ in $\mathbb{X}$; thus, $\tilde{C} = C$, which means that $C \subseteq \overline{V}$.

At this point, we have that $\Phi_i(C)$ are pairwise disjoint. Since C is closed, then $\Phi_i(C)$ is closed; moreover, $\Phi_i(C)$ is open in C, since we have

$$C \setminus \Phi_i(C) = \bigcup_{j\neq i, j=1}^N \Phi_j(C).$$

With the notation introduced in the second step, we can similarly show that $(\Phi_{\underline{i}}(C))_{\underline{i}\in I^m}$ is a disjoint partition of C and $\Phi_{\underline{i}}(C)$ is open and closed in C for all $\underline{i} \in I^m$ for all $m \in \mathbb{N}$. Having said that, our aim is to construct a probability measures μ on C s.t.

$$\mu(C) = 1,\ \mu(\Phi_i(C)) = \lambda_i^d,\ \ldots,\ \mu(\Phi_{\underline{i}}(C)) = \lambda_{\underline{i}}^d \ \ \forall m \in \mathbb{N}\ \forall \underline{i} \in I^m.$$

We construct such a probability measure as weak limit of the following sequence of probability measure on C. Take x any point in C and set

$$\mu_0 := \delta_x,\ \mu_1 := \sum_{i=1}^N \lambda_i^d \delta_{\Phi_i(x)},\ \ldots,\ \mu_m := \sum_{\underline{i}\in I^m} \lambda_{\underline{i}}^d \delta_{\Phi_{\underline{i}}(x)}\ \forall m \in \mathbb{N}.$$

We claim that $\mu_m(C_{\underline{i}}) = \lambda_{\underline{i}}^d$ for all $m' \leq m$ for all $\underline{i} \in I^{m'}$. Given $m, m', \underline{i}$ as above, notice that

$$\Phi_{\underline{i}}(C) = \bigcup_{\underline{j} \in I^{m-m'}} \Phi_{(\underline{i},\underline{j})}(C),$$

where $(\underline{i}, \underline{j})$ is the concatenation of two multi-indixes. Moreover, the union is disjoint. Thus, we have

$$\begin{aligned}\mu_m(\Phi_{\underline{i}}(C)) &= \sum_{\underline{j} \in I^{m-m'}} \mu_m(\Phi_{(\underline{i},\underline{j})}(C)) = \sum_{\underline{j} \in I^{m-m'}} \lambda_{\underline{i}}^d \lambda_{\underline{j}}^d \\ &= \lambda_{\underline{i}}^d \sum_{\underline{j} \in I^{m-m'}} \lambda_{\underline{j}}^d = \lambda_{\underline{i}}^d \left(\sum_{j=1}^N \lambda_j^d \right)^{m-m'} = \lambda_{\underline{i}}^d.\end{aligned}$$

The existence of a probability measure μ with the required properties follows immediately from the compactness theorem 1.1.33; since C is compact, up to subsequences, $(\mu_n)_n$ converges in the sense of measures to a probability measure μ in C, that is

$$\lim_{n \to +\infty} \int_C f \; d\mu_n = \int_C f \; d\mu \quad \forall f \in C(C).$$

Since $\Phi_{\underline{i}}(C)$ is open and closed in C, $\mathbb{1}_{\Phi_{\underline{i}}(C)}$ is an admissible test function. Hence, we have that

$$\lambda_{\underline{i}}^d = \lim_{n \to +\infty} \mu_n(\Phi_{\underline{i}}(C)) = \mu(\Phi_{\underline{i}}(C)).$$

To conclude, we show that $\Theta_d^*(\mu, x) < +\infty$ for all $x \in C$. Fix $x \in C$; there exists $\underline{i} \in I^{\mathbb{N}}$ s.t. $x \in \Phi_{\underline{i}_m}(C)$ for all $m \in \mathbb{N}$, where $\underline{i}_m = (i_1, \ldots, i_m)$ is the truncation of $\underline{i}$. Fix $\delta > 0$ s.t.

- $\delta < \min_{i=1,\ldots,N} \operatorname{diam}(\Phi_i(\overline{V}))$,
- $\delta \leq \operatorname{dist}(\Phi_i(\overline{V}), \Phi_j(\overline{V}))$ for all $i \neq j$

We remark that the second condition can be fulfilled, since we are assuming that $\Phi_i(\overline{V})$ are disjoint closed sets. Given $r > 0$, choose $m \in \mathbb{N}$ s.t.

$$\delta \lambda_{\underline{i}_{m+1}} < r \leq \delta \lambda_{\underline{i}_m}.$$

Such m exists, since $\lambda_{\underline{i}_m} \leq \lambda_{max}^m$ that goes to 0 as $m \to \infty$. By the choice of m, it follows that $\overline{B}(x, r) \cap C \subseteq C_{\underline{i}_m}$; thus, we have

$$\mu(\overline{B}(x, r)) \leq \mu(\Phi_{\underline{i}_m}(C)) = \lambda_{\underline{i}_m}^d.$$

Moreover, we have

$$r \geq \delta \lambda_{\underline{i}_{m+1}} = \delta \lambda_{\underline{i}_m} \lambda_{i_{m+1}}.$$

Hence, we obtain that

$$\frac{\mu(\overline{B}(x, r))}{r^d} \leq \frac{\lambda_{\underline{i}_m}^d}{\delta^d \lambda_{\underline{i}_m}^d \lambda_{i_{m+1}}} = \frac{1}{\delta^d \lambda_{i_{m+1}}^d} \leq \frac{1}{\delta^d \lambda_{min}^d}.$$

We deduce that

$$\Theta_d^*(\mu, x) \leq \frac{1}{c_d \delta^d \lambda_{min}^d} < +\infty.$$

Then, the proof is concluded. $\square$

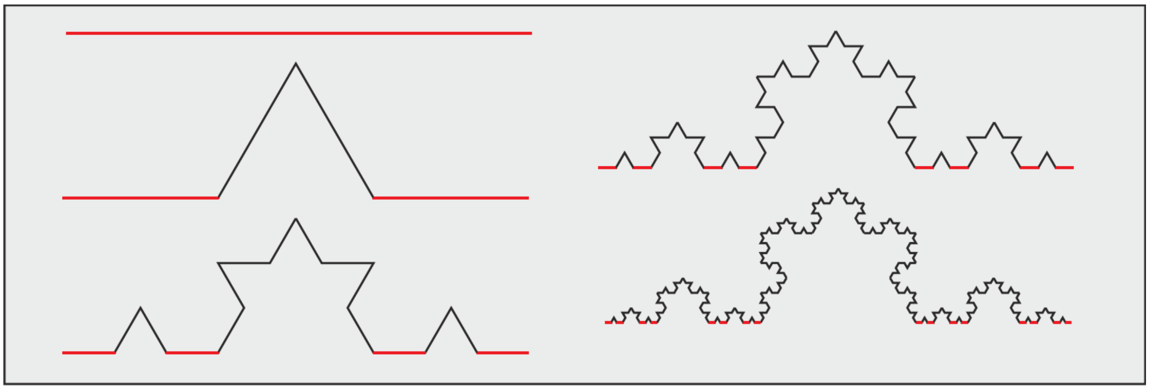

Figure 3.1: The first iterations in the construction of the Von Koch curve.

We conclude this section with some famous examples of self-similar fractals.

Example 3.3.9. The Cantor set described in 1.2.8 is a self-similar fractal in $\mathbb{R}$, obtained by two self-similarities of scaling factor $\frac{1}{3}$; since theorem 3.3.8 applies, we prove immediately that its Hausdorff dimension is $\frac{\log(2)}{\log(3)}$.

Example 3.3.10. The Von Koch curve is the self-similar fractal in $\mathbb{R}^2$ obtained by four self-similarities of scaling factor $\frac{1}{3}$, as explained in figure 3.1. Moreover, theorem 3.3.8 applies and we obtain that the Von Koch curve has Hausdorff dimension equal to $\frac{\log(4)}{\log(3)}$.

Example 3.3.11. The Cantor dust is the self-similar fractal in $\mathbb{R}^2$ obtained by four self-similarities of scaling factor $\frac{1}{4}$, as explained in figure 3.2. Since theorem 3.3.8 applies, we get that the Hausdorff dimension of this set is 1. Similarly, for any $\lambda \in (0, 1/2)$, one can consider the self-similar fractal K_λ in $\mathbb{R}^2$ obtained by four self-similarities of scaling factor λ (in other words, at the first step, we have four squares of side-length λ; at the second step, we have sixteen squares of side-length λ^2 and so on). By theorem 3.3.8, we can easily compute that

$$\dim_{\mathcal{H}}(K_\lambda) = \left| \frac{\log(4)}{\log(\lambda)} \right|.$$

Example 3.3.12. The Sierpisky triangle is the self-similar fractal in $\mathbb{R}^2$ obtained by three self-similarities of scaling factor $\frac{1}{3}$, as explained in figure 3.3. Since theorem 3.3.8 applies, we get that its Hausdorff dimension is 1.

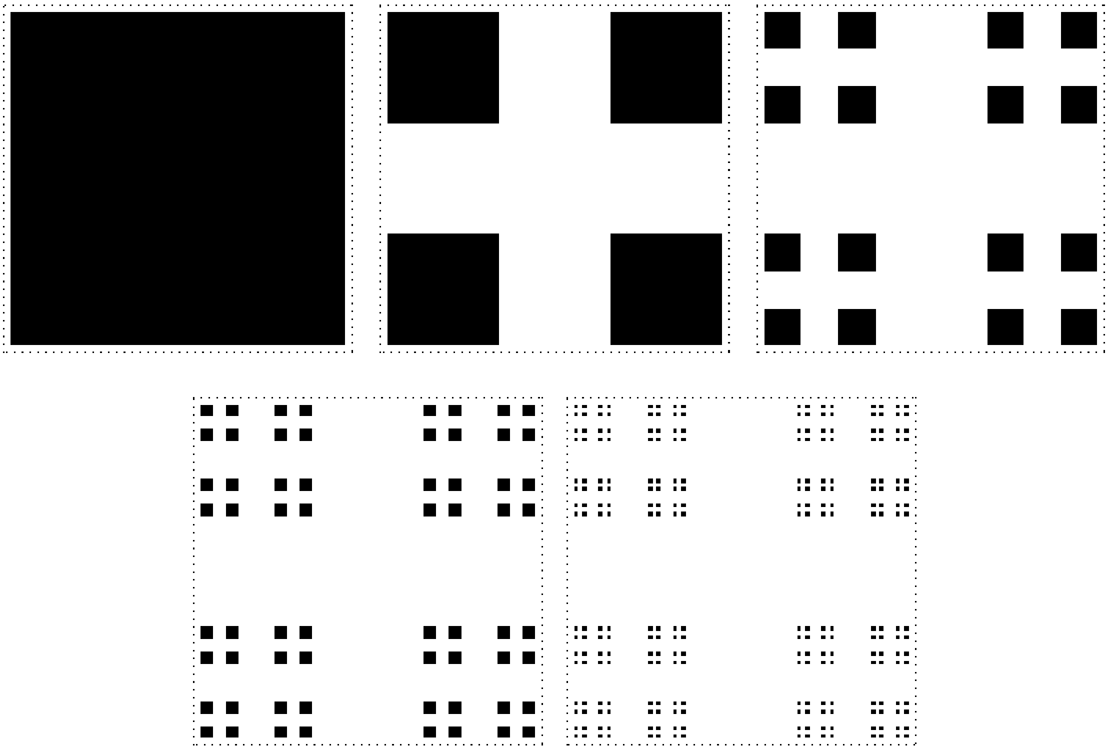

Figure 3.2: The first iterations in the construction of the Cantor dust.

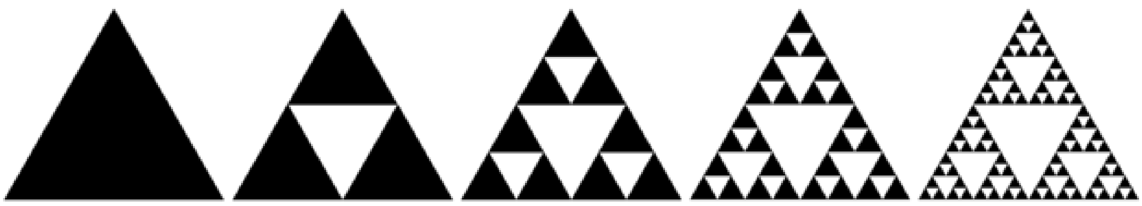

Figure 3.3: The first iterations in the construction of the Sierpisky triangle.

Chapter 4

Haar measures

In this chapter we consider a topological group $\mathbb{G}$ acting on a topological space $\mathbb{X}$. Haar measures are the natural measures to consider in some contexts (see [7]).

4.1 Construction

Definition 4.1.1 (Action)**.** An action of $\mathbb{G}$ on $\mathbb{X}$ is a continuous map $\mathcal{T} : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ s.t.

$$\mathcal{T}_y(\mathcal{T}_{y'}x) = \mathcal{T}_{yy'}x \quad \forall y, y' \in \mathbb{G}\ \forall x \in \mathbb{X}.$$

To be precise, $\mathcal{T}$ is a left action of $\mathbb{G}$ on $\mathbb{X}$. The following theory can be also stated for right actions, being completely similar.

Example 4.1.2. A topological group $\mathbb{G}$ acts on itself by left-multiplication, that is $\mathcal{T} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is given by $\mathcal{T}_g(g') = gg'$.

Definition 4.1.3 (Translation of a measure)**.** Given a Borel measure μ on $\mathbb{X}$ and $y \in \mathbb{G}$, we define the Borel measure $\mathcal{T}_y\mu$ as

$$\mathcal{T}_y\mu(E) := \mu(\mathcal{T}_yE) \quad \forall E \in \mathcal{B}(\mathbb{X}).$$

Definition 4.1.4 (Left-invariant measure)**.** A Borel measure μ on $\mathbb{X}$ is said to be left-invariant if $\mu(E) = \mu(\mathcal{T}_y(E))$ for all $y \in \mathbb{G}$ for all $E \in \mathcal{B}(\mathbb{X})$.

In some sense, a left-invariant measure is the natural measure to put on a topological space $\mathbb{X}$, provided a group $\mathbb{G}$ acting on $\mathbb{X}$. At this stage it is interesting to studying existence and uniqueness of these measures. We will show the existence of an invariant measure in a topological group $\mathbb{G}$ acting on itself (under some additional assumptions), provided some compactness properties of $\mathbb{G}$. In the general setting of a group $\mathbb{G}$ acting on a topological space $\mathbb{X}$, the compactness of $\mathbb{G}$ and $\mathbb{X}$ is not sufficient to provide the existence of an invariant measure, as we sketch in the following example.

Example 4.1.5. Consider the topological space $\mathbb{X} = \mathbb{P}^1\mathbb{R}$ and $\mathbb{G}$ the group of the projectivies on $\mathbb{P}^1\mathbb{R}$. One can show the following results:

- since the translation group is included in $\mathbb{G}$, the only possible invariant measure is the Lebesgue measure $\mathscr{L}^1$;
- on the other hand, $\mathscr{L}^1$ is not invariant under homothety.

Thus, one could conclude that there are no $\mathbb{G}$-invariant measures.

Example 4.1.6.
- Recall that $\mathbb{R}^n$ is a topological group with the sum. Since the Lebesgue measure is translation invariant, it is an invariant measure on $\mathbb{R}^n$ in the sense of definition 4.1.4.
- Recall that $\mathbb{S}^1 \subseteq \mathbb{C}^*$ is a topological group with the complex multiplication. The measure $\mathcal{H}^1$ on $\mathbb{S}^1$ is rotation invariant; hence, it is an invariant measure in the sense of definition 4.1.4.
- Similarly, $\mathcal{H}^3$ on $\mathbb{S}^3$ is an invariant measure in the sense of definition 4.1.4, where $\mathbb{S}^3$ is a topological group as a subset of $\mathbb{H}^*$ (with the multiplication of quaternions).

We state and show some lemmas that goes in the direction of proving existence and uniqueness of invariant measures. Then, we give the general statements.

In the proof of the following lemmas, we will use the theorems established in 1.1, adapted in the context of locally compact Hausdorff spaces. They still hold true, without modifications.

Lemma 4.1.7. *Assume that $\mathbb{X}$ is compact, $\mathbb{G}$ is compact and commutative. Then, there exists a $\mathbb{G}$-invariant probability measure μ on $\mathbb{X}$.*

Proof. Denote by $\mathbb{P}(\mathbb{X})$ the set of Borel probability measures on $\mathbb{X}$. Given $\mathbb{F} \subseteq \mathbb{G}$, we define

$$\mathbb{P}_{\mathbb{F}} := \{\mu \in \mathbb{P}(\mathbb{X}) \mid \mathcal{T}_y\mu = \mu \ \forall y \in \mathbb{F}\}.$$

We want to show that $\mathbb{P}_{\mathbb{G}}$ is non-empty.

Step 1: We claim that $\mathbb{P}_{\mathbb{F}}$ is closed with respect to the convergence of measure. Given $(\mu_n)_n$ a sequence in $\mathbb{P}_{\mathbb{F}}$ that converges to μ in the sense of measures, we claim that μ is in $\mathbb{P}_{\mathbb{F}}$. We already know (see 1.1.33) that μ is a Borel probability measure on $\mathbb{X}$. Take $y \in \mathbb{F}$ and $\nu \in \mathbb{P}_{\mathbb{F}}$; it is easy to check the following "change of variable" formula:

$$\int_{\mathbb{X}} g \, d\mathcal{T}_y\nu = \int_{\mathbb{X}} \mathcal{T}_{y^{-1}} g \, d\nu \quad \forall g \in L^1(\mathbb{X}, \nu)$$

It is immediate if $g = \mathbb{1}_A$; by linearity, we obtain the formula for step functions; by Beppo Levi theorem, we deduce the formula for non-negative functions; then, it is immediate to extend at $L^1(\mathbb{X}, \nu)$ functions.

Fix $y \in \mathbb{F}$ and $g \in C(\mathbb{X})$ a test function. There holds

$$\begin{aligned}\int_{\mathbb{X}} g \, d\mu &= \lim_{n\to+\infty} \int_{\mathbb{X}} g \, d\mu_n = \lim_{n\to+\infty} \int_{\mathbb{X}} g \, d\mathcal{T}_y\mu_n \\ &= \lim_{n\to+\infty} \int_{\mathbb{X}} \mathcal{T}_{y^{-1}} g \, d\mu_n = \int_{\mathbb{X}} \mathcal{T}_{y^{-1}} g \, d\mu \\ &= \int_{\mathbb{X}} g \, d\mathcal{T}_y\mu.\end{aligned}$$

By Riesz's representation theorem (see 1.1.27), we conclude that $\mu = \mathcal{T}_y\mu$; thus $\mu \in \mathbb{P}_{\mathbb{F}}$.

Step 2: We claim that for all $y \in \mathbb{G}$, then $\mathbb{P}_{\{y\}}$ is non-empty. Take μ_0 any Borel probability measure on $\mathbb{X}$. Given $n \in \mathbb{N}$, we define

$$\mu_n := \frac{1}{n+1} \sum_{m=0}^{n} (\mathcal{T}_y)^m \mu_0 = \sum_{m=0}^{n} \mathcal{T}_{my}\mu_0.$$

Up to subsequences, theorem 1.1.33 guarantees the existence of a Borel probability measure μ on $\mathbb{X}$ s.t. $(\mu_n)_n$ converges to μ in the sense of measures. We claim that $\mathcal{T}_y\mu = \mu$. In deed, we have

$$\|\mathcal{T}_y\mu_n - \mu_n\|_1 = \frac{1}{n+1}\left\|\mathcal{T}_{(n+1)y}\mu_0 - \mu_0\right\|_1 \leq \frac{2}{n+1}.$$

Notice that $\mathcal{T}_y\mu_n$ converges to $\mathcal{T}_y\mu$ in the sense of measures, as one can easily check. Then, by 1.1.30 we infer that

$$\|\mathcal{T}_y\mu - \mu\|_1 \leq \liminf_{n\to+\infty} \|\mathcal{T}_y\mu_n - \mu_n\|_1 \leq \lim_{n\to+\infty} \frac{2}{n+1} = 0.$$

Then, we deduce that $\mathcal{T}_y\mu = \mu$.

Step 3: We claim that if $\mathbb{P}_\mathbb{F} \neq \emptyset$, then $\mathbb{P}_{\mathbb{F}\cup\{y\}} \neq \emptyset$ for all $y \in \mathbb{G}$. This can be proved as in the previous step. Take $y \in \mathbb{G}$ and $\mu_0 \in \mathbb{P}_\mathbb{F} \neq \emptyset$. Given $n \in \mathbb{N}$, define

$$\mu_n := \frac{1}{n+1}\sum_{m=0}^{n}(\mathcal{T}_y)^m\mu_0 = \sum_{m=0}^{n}\mathcal{T}_{my}\mu_0.$$

Since $\mathbb{G}$ is commutative, $(\mu_n)_n$ is a sequence in $\mathbb{P}_\mathbb{F}$. Up to subsequences, theorem 1.1.33 guarantees the existence on a Borel probability measure μ on $\mathbb{X}$ s.t. $(\mu_n)_n$ converges to μ in the sense of measures. Since $\mathbb{P}_\mathbb{F}$ is closed, then $\mu \in \mathbb{P}_\mathbb{F}$. We claim that $\mathcal{T}_y\mu = \mu$. This can be checked as in the previous step. Then, we conclude that $\mu \in \mathbb{P}_{\mathbb{F}\cup\{y\}}$.

Step 4: The previous steps implies that $\mathbb{P}_\mathbb{F} \neq \emptyset$ whenever $\mathbb{F}$ is a finite set. Notice that

$$\mathbb{P}_\mathbb{G} = \bigcap_{y\in\mathbb{G}} \mathbb{P}_{\{y\}};$$

moreover, we have shown that $\mathbb{P}_{\{y\}}$ has the finite intersection property; since $\mathbb{G}$ is compact, we deduce that

$$\bigcap_{y\in\mathbb{G}} \mathbb{P}_{\{y\}} \neq \emptyset.$$

□

Lemma 4.1.8. *Assume that $\mathbb{G}$ is locally compact and commutative; let μ_1, μ_2 invariant measures on $\mathbb{G}$. Then $\mu_1 = \lambda\mu_2$ for some $\lambda \geq 0$.*

Proof. Take a function $g \in C_c(\mathbb{G})$ s.t. $\int_\mathbb{G} g\ d\mu_1 = 1$. Denote

$$\lambda := \int_\mathbb{G} g(-x)\ d\mu_2.$$

Take a test function $f \in C_0(\mathbb{G})$; the invariant property of μ_1, μ_2 (that justify the "change of variable") and the Fubini's theorem guarantee that

$$\begin{aligned}
\int_{\mathbb{G}} f(x)\ d\mu_2(x) &= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(y)\ d\mu_1(y) \right) f(x)\ d\mu_2(x) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(y) f(x+y)\ d\mu_1(y) \right)\ d\mu_2(x) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(y-x) f(y)\ d\mu_1(y) \right)\ d\mu_2(x) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(y-x) f(y)\ d\mu_2(x) \right)\ d\mu_1(y) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(y-x)\ d\mu_2(x) \right) f(y)\ d\mu_1(y) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} g(-x)\ d\mu_2(x) \right) f(y)\ d\mu_1(y) \\
&= \lambda \int_{\mathbb{G}} f(y)\ d\mu_1(y).
\end{aligned}$$

Then, by Riesz's representation theorem (see 1.1.27), we conclude that $\mu_2 = \lambda\mu_1$. We remark that the locally compactness assumption is needed for the Riesz's representation theorem; the role of the commutativity of $\mathbb{G}$ is hidden in the computation above. □

Lemmas 4.1.7 and 4.1.8 give partial proofs of the general theorems stated below.

Theorem 4.1.9. *If $\mathbb{G}$ is a compact group (not necessarily commutative), there exists a unique left-invariant probability measure.*

If $\mathbb{G}$ is a locally compact group (not necessarily commutative), there exists a left-invariant measures, which is unique up to a constant.

Definition 4.1.10 (Haar measure)**.** The left invariant measures on $\mathbb{G}$ provided by the theorem 4.1.9 are called Haar measures.

In the setting of a group $\mathbb{G}$ (not necessarily commutative) acting on a topological space $\mathbb{X}$, stronger assumptions are needed to guarantee the existence of invariant measure.

Theorem 4.1.11. *There exists a $\mathbb{G}$-invariant probability measure μ on $\mathbb{X}$ in the following cases:*

- *$\mathbb{G}$ commutative and compact, $\mathbb{X}$ compact;*
- *$\mathbb{G}$ compact, $\mathbb{X} = \mathbb{G}/H$, where H is a closed subgroup of $\mathbb{G}$. In this case, μ is also unique;*
- *$\mathbb{G}$ is compact and satisfies the so called "Weyl condition".*

We will not explain what is the "Weyl condition", because we will never use it in this context.

Integal-geometric measure (Favard)

We skecth how to apply the theorem 4.1.9 to define the integral-geometric measure.

Let $\mathrm{Gr}(m,n)$ be the Grassmannian of the n-dimensional plane in $\mathbb{R}^m$. We can identify $\mathrm{Gr}(m,n)$ as the quotient

$$\frac{O(m)}{O(n)\times O(m-n)},$$

where $O(n)\times O(m-n)$ the a closed subgroup of $O(n)$ given by the orthogonal matrix in $O(m)$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim (A,B) \in O(n)\times O(m-n).$$

Theorem 4.1.11 provides the existence of a unique probability measure μ on $\mathrm{Gr}(m,n)$, which is $O(m)$-invariant.

Definition 4.1.12 (Integral-geometric measure)**.** The n-dimensional integral-geometric measure (of parameter 1) in $\mathbb{R}^m$ is defined as

$$\mathcal{I}_1^m(E) := c_{n,m} \int_{V\in \mathrm{Gr}(m,n)} \left(\int_{y\in V} \#(p_V^{-1}(y)\cap E)\, d\mathcal{H}^n(y) \right)\, d\mu(V),$$

where $c_{n,m}$ is a normalization constant, p_V is the orthogonal projection to V and μ is the invariant measure on $\mathrm{Gr}(m,n)$.

It can be proved that $\mathcal{I}_1^m$ is invariant under affine isometries and agrees with $\mathcal{H}^n$ on every $V \in \mathrm{Gr}(m,n)$; in deed, it agrees with $\mathcal{H}^n$ on every C^1-surface of dimension n in $\mathbb{R}^m$.

Chapter 5

Lipschitz maps

In this chapter, we deal with Lipschitz maps. For further references, see [4] and [5].

5.1 Definition and main properties

Definition 5.1.1 (Lipschitz map)**.** Let $\mathbb{X}, \mathbb{Y}$ be metric spaces; a map $f : \mathbb{X} \to \mathbb{Y}$ is Lipschitz if there exists $L > 0$ s.t.

$$d_{\mathbb{Y}}(f(x_1), f(x_2)) \leq L d_{\mathbb{X}}(x_1, x_2) \quad \forall x_1, x_2 \in \mathbb{X}. \tag{5.1}$$

The Lipschitz constant is defined as

$$\mathrm{Lip}(f) \coloneqq \inf\{L > 0 \mid (5.1) \text{ holds}\}.$$

Remark 5.1.2. The infimum that defines the Lipschitz constant is actually a minimum.

Compactness properties

Lipschitz maps have good compactness properties. The statement below is a particular instance of the Ascoli-Arzelà theorem.

Theorem 5.1.3. *Let $(f_n)_n$ be a sequence of Lipschitz functions defined on a metric space $\mathbb{X}$ with values in another metric space $\mathbb{Y}$. Assume that*

- *$\mathbb{X}$ is compact,*
- *$\sup_n Lip(f_n) < +\infty$,*
- *for all $x \in \mathbb{X}$ the set $\{f_n(x) \mid n \in \mathbb{N}\}$ is relatively compact in $\mathbb{Y}$.*

Then, up to subsequences, $(f_n)_n$ converges uniformly to a Lipschitz map f s.t.

$$Lip(f) \leq \liminf_{n \to +\infty} Lip(f_n).$$

Extension properties

We briefly mention some of the extension properties of the Lipschitz maps.

Lemma 5.1.4 (McShane)**.** *Let $\mathbb{X}$ be a metric space and $E \subseteq \mathbb{X}$ any subset. Let $f : E \to \mathbb{R}$ be a Lipschitz function. There exists $\tilde{f} : \mathbb{X} \to \mathbb{R}$ s.t. $\tilde{f}$ and f agree on E and $Lip(\tilde{f}) = Lip(f)$.*

Proof. It is enough to define $\tilde{f}$ as the lower affine envelope of f, that is

$$\tilde{f}(x) := \inf_{y \in E}\{f(y) + \mathrm{Lip}(f)d(x,y)\}.$$

The function $\tilde{f} : \mathbb{X} \to \mathbb{R}$ is well defined and $\mathrm{Lip}(\tilde{f}) \leq \mathrm{Lip}(f)$, since it is pointwise infimum of a family of $\mathrm{Lip}(f)$-Lipschitz functions. Moreover, $\tilde{f}$ agrees on f in E. Then $\mathrm{Lip}(\tilde{f}) \geq \mathrm{Lip}(f)$. □

Remark 5.1.5. In the proof of McShane lemma (see 5.1.4), also the upper affine envelope works

$$\tilde{f}(y) := \sup_{y \in E}\{f(y) - \mathrm{Lip}(f)d(x,y)\}.$$

We also remark that McShane lemma provides an extension $\tilde{f}$ for maps $f : E \to \mathbb{R}^n$. In this case, there holds $\mathrm{Lip}(\tilde{f}) < +\infty$; since we work componentwise, it is not guaranteed that $\mathrm{Lip}(f) = \mathrm{Lip}(\tilde{f})$. This statement can be proved, but it is much harder than the McShane lemma.

Theorem 5.1.6 (Kirszbraun)**.** *If $\mathbb{X}, \mathbb{Y}$ are Hilbert spaces, $E \subseteq \mathbb{X}$ and $f : E \to \mathbb{Y}$ is a Lipschitz map, then there exists an extension $\tilde{f} : \mathbb{X} \to \mathbb{Y}$ s.t. $Lip(f) = Lip(\tilde{f})$.*

Remark 5.1.7. The extension provided by Kirszbraun theorem (see 5.1.6) may not be the affine one.

Remark 5.1.8. The assumptions of theorem 5.1.6 on $\mathbb{X}, \mathbb{Y}$ cannot be dropped. The identity map from S^1 to S^1 cannot be extended to a continuous map from $\mathbb{R}^2$ to S^1, since topological obstructions occur.

Differentiability properties

Lipschitz maps have good properties of differentiability, which makes a huge difference from continuous (and even Hölder continuous) maps.

Theorem 5.1.9. *Consider $f : \mathbb{R}^n \to \mathbb{R}^m$.*

- *If f is Lipschitz, then $f \in W^{1,\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$.*
- *If $n < p < +\infty$ and $f \in W^{1,p}_{loc}(\mathbb{R}^n, \mathbb{R}^m) \cap C(\mathbb{R}^n, \mathbb{R}^m)$, then f is differentiable at almost every point $x \in \mathbb{R}^n$. Moreover, the gradient of f agrees with the distributional gradient a.e.; hence, we use ∇f to denote both.*

Proof. **Step 1**: The proof of the first statement is based on the technique of the difference quotient. Pick $h \in \mathbb{R}, h \neq 0$ and e_i the i-th element of the canonical basis of $\mathbb{R}^n$. We denote the difference quotient

$$D^h_i f(x) := \frac{f(x + he_i) - f(x)}{h}.$$

Since f is Lipschitz, then the sequence $(D^h_i f)_h$ is uniformly bounded in $\mathbb{R}^n$; hence, up to subsequence, it converges to a function g_i weakly* in $L^\infty(\mathbb{R}^n, \mathbb{R}^m)$. We claim that g_i is the i-th distributional derivative of f. Given a test function $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^m)$, the discrete integration by parts formula guarantees that

$$\int_{\mathbb{R}^n} \varphi(x) D^h_i f(x)\ dx = -\int_{\mathbb{R}^n} f(x) D^{-h}_i \varphi(x)\ dx.$$

From the weak* convergence, it follows that

$$\lim_{h\to 0}\int_{\mathbb{R}^n}\varphi(x)D_i^h f(x)\ dx=\int_{\mathbb{R}^n}\varphi(x)g_i(x)\ dx.$$

Since $\varphi\in C_c^\infty(\mathbb{R}^n,\mathbb{R}^m)$, the sequence $(D_i^{-h}\varphi)_h$ converges to $\partial_i\varphi$ uniformly in $\mathbb{R}^n$; moreover, $f\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^m)$. Thus, we have

$$\lim_{h\to 0}\int_{\mathbb{R}^n}f(x)D_i^{-h}\varphi(x)\ dx=\int_{\mathbb{R}^n}\partial_i\varphi(x)f(x)\ dx.$$

In other words, $g_i=\partial_i f$ in distributional sense.

Step 2: The second statement relies on the classical theory of Sobolev spaces. Let ∇f be the distributional gradient of f. Fix a ball $\overline{B}(\overline{x},r)\subseteq\mathbb{R}^n$. Since $p>n$ we can choose the continuous representative of $f\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^m)$. Sobolev embedding ($p>n$) and Poincarè-Wirtinger inequality imply that

$$\begin{aligned}\sup_{x\in\overline{B}(\overline{x},r)}\left|f(x)-\fint_{\overline{B}(\overline{x},r)}f(y)\ dy\right| &\le C(\overline{x},r,p,n,m)\left\|f-\fint_{\overline{B}(\overline{x},r)}f(y)\ dy\right\|_{W^{1,p}(B(\overline{x},r))}\\ &\le C(\overline{x},r,p,n,m)\left(\int_{\overline{B}(\overline{x},r)}|\nabla f(y)|^p\ dy\right)^{\frac{1}{p}}\\ &\le C(\overline{x},r,p,n,m)\left(\fint_{\overline{B}(\overline{x},r)}|\nabla f(y)|^p\ dy\right)^{\frac{1}{p}}\end{aligned}$$

By a translation argument, we deduce that the constant C is independent of $\overline{x}$; by a rescaling argument, it is easy to see that C scales linearly with respect to r, that is $C(\overline{x},r,p,n,m)=rC(p,n,m)$. Thus, we obtain that

$$\sup_{x,y\in\overline{B}(\overline{x},r)}|f(x)-f(y)|\le rC(p,n,m)\left(\fint_{\overline{B}(\overline{x},r)}|\nabla f(y)|^p\ dy\right)^{\frac{1}{p}}.$$

In particular, we deduce that for all $\overline{x},h\in\mathbb{R}^n$ there holds

$$|f(\overline{x}+h)-f(\overline{x})|\le C(p,n,m)\,|h|\left(\fint_{\overline{B}(\overline{x},|h|)}|\nabla f(y)|^p\ dy\right)^{\frac{1}{p}}. \tag{5.2}$$

If we apply the inequality (5.2) with $g(x):=f(x)-\nabla f(\overline{x})x$, we obtain

$$|f(\overline{x}+h)-f(\overline{x})-\nabla f(\overline{x})x|\le C(p,n,m)\,|h|\left(\fint_{\overline{B}(\overline{x},|h|)}|\nabla f(y)-\nabla f(\overline{x})|^p\ dy\right)^{\frac{1}{p}}, \tag{5.3}$$

that is

$$\frac{|f(\overline{x}+h)-f(\overline{x})-\nabla f(\overline{x})x|}{|h|}\le C(p,n,m)\left(\fint_{\overline{B}(\overline{x},|h|)}|\nabla f(y)-\nabla f(\overline{x})|^p\ dy\right)^{\frac{1}{p}}, \tag{5.4}$$

Take a point $\overline{x}$ of L^p-approximate continuity for ∇f (see 3.2.11); deduce that the right hand side in (5.4) goes to 0 as $|h|$ goes to 0. Then, f is classically differentiable at every point $\overline{x}$ of L^p-approximate continuity; moreover, the classical gradient agrees in these points with the distributional gradient. To conclude, we recall that ∇f is L^p-approximate continuous at a.e. $x\in\mathbb{R}^n$ (see 3.2.12). □

As a corollary, we obtain the following famous theorem.

Theorem 5.1.10 (Rademacher). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Then f is differentiable at a.e. $x \in \mathbb{R}^n$.*

Proof. It is a direct application of theorem 5.1.9. □

Lusin properties

Lipschitz functions have good Lusin properties, in a sense that will be clear later.

Lemma 5.1.11. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $K \subseteq \Omega$ be a compact set and $f, g : \Omega \to \mathbb{R}$ a.e. differentiable in K. Assume that $f = g$ a.e. in K. Then $\nabla f = \nabla g$ a.e. in K.*

Proof. By taking the difference, it is equivalent to show that $f : \Omega \to \mathbb{R}$ a.e. differentiable in K s.t. $f = 0$ a.e. in K implies that $\nabla f = 0$ a.e. on K. Take $x \in K$ s.t. $f(x) = 0$, f is differentiable at x and $\Theta_{\mathscr{L}^n}(K, x) = 1$. The assumptions and theorem 3.1.2 guarantee that the set of the point $x \in K$ with the property required has full measure $\mathscr{L}^n$. Fix $i \in \{1, \dots, n\}$. Since $\Theta_{\mathscr{L}^d}(K, x) = 1$, it is easy to see that there exists a sequence $(x_n)_n \subseteq K$ s.t. $f(x_n) = 0$, $x_n \neq x$ and $\frac{x_n - x}{|x_n - x|}$ goes to the i-th element of the canonical basis of $\mathbb{R}^n$. The first properties are easy to fulfill; the last can be proved by contradiction, looking at the definition of upper density (see 3.2.4) and using the fact that $\Theta_{\mathscr{L}^n}(K, x) = 1$.

By the differentiability at x, it follows that

$$0 = \lim_{n \to +\infty} \frac{|f(x_n) - f(x) - \nabla f(x)(x_n - x)|}{||x_n - x||} = \lim_{n \to +\infty} \frac{|\nabla f(x)(x_n - x)|}{|x_n - x|} = |\nabla f(x) e_i| \, .$$

Thus, we conclude that $\nabla f(x) = 0$. □

Theorem 5.1.12. *Let Ω be a bounded open set in $\mathbb{R}^n$; let $f : \Omega \to \mathbb{R}$ continuous and a.e. differentiable in Ω with gradient ∇f. Then, for all $\varepsilon > 0$ there exists a compact set $K \subseteq \Omega$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 with the following properties:*

- *$\mathscr{L}^n(\Omega \setminus K) \leq \varepsilon$;*
- *$f = g$ on K and $\nabla f = \nabla g$ a.e. on K;*
- *Lip$(g) \leq$ Lip(f) (eventually Lip$(f) \in [0, +\infty]$).*

Proof. We sketch the proof. Let D the set of points $x \in \Omega$ s.t. f is differentiable at x; that is

$$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - \nabla f(x) h|}{|h|} = 0 \quad \forall x \in D. \tag{5.5}$$

Clearly, D has full measure $\mathscr{L}^n$ in Ω. Fix $\varepsilon > 0$; by Severini-Egorov theorem, we can find a compact $K \subseteq D$ s.t. $\mathscr{L}^n(D \setminus K) \leq \varepsilon$ and the convergence in (5.5) is uniform with respect to $x \in K$. By Lusin theorem, up to shrink K, we can assume that ∇f is continuous on K, that is

$$f(x+h) = f(x) + \nabla f(x) h + R_x(h),$$

where the reminder $R_x(h)$ is s.t. $|R_x(h)| \leq |h| \, \omega(|h|)$ and ω is independent on $x \in K$ (this follows from the uniform continuity in K). By Whitney's extension theorem, $f_{|K}$ can be extended to a map $g : \mathbb{R}^n \to \mathbb{R}$. Hence, lemma 5.1.11 implies that $\nabla f = \nabla g$ a.e. in K; moreover, if f is Lipschitz, then we can also obtain $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$. □

5.2 Area formula

We state many version of the area formula: the first is the simplest to prove, the last is the most general and it is usually applied.

5.2.1 Hausdorff measure on regular surfaces in $\mathbb{R}^m$

The first version of the area formula has a strong relation with the Hausdorff measure on regular surfaces in $\mathbb{R}^m$. Thus, we need the following characterization.

Definition 5.2.1 (Isometry defect)**.** Take $U \subseteq \mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ a map. Given $\delta > 0$, we say that f has isometry defect at most δ if it holds that

$$\frac{1}{1+\delta}\,|x-x'| \le |f(x)-f(x')| \le (1+\delta)\,|x-x'| \quad \forall x, x' \in U.$$

Remark 5.2.2. In some sense, a map with isometry defect at most $\delta > 0$ (see 5.2.1) differs from an isometry of a factor $1+\delta$. We also notice that such a map is injective.

The following results are the fundamental steps toward the area formula.

Theorem 5.2.3. *Fix an integer d; let Σ be a d-dimensional surface of class C^1 in $\mathbb{R}^m$; let λ be a measure on Σ with the following property: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all map $f : U \to \mathbb{R}^d$ (where U is any open set in Σ) of class C^1 with isometry defect at most δ (see 5.2.1), then it holds that*

$$\frac{1}{1+\varepsilon}\lambda(E) \le \mathscr{L}^d(f(E)) \le (1+\varepsilon)\lambda(E) \quad \forall E \subseteq U \ \textit{Borel.}$$

Then, λ is unique.

Proof. Take λ, λ' measures on Σ as above. Given a Borel set $E \subseteq S$, we claim that $\lambda(E) = \lambda'(E)$. Fix $\varepsilon > 0$ and take $\delta > 0$ corresponding to ε in the assumptions. Since Σ is a C^1 d-dimensional surface, we can cover Σ with a countable (at most) family of open sets U_i s.t. there exist maps $f_i : U_i \to \mathbb{R}^d$ with isometry defect at most δ. Hence, we can write

$$E = \bigcup_i E_i$$

where the union is disjoint and $E_i \subseteq U_i$ is Borel. The assumptions on λ, λ' guarantee that

$$\frac{1}{1+\varepsilon}\,\mathscr{L}^d(f_i(E_i)) \le \lambda(E_i), \quad \lambda'(E_i) \le (1+\varepsilon)\,\mathscr{L}^d(f_i(E_i)) \quad \forall i.$$

Hence, we deduce that

$$\frac{1}{(1+\varepsilon)^2}\lambda(E_i) \le \lambda'(E_i) \le (1+\varepsilon)^2\lambda(E_i) \qquad \forall i;$$

thus, summing over i, we have that

$$\frac{1}{(1+\varepsilon)^2}\lambda(E) \le \lambda'(E) \le (1+\varepsilon)^2\lambda(E).$$

If $\lambda(E) = +\infty$ or $\lambda'(E) = +\infty$, we immediately obtain that $\lambda(E) = \lambda(E') = +\infty$. Hence, we can assume that either $\lambda(E)$ and $\lambda'(E)$ are finite. Thus, the conclusion follows immediately taking the limit as $\varepsilon \to 0$. $\square$

Proposition 5.2.4. *Let Σ be a d-dimensional surface of class C^1 in $\mathbb{R}^m$. The measure $\mathcal{H}^d \llcorner \Sigma$ has the property assumed in theorem 5.2.3.*

Proof. Fix $\delta > 0$; take $U \subseteq \Sigma$ open and $f : U \to \mathbb{R}^d$ a map with isometry defect at most δ (see 5.2.1). Notice that $f : U \to f(U)$ and $f^{-1} : f(U) \to U$ are $(1+\delta)$-Lipschitz (and well defined). Take $E \subseteq U$ Borel set. Thanks to 1.2.3, 2.3.1 and 2.3.8, we deduce that

$$\begin{aligned} \mathscr{L}^d(f(E)) = \mathcal{H}^d(f(E)) \leq (1+\delta)^d \mathcal{H}^d(E); \\ \mathcal{H}^d(E) = \mathcal{H}^d(f^{-1}(f(E))) \leq (1+\delta)^d \mathcal{H}^d(f(E)) = (1+\delta)^d \mathscr{L}^d(f(E)). \end{aligned}$$

Thus, we have that

$$\frac{1}{(1+\delta)^d}\mathcal{H}^d(E) \leq \mathscr{L}^d(f(E)) \leq (1+\delta)^d \mathcal{H}^d(E).$$

So, it suffices to choose $\varepsilon > 0$ s.t. $(1+\delta)^d = 1+\varepsilon$. □

Definition 5.2.5 (Jacobian of a parametrization)**.** Let Σ be a d-dimensional surface of class C^1 in $\mathbb{R}^n$ and $\Phi : \Omega \to \Sigma$ be a local parameterization of Σ, where Ω is an open set in $\mathbb{R}^d$ and $\Phi \in C^1(\Omega, \mathbb{R}^n)$. For all $x \in \Omega$, denote

$$d\Phi_x : \mathbb{R}^d \to \mathrm{Tan}_{\Phi(x)}\Sigma$$

the differential of Φ at x. We define the jacobian of Φ at x as

$$J\Phi(x) := \left|\det(d\Phi_x)\right|.$$

Remark 5.2.6. The definition 5.2.5 is clearly well posed in the sense that it does not depend on the choice of the basis used to represent the linear map $d\Phi_x$.

Proposition 5.2.7. *Let Σ be a d-dimensional surface of class C^1 in $\mathbb{R}^n$ and $\Phi : \Omega \to \Sigma$ be a local parameterization of Σ, where Ω is an open set in $\mathbb{R}^d$ and $\Phi \in C^1(\Omega, \mathbb{R}^n)$. For all $x \in \Omega$, denote $\nabla\Phi(x)$ the $n \times d$ matrix which represents $d\Phi_x$ as a linear map from $\mathbb{R}^d$ to $\mathbb{R}^n$ with respect to the canonical basis. In other words, we have*

$$(\nabla\Phi(x))_{i,j} = \frac{\partial \Phi_i}{\partial x_j}(x).$$

Then, the following holds:

$$J\Phi(x) = \sqrt{\det((\nabla\Phi(x))^T \cdot \nabla\Phi(x))} = \sqrt{\sum_{d\times d \textit{ minor of } \nabla\Phi(x)} \det(M)^2}.$$

Proof. Let M be the $d \times d$ matrix that represents $d\Phi_x : \mathbb{R}^d \to \mathrm{Tan}_{\Phi(x)}\Sigma$ with respect to orthonormal basis. There exists an $n \times d$ matrix R s.t. $\nabla\Phi(x) = R \cdot M$ and $R^T \times R = Id_{d\times d}$. Hence, we have that

$$\begin{aligned} \det((\nabla\Phi(x))^T \cdot \nabla\Phi(x))) &= \det((RM)^T \cdot (RM)) = \det(M^T \cdot (R^T R) \cdot M) \\ &= \det(M^T \cdot M) = (\det(M))^2. \end{aligned}$$

The identity

$$\sqrt{\det((\nabla\Phi(x))^T \cdot \nabla\Phi(x))} = \sqrt{\sum_{d\times d \text{ minor of } \nabla\Phi(x)} \det(M)^2}$$

follows from the Cauchy-Binet formula (see 7.2.8). □

Now we are in the position to state prove the first version of the area formula.

Theorem 5.2.8 (Area formula - 1)**.** *Let Σ be a d-dimensional surface of class C^1 in $\mathbb{R}^n$ and $\Phi : \Omega \to \Sigma$ be a global parameterization of Σ, where Ω is an open set in $\mathbb{R}^d$ and $\Phi \in C^1(\Omega, \mathbb{R}^n)$. For all Borel set $F \subseteq \Sigma$ there holds*

$$\mathcal{H}^d(F) = \int_{\Phi^{-1}(F)} J\Phi(x)\ dx, \tag{5.6}$$

where the jacobian $J\Phi$ is defined as in 5.2.5. In other words, the Hausdorff dimension on Σ is the d-dimensional volume.

Proof. Notice that (5.6) can be stated equivalently as follows:

$$\mathcal{H}^d \llcorner \Sigma = \Phi_\#(J\Phi \cdot \mathscr{L}^d \llcorner \Omega), \tag{5.7}$$

where $\Phi_\#$ is the push-forward according to Φ. Define

$$\lambda := \Phi_\#(J\varphi \cdot \mathscr{L}^d \llcorner \Omega);$$

thanks to the characterization of the Hausdorff dimension on regular surfaces (see 5.2.3 and 5.2.4), formula (5.7) follows if we show that λ has the property stated in theorem 5.2.3.

Fix $\delta > 0$; we claim that there exists $\varepsilon > 0$ s.t. for all $f : U \to \mathbb{R}^d$ of class C^1 with isometry defect (see 5.2.1) at most δ (U is any open set in Σ) there holds

$$\frac{1}{1+\varepsilon}\lambda(E) \le \mathscr{L}^d(f(E)) \le (1+\varepsilon)\lambda(E) \quad \forall E \subseteq U \text{ Borel.}$$

Denote $\tilde{U} := \Phi^{-1}(U)$ and $g := f \circ \Phi : \tilde{U} \to \mathbb{R}^d$. Notice that $\tilde{U}$ is an open set in $\mathbb{R}^d$ and $g \in C^1(\tilde{U}, \mathbb{R}^d)$. Using the definition of differential (for instance, via curves) it is easy to check that

$$\frac{1}{1+\delta}\,|h| \le |df_y(h)| \le (1+\delta)\,|h| \quad \forall y \in \Sigma\ \forall h \in \mathrm{Tan}_y\Sigma.$$

Thus, we have

$$(1+\delta)^{-d} \le |\det(df_y)| \le (1+\delta)^d \quad \forall y \in \Sigma.$$

By the chain rule it follows that

$$(1+\delta)^{-d} J\Phi(x) \le |\det(dg_x)| \le (1+\delta)^d J\Phi(x) \quad \forall x \in \tilde{U}.$$

Since $g \in C^1(\tilde{U}, \mathbb{R}^d)$ and $\tilde{U}$ is an open set in $\mathbb{R}^d$, we have that

$$|\det(dg_x)| = |\det(\nabla g(x))| \quad \forall x \in \tilde{U},$$

where $(\nabla g(x))_{i,j} = \frac{\partial g_i}{\partial x_j}(x)$. Then, given $E \subseteq U$ Borel and defined $\tilde{E} := \Phi^{-1}(E)$, by the change of variable formula in multiple integrals, we have that

$$\mathscr{L}^d(f(E)) = \mathscr{L}^d(g(\tilde{E})) = \int_{\tilde{E}} |\det(\nabla g(x))|\ dx.$$

In conclusion, we have that

$$\int_{\tilde{E}} (1+\delta)^{-d} J\Phi(x)\ dx \le \int_{\tilde{E}} |\det(\nabla g(x))|\ dx \le \int_{\tilde{E}} (1+\delta)^d J\Phi(x)\ dx,$$

that is

$$(1+\delta)^{-d}\lambda(E) \le \mathscr{L}^d(f(E)) \le (1+\delta)^d\lambda(E).$$

In conclusion, it suffices to choose $\varepsilon > 0$ s.t. $(1+\delta)^d = 1+\varepsilon$. $\square$

5.2.2 Area formula for Lipschitz maps

We state further versions of the area formula, which are more general than the one stated in 5.2.8. As for the proof, we will not give them in full details.

Definition 5.2.9 (Jacobian of a Lipschitz map)**.** Let $f : \Omega \to \mathbb{R}^n$ be a Lipschitz map defined in an open set $\Omega \subseteq \mathbb{R}^d$. Let $x \in \Omega$ be a point where f is differentiable (recall that this happens $\mathscr{L}^d$-a.e. in Ω as stated in 5.1.9); we denote as $df_x : \mathbb{R}^d \to \mathbb{R}^n$ the differential of f at x. We define

$$Jf(x) := \begin{cases} 0 & \text{if } \mathrm{rk}(df_x) < d, \\ \sqrt{\det((\nabla f(x))^T \cdot \nabla f(x))} & \text{if } \mathrm{rk}(df_x) = d. \end{cases}$$

Theorem 5.2.10 (Area formula - 2)**.** *Given $f : \Omega \to \mathbb{R}^n$ a Lipschitz map ($\Omega \subseteq \mathbb{R}^d$ is an open set), then for all $F \subseteq \mathbb{R}^n$ Borel, there holds that*

$$\int_F \# f^{-1}(y)\ d\mathcal{H}^d(y) = \int_{f^{-1}(F)} Jf(x)\ dx, \tag{5.8}$$

where $J\Phi(x)$ is the jacobian of a Lipschitz map defined in 5.2.9.

Remark 5.2.11. The function $\# f^{-1}(\cdot) : F \to \mathbb{N} \cup \{\infty\}$ in (5.8) turns out to be Borel. The left hand side in (5.8) is the so called $\mathcal{H}^d$ measure of F counted with multiplicity.

We state a more general version of the area formula, which implies 5.2.10.

Theorem 5.2.12 (Area formula - 3)**.** *Given $f : \Omega \to \mathbb{R}^n$ a Lipschitz map ($\Omega \subseteq \mathbb{R}^d$ is an open set), then for all $E \subseteq \Omega$ Borel, there holds that*

$$\int_{f(E)} \#(f^{-1}(y) \cap E)\ d\mathcal{H}^d(y) = \int_E Jf(x)\ dx, \tag{5.9}$$

where $Jf(x)$ is the jacobian of a Lipschitz map defined in 5.2.9.

Remark 5.2.13. The function $\#(f^{-1}(\cdot) \cap E) : f(E) \to \mathbb{N} \cup \{\infty\}$ in (5.9) is not necessarily Borel, but agrees $\mathcal{H}^d$-a.e. with a Borel one. The left hand side in (5.8) is the so called $\mathcal{H}^d$ measure of $f(E)$ counted with multiplicity.

Here, we prove 5.2.10 provided 5.2.12.

Proof. Given $F \subseteq \mathbb{R}^n$ Borel, choosing $E := f^{-1}(E)$ (which is a Borel set in Ω), (5.8) is an immediate consequence of (5.9). □

Having said that, we prove 5.2.12.

Proof. **Step 1**: We claim that formula (5.9) holds true if $f \in C^1(\Omega)$ and $Jf(x) \neq 0$ for all $x \in \Omega$ (i.e. df_x has rank d at every $x \in \Omega$). In this case, we can cover Ω with open sets (not necessarily disjoint) Ω_i s.t. f restrict to a parameterization in Ω_i. Take $E \subseteq \Omega$ a Borel set and split E as disjoint union of Borel sets $(E_i)_i$ s.t. $E_i \subseteq \Omega_i$. Call $f_i := f_{|\Omega_i}$, $\Sigma_i := f(\Omega_i)$ and $F_i := f_i(E_i) = f(E_i)$. Note that F_i is Borel; moreover, the map $\#(f^{-1}(\cdot) \cap E) : f(E) \to \mathbb{N} \cup \{\infty\}$ is Borel. In deed, we have that

$$\#(f^{-1}(y) \cap E) = \sum_i \mathbb{1}_{F_i}(y).$$

Then, we can apply 5.2.8 to f_i and Σ_i to deduce that

$$\mathcal{H}^d(f_i(E_i)) = \int_{E_i} Jf_i(x)\ dx = \int_{E_i} Jf(x)\ dx.$$

If we sum with respect to i we deduce that

$$\begin{aligned}\int_{f(E)} \#(f^{-1}(y)\cap E)\ d\mathcal{H}^d(y) &= \int_{f(E)} \sum_i \mathbb{1}_{f_i(E_i)}(y)\ d\mathcal{H}^d(y)\\ &= \sum_i \mathcal{H}^d(f_i(E_i))\\ &= \sum_i \int_{E_i} Jf(x)\ dx\\ &= \int_E Jf(x)\ dx.\end{aligned}$$

Step 2: We claim that if $f \in C^1(\Omega)$ and $Jf(x) = 0$ for all $x \in \Omega$ (i.e. df_x has rank at most $d-1$ at every $x \in \Omega$), then $\mathcal{H}^d(f(E)) = 0$. In this case, the function $\#(f^{-1}(\cdot)\cap E) \to \mathbb{N}\cup\{\infty\}$ is not necessarily Borel; however, it can be integrated over $f(E)$, since $\mathcal{H}^d(f(E)) = 0$. In particular, formula (5.9) holds true. We can easily reduce to the case in which E is bounded and ∇f is bounded in E (otherwise split E in countably many pieces of finite $\mathscr{L}^d$ measure in which ∇f is bounded). Given $\varepsilon > 0$, define $g_\varepsilon : \Omega \to \mathbb{R}^d \times \mathbb{R}^n$ s.t.

$$g_\varepsilon(x) = (\varepsilon x, f(x)).$$

Let $p : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ the projection onto the last n coordinates. Then, we have $f = p \circ g_\varepsilon$. Since the differential of g_ε as maximal rank (i.e. d) at every $x \in \Omega$ and p is a 1-Lipschitz map, we can use 5.2.8 and the property stated in 1.2.3. Thus, we obtain

$$\mathcal{H}^d(f(E)) = \mathcal{H}^d(p(g_\varepsilon(E))) \le \mathcal{H}^d(g_\varepsilon(E)) = \int_E Jg_\varepsilon(x)\ dx.$$

Since $\mathscr{L}^d(E) < +\infty$, it suffices to show that Jg_ε is uniformly bounded in x and that $Jg_\varepsilon(x) \to 0$ for every $x \in E$. Then, by the dominated convergence theorem, we obtain that

$$\mathcal{H}^d(f(E)) \le \lim_{\varepsilon\to 0} \int_E Jg_\varepsilon(x)\ dx = 0.$$

Recall that

$$\nabla g_\varepsilon(x) = \begin{pmatrix} \varepsilon Id_{d\times d} \\ \nabla f(x) \end{pmatrix}.$$

Then, we have that

$$\begin{aligned} Jg_\varepsilon(x) &= \sqrt{\det((\nabla g_\varepsilon(x))^T \cdot \nabla g_\varepsilon(x))}\\ &= \sqrt{\det(\varepsilon^2 Id_{d\times d} + (\nabla f(x))^T \cdot \nabla f(x))}.\end{aligned}$$

In particular, we deduce that $Jg_\varepsilon(x)$ is uniformly bounded for $x \in E$ and $\varepsilon \in (0,1)$; moreover, for all $x \in E$ we have that

$$\lim_{\varepsilon\to 0} Jg_\varepsilon(x) = \sqrt{\det((\nabla f(x))^T \cdot \nabla f(x))} = Jf(x) = 0.$$

Step 3: We claim that (5.9) holds true if f is any map in $C^1(\Omega, \mathbb{R}^n)$. Split Ω in $\Omega_0 \cup \Omega_1$, where

$$\Omega_0 := \{x \in \Omega \mid Jf(x) = 0\}, \quad \Omega_1 := \{x \in \Omega \mid Jf(x) \neq 0\}.$$

Then, we can split E as

$$E = (E \cap \Omega_0) \cup (E \cap \Omega_1).$$

Notice that for all $y \in f(E)$ there holds

$$\#(f^{-1}(y) \cap E) = \#(f^{-1}(y) \cap E \cap \Omega_0) + \#(f^{-1}(y) \cap E \cap \Omega_1).$$

If $y \in f(E) \setminus f(E \cap \Omega_0)$, then we have

$$\#(f^{-1}(y) \cap E) = \#(f^{-1}(y) \cap E \cap \Omega_1).$$

Hence, the function $\#(f^{-1}(\cdot) \cap E) : f(E) \setminus f(E \cap \Omega_0) \to \mathbb{N} \cup \{\infty\}$ is Borel. Since $\mathcal{H}^d(f(E \cap \Omega_0)) = 0$, we deduce that the function $\#(f^{-1}(\cdot) \cap E) : f(E) \to \mathbb{N} \cup \{\infty\}$ agrees up to an $\mathcal{H}^d$-null set with a Borel one; hence, it can be integrated (even if it is not necessarily Borel). Applying the first two steps, we have that

$$\begin{aligned}
\int_{f(E)} \#(f^{-1}(y) \cap E)\, d\mathcal{H}^d(y) &= \int_{f(E\cap\Omega_1)} \#(f^{-1}(y) \cap E)\, d\mathcal{H}^d(y) \\
&= \int_{f(E\cap\Omega_1)} \#(f^{-1}(y) \cap E \cap \Omega_1)\, d\mathcal{H}^d(y) + \int_{f(E\cap\Omega_1)} \#(f^{-1}(y) \cap E \cap \Omega_0)\, d\mathcal{H}^d(y) \\
&= \int_{E\cap\Omega_1} Jf(x)\, dx + \int_{f(E\cap\Omega_1)\cap f(\Omega_0)} \#(f^{-1}(y) \cap E \cap \Omega_0)\, d\mathcal{H}^d(y) \\
&= \int_{E\cap\Omega_1} Jf(x)\, dx \\
&= \int_E Jf(x)\, dx.
\end{aligned}$$

Step 4: If f is Lipschitz and $\mathscr{L}^d(E) = 0$, we already know that $\mathcal{H}^d(f(E)) = 0$. To conclude, we only sketch the general case. By Lusin property of Lipschitz maps (see 5.1.12), we can find maps $f_n \in C^1(\Omega, \mathbb{R}^n)$ and compact sets $K_n \subseteq \Omega$ with the following properties:

- $f_n = f$ for $\mathscr{L}^d$-a.e. $x \in K_n$;
- $Jf_n = Jf$ for $\mathscr{L}^d$-a.e. $x \in K_n$;
- $\mathscr{L}^d(\Omega \setminus K_n) \to 0$.

Then, we split E as disjoint union of E_n, where $E_n \subseteq K_n$ for $n \geq 1$ and $\mathscr{L}^d(E_0) = 0$. The previous step guarantees that (5.9) holds for E_n for all $n \geq 1$; moreover, we have already remarked that (5.9) holds for E_0. Then, we have

$$\begin{aligned}
\int_{f(E)} \#(f^{-1}(y) \cap E)\ d\mathcal{H}^d(y) &= \sum_{n=0}^{\infty} \int_{f_n(E_n)} \#(f_n^{-1}(y) \cap E_n)\ d\mathcal{H}^d(y) \\
&= \sum_{n=0}^{\infty} \int_{E_n} Jf_n(x)\ dx \\
&= \sum_{n=0}^{\infty} \int_{E_n} Jf(x)\ dx \\
&= \int_E Jf(x)\ dx.
\end{aligned}$$

□

Corollary 5.2.14 (Area formula - 4)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $f : \Omega \to \mathbb{R}^n$ be a Lipschitz map and $h : \Omega \to [0, +\infty]$ be a Borel map. Then, there holds*

$$\int_{f(\Omega)} \left(\sum_{x \in f^{-1}(y)} h(x) \right) d\mathcal{H}^d(y) = \int_\Omega h(x) Jf(x)\ dx. \tag{5.10}$$

Proof. We only sketch the proof of the statement above. If $h = \mathbb{1}_E$, for some Borel set $E \subseteq \Omega$, then (5.10) reduces to (5.9); by linearity, (5.10) holds true for nonnegative step functions. By approximation (Beppo Levi's theorem is needed), we obtain (5.10) for nonnegative Borel functions. □

Chapter 6

Rectifiable sets

This chapter deals with rectifiable sets (see [4], [5] and [2]). Let $(\mathbb{X}, d)$ be a metric space.

6.1 Definition and main properties

Definition 6.1.1 (Rectifiable set)**.** Given $d \in \mathbb{N}$, a set $E \subseteq \mathbb{X}$ is d-rectifiable if it can be decomposed as countable union of Borel sets E_i, namely $E = \bigcup_{i=0}^{\infty} E_i$, where

1. $\mathcal{H}^d(E_0) = 0$,
2. for all $i \geq 1$, we have that $E_i = f(F_i)$ where $F_i \subseteq \mathbb{R}^d$ is a Borel set and $f_i : F_i \to \mathbb{X}$ is a Lipschitz map.

If the metric space $\mathbb{X}$ is $\mathbb{R}^n$, we can equivalently define rectifiable sets with regular maps, as stated below. We will not prove this results.

Proposition 6.1.2. *Assume $\mathbb{X} = \mathbb{R}^n$. The second condition in 6.1.1 can be replaced with one of the followings:*

2' $E_i \subseteq f(A_i)$, where $A_i \subseteq \mathbb{R}^d$ is an open set and $f_i \in C^1(A_i, \mathbb{R}^n)$;

2" $E_i \subseteq f_i(A_i)$, where $A_i \subseteq \mathbb{R}^d$ is an open set and $f_i \in C^1(A_i, \mathbb{R}^n)$ is a regular parameterization;

2"' $E_i \subseteq \Sigma_i$, where Σ_i is a d-dimensional surface of class C^1 in $\mathbb{R}^n$.

The proof, which will be omitted, is based on the following lemma.

Lemma 6.1.3. *If $F \subseteq \mathbb{R}^d$ is a Borel set and $f : F \to \mathbb{R}^n$ is a Lipschitz map, then $f(F)$ can be written as $f(F) = \bigcup_{n=0}^{\infty} E_n$, where $\mathcal{H}^d(E_0) = 0$ and $E_n \subseteq f_n(A_n)$ for all $n \geq 1$, with $A_n \subseteq \mathbb{R}^d$ open set and $f_n \in C^1(A_n, \mathbb{R}^n)$.*

Proof. We only sketch the proof. Kirszbraun theorem (see 5.1.6) or McSchane lemma (see 5.1.4) provide a Lipschitz extension of f to a $\mathbb{R}^d$. By Lusin property of Lipschitz maps (see 5.1.12), there exist maps $f_n \in C^1(\mathbb{R}^d, \mathbb{R}^n)$ and compact sets K_n s.t. $f = f_n$ on K_n and $\bigcup_n K_n$ covers a.a. of $\mathbb{R}^d$. Having said that, we define $E_n = f_n(K_n)$ for all $n \geq 1$ and $E_0 = f\left(F \setminus \bigcup_n K_n\right)$. Since f is Lipschitz, we have that

$$\mathcal{H}^d\left(f\left(F \setminus \bigcup_n K_n\right)\right) \leq \operatorname{Lip}(f)\, \mathscr{L}^d\left(F \setminus \bigcup_n K_n\right) = 0.$$

□

Definition 6.1.4 (Purely unrectifiable set)**.** Given $d \in \mathbb{N}$, a set $E \subseteq \mathbb{X}$ is d-purely unrectifiable if $\mathcal{H}^d(E \cap f(F)) = 0$ for all map $f : F \to \mathbb{X}$ Lipschitz, where $F \subseteq \mathbb{R}^d$ is a Borel set.

As in the case of rectifiable sets (see 6.1.2), if $\mathbb{X} = \mathbb{R}^n$ there are equivalent definitions for purely unrectifiable sets. We will not prove this result.

Proposition 6.1.5. *Let $E \subseteq \mathbb{R}^n$ be a Borel set. The followings are equivalent:*

- *E is d-purely unrectifiable in the sense of definition 6.1.4, i.e. for all $f : F \to \mathbb{R}^n$ Lipschitz ($F \subseteq \mathbb{R}^d$ is a Borel set), there holds*
$$\mathcal{H}^d(E \cap f(F)) = 0;$$
- *for every map $f \in C^1(\mathbb{R}^d, \mathbb{R}^n)$ there holds*
$$\mathcal{H}^d(E \cap f(\mathbb{R}^d)) = 0;$$
- *for every regular parameterization $f \in C^1(A, \mathbb{R}^n)$ ($A \subseteq \mathbb{R}^d$ is an open set) there holds*
$$\mathcal{H}^d(E \cap f(A)) = 0;$$
- *for every Σ d-dimensional surface of class C^1 in $\mathbb{R}^n$ there holds*
$$\mathcal{H}^d(E \cap \Sigma) = 0.$$

Remark 6.1.6.

- If E is d-rectifiable, then $\dim_{\mathcal{H}}(E) \leq d$, because E is $\mathcal{H}^d$ σ-finite.
- If $\mathcal{H}^d(E) = 0$, then E is clearly d-rectifiable and d-purely unrectifiable. On the other hand, assume that E is d-rectifiable and d-purely unrectifiable. By rectifiability, write $E = \bigcup_{n=0}^{\infty} E_n$, where $\mathcal{H}^d(E_0) = 0$ and E_n is the image of a Lipschitz map defined on a Borel subset of $\mathbb{R}^d$ for all $n \geq 1$; since E is purely unrectifiable, then $\mathcal{H}^d(E_n) = 0$ for all $n \geq 1$. Then, $\mathcal{H}^d(E) = 0$.
- $\mathcal{H}^d(E) = 0$ does not imply that E can be covered by countably many Lipschitz images of sets in $\mathbb{R}^d$. In deed, there exists $K \subseteq \mathbb{R}^2$ compact s.t. $\dim_{\mathcal{H}} K = 0$ (which implies that $\mathcal{H}^1(K) = 0$) that cannot be covered by countably many Lipschitz images of $\mathbb{R}$. Hence, it is interesting to characterize the compact sets $K \subseteq \mathbb{R}^2$ that can be covered by a Lipschitz curve of finite length. This is known as "travelling salesman problem" (due to P.Jones). In other words, the set E_0 in the definition of rectifiability (see 6.1.1) cannot be removed.
- We cannot replace surfaces of class C^1 in the definition of rectifiability (see 6.1.2) with surfaces of class C^2 or even $C^{1,\alpha}$ for some $\alpha > 0$. In fact, C^1 functions do not have Lusin properties with functions of class $C^{1,\alpha}$, with $\alpha > 0$. In deed, there exists $f \in C^1([0,1])$ s.t. for all $\alpha > 0$ for all $g \in C^{1,\alpha}([0,1])$ there holds $\mathscr{L}^1(\{x \mid f(x) = g(x)\}) = 0$.

Here we provide a useful criterion to establish unrectifiability; we only sketch the proof.

Lemma 6.1.7. *Given $E \subseteq \mathbb{R}^2$ with projections E_1, E_2 on the axis, if $\mathscr{L}^1(E_1) = \mathscr{L}^1(E_2) = 0$, then E is 1-purely unrectifiable.*

Proof. Given a curve Σ in $\mathbb{R}^2$ of class C^1, we have to show that $\mathcal{H}^1(E \cap \Sigma) = 0$. First of all, suppose that Σ is the graph of a C^1 function $g : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. In other words, we have that

$$\Sigma = \{(x_1, g(x_1)) \mid x_1 \in I\}.$$

We have that

$$\mathcal{H}^1(E \cap \Sigma) = \int_{E_1 \cap I} \sqrt{1 + g'(x)^2}\, dx = 0,$$

since $\mathscr{L}^1(E_1) = 0$. Similarly, if Σ is the image of a C^1 function of the second variable, we obtain $\mathcal{H}^d(E \cap \Sigma) = 0$. To conclude, we recall that every curve Σ of class C^1 can be covered by countably many pieces of C^1 graphs (in the first or in the second variable). □

Remark 6.1.8. The criterion given in 6.1.7 can be generalized to the case of d-purely unrectifiable sets in $\mathbb{R}^n$. However, it is enough to understand that the property of rectifiability is not related to the Hausdorff dimension, as shown in the example below. In deed, for all $d \in (0, 2)$ there exists a compact set $K \subseteq \mathbb{R}^2$ s.t. $\dim_{\mathcal{H}}(K) = d$ and K is 1-purely unrectifiable.

Example 6.1.9. Given $\lambda \in \left(0, \frac{1}{2}\right)$, consider the self-similar fractal K_λ associated to the four similarities with scaling factor λ, as described in 3.3.11. Recall that $d = \dim_{\mathcal{H}}(K_\lambda)$ is s.t. $4\lambda^d = 1$; hence, $d = \frac{\log 4}{\log \lambda}$, which can be any number in $(0, 2)$. Clearly, K_λ has projections on the coordinate axis that are $\mathscr{L}^1$-null. Hence, by lemma 6.1.7, K_λ is 1-purely unrectifiable.

Example 6.1.10. It can be shown that the Von Koch curve (see 3.3.10) is 1-purely unrectifiable, but the previous lemma does not apply.

The following criterion holds true; we only sketch the proof.

Proposition 6.1.11. *If $E \subseteq \mathbb{R}^n$ is an $\mathcal{H}^d$ σ-finite Borel set, then it can be decomposed as $E = E_r \cup E_{pu}$, where*

- *E_r is d-rectifiable,*
- *E_{pu} is d-purely unrectifiable.*

The decomposition is unique up to $\mathcal{H}^d$-null sets.

Proof. Let $\mathcal{F} := \{F \subseteq E \mid F \text{ is } d\text{-rectifiable}\}$. $\mathcal{F}$ is closed under countable union; it can be proved that $\mathcal{F}$ has an element E_r which maximizes $\mathcal{H}^d$. Then, it can be checked that $E \setminus E_r$ is d-purely unrectifiable. □

6.2 Tangent space to rectifiable sets

6.2.1 Weak tangent bundle

We want to provide a definition of tangent bundle for rectifiable sets. In the following, denote by $\mathrm{Gr}(n, d)$ the Grassmannian of the d-dimensional hyperplanes in $\mathbb{R}^n$.

Definition 6.2.1 (Weak tangent bundle)**.** Given a set $E \subseteq \mathbb{R}^n$ Borel, we say that $V : E \to \mathrm{Gr}(n,d)$ is a weak tangent bundle to E if the following holds true: for all Σ d-dimensional surface of class C^1 in $\mathbb{R}^n$, there holds $\mathrm{Tan}_x\Sigma = V(x)$ for $\mathcal{H}^d$-a.e. $x \in \Sigma \cap E$.

We start with a fundamental remark.

Remark 6.2.2. Given Σ, Σ' d-dimensional surfaces of class C^1 in $\mathbb{R}^n$, then $\mathrm{Tan}_x\Sigma = \mathrm{Tan}_x\Sigma'$ for $\mathcal{H}^d$-a.e. $x \in \Sigma \cap \Sigma'$. This follows from the fact that, given $f, g : \mathbb{R}^d \to \mathbb{R}^m$ of class C^1, then $\nabla f(x) = \nabla g(x)$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ s.t. $f(x) = g(x)$ (in deed, this is a consequence of the Sard's lemma). In other words, the classical tangent bundle to a C^1 surface is also a weak tangent bundle, which is unique in this case.

Proposition 6.2.3. *If E is a d-rectifiable set in $\mathbb{R}^n$, then E admits a weak tangent bundle $V : E \to Gr(n,d)$ (see 6.2.1), which is unique up to $\mathcal{H}^d$ null sets.*

Proof. We can write $E = \bigcup_{n=0}^\infty E_n$, where the union is disjoint, E_0 is $\mathcal{H}^d$ null and for all $i \geq 1$ there holds $E_i \subseteq \Sigma_i$, where Σ_i is a d-dimensional surface of class C^1 in $\mathbb{R}^n$. We set

$$V(x) := \begin{cases} \mathrm{Tan}_x\Sigma_i & \text{if } x \in E_i,\ i \geq 1 \\ \text{any } V \in \mathrm{Gr}(n,d) & \text{if } x \in E_0. \end{cases}$$

Take Σ a d-dimensional surface of class C^1 in $\mathbb{R}^n$. Given $i \geq 1$, as remarked in 6.2.2, we have that $\mathrm{Tan}_x\Sigma = \mathrm{Tan}_x\Sigma_i = V(x)$ for $\mathcal{H}^d$-a.e. $x \in E_i \cap \Sigma$. Then, we obtain that $\mathrm{Tan}_x\Sigma = V(x)$ for $\mathcal{H}^d$-a.e. $x \in \Sigma \cap E$. In other words, we have shown that V is a weak tangent bundle to E. In deed, this argument also shows that V is unique up to $\mathcal{H}^d$ null sets; in particular, V does not depend on the decomposition of E chosen, up to $\mathcal{H}^d$ null sets. $\square$

Definition 6.2.4. Given $E \subseteq \mathbb{R}^n$ a d-rectifiable set, we denote as $\mathrm{Tan}^w E$ the weak tangent bundle to E provided by the proposition 6.2.3.

6.2.2 Approximate tangent space

We have shown that any d-rectifiable sets E has a weak tangent bundle, which is well defined and unique up to $\mathcal{H}^d$-null sets. If E is also $\mathcal{H}^d$-locally finite, than we have stronger tangential properties. We introduce the notion of a approximate tangent space, which is bases on a blow-up procedure.

Definition 6.2.5 (Approximate tangent space)**.** Take $V \in \mathrm{Gr}(n,d)$ and $E \subseteq \mathbb{R}^n$ Borel and $\mathcal{H}^d$ locally finite. For all $x \in E$ for all $r > 0$, we denote

$$E_{x,r} := \frac{1}{r}(E - x).$$

Given $x \in E$, we say that V is an approximate tangent space to E at x if $\mathcal{H}^d \llcorner E_{x,r} \xrightarrow{*} \mathcal{H}^d \llcorner V$ locally in the sense of measures as $r \to 0$, that is

$$\lim_{r\to 0} \int_{E_{x,r}} g(y)\ d\mathcal{H}^d(y) = \int_V g(y)\ d\mathcal{H}^d(y) \quad \forall g \in C_c(\mathbb{R}^n).$$

Remark 6.2.6. In the framework of the definition 6.2.5, if E admits an approximate tangent space V at x, then V is unique. In deed, $\mathcal{H}^d \llcorner V$ is uniquely determined, since it is a weak-* limit of measures. Then, we denote V as $\mathrm{Tan}^a_x E$.

Here we show that the notion of approximate tangent space agrees with the classical one in the setting of C^1 submanifold of $\mathbb{R}^n$; this result will be useful to extend the notion of approximate tangent space to rectifiable sets.

Lemma 6.2.7. *Let Σ be k-submanifold of $\mathbb{R}^n$ of class C^1 (without boundary). For all $x \in \Sigma$, we have*

$$\mathcal{H}^k \llcorner \left(\frac{\Sigma - x}{r}\right) \stackrel{*}{\rightharpoonup} \mathcal{H}^k \llcorner \, Tan_x\Sigma \quad \text{as } r \downarrow 0.$$

Proof. Fix $x \in \Sigma$. We denote as B^k_δ the k-dimensional ball centered at the origin in $\mathbb{R}^k$ of radius δ. We can reduce to the following case:

- $x = 0$;
- $\mathrm{Tan}_x\Sigma = \mathrm{Span}(e_1, \dots, e_k) = \mathbb{R}^k$, where $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$;
- there exist $\delta > 0$ and a C^1 map $\Phi : B^k_\delta \to B^{n-k}_\delta$ s.t. $\Sigma \cap (B^k_\delta \times B^{n-k}_\delta)$ is the graph of Φ, that is

$$\Sigma \cap (B^k_\delta \times B^{n-k}_\delta) = \{(x, \Phi(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid x \in B^k_\delta\}.$$

Let g be a test function in $C_c(\mathbb{R}^n)$; we have to check that

$$\lim_{r\to 0} \int_{\Sigma/r} g(y)\ d\mathcal{H}^k(y) = \int_{\mathbb{R}^k} g(x, 0)\ dx$$

Notice that Σ/r can be parameterized in $B_{\delta/r} \times B^{n-k}_{\delta/r}$ as follows:

$$\Sigma/r \cap (B^k_{\delta/r} \times B^{n-k}_{\delta/r}) = \left\{ \left(z, \frac{\Phi(rz)}{r}\right) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \;\middle|\; z \in B^k_{\delta/r} \right\}.$$

Since we are interested in the limiting behaviour as r approaches 0, there exists $\eta > 0$ s.t. for all $r \in (0, \eta)$ there holds

$$\mathrm{supp}(g) \subseteq B^k_{\delta/r} \times B^{n-k}_{\delta/r}.$$

Given $x \in B^k_\delta$, denote by $J\Phi(x)$ the jacobian determinant of Φ; by the area formula (see 5.2.14), for $r \in (0, \eta)$ there holds

$$\int_{\Sigma/r} g(y)\ d\mathcal{H}^k(y) = \int_{B^k_{\delta/r}} g\left(z, \frac{\Phi(zr)}{r}\right) \sqrt{1 + J\Phi(rz)}\ dx.$$

Under our assumptions, notice that the map Φ as differential $d\Phi$ that vanishes at zero; hence, $J\Phi(0) = 0$. As $r \downarrow 0$, for all $z \in \mathbb{R}^k$, there holds that

$$\lim_{r\to 0} \frac{\Phi(zr)}{r} = d\Phi_0(z) = 0.$$

Since Φ is a map of class C^1, we have that

$$\lim_{r\to 0} J\Phi(rz) = 0.$$

It is easy to show that the pointwise limit above are are uniformly bounded with respect to $r \in (0, \eta)$ and z in any compact set of $\mathbb{R}^k$. Then, by the dominated convergence theorem, we deduce that

$$\lim_{r\to 0} \int_{B^k_{\delta/r}} g\left(z, \frac{\Phi(zr)}{r}\right) \sqrt{1 + J\Phi(rz)}\, dx = \int_{\mathbb{R}^k} g(x, 0)\, dx.$$

$\square$

Proposition 6.2.8. *If $E \subseteq \mathbb{R}^n$ is d-rectifiable and $\mathcal{H}^d$ locally finite, then for $\mathcal{H}^d$-a.e. $x \in E$ we have that $Tan^w_x E$ is the approximate tangent space.*

Proof. We can write $E = \bigcup_{n=0}^{\infty} E_n$, where the union is disjoint, E_0 is $\mathcal{H}^d$ null and for all $i \geq 1$ there holds $E_i \subseteq \Sigma_i$, where Σ_i is a d-dimensional surface of class C^1 in $\mathbb{R}^n$. We fix $i \geq 1$ and we claim that $\mathcal{H}^d \llcorner E_{x,r} \to \mathcal{H}^d \llcorner \mathrm{Tan}_x \Sigma_i$ for $\mathcal{H}^d$-a.e. $x \in E_i$. This is enough to conclude, since $\mathrm{Tan}_x \Sigma_i = \mathrm{Tan}^w_x E$ for $\mathcal{H}^d$-a.e. $x \in E_i$.

Notice that

$$E_{x,r} = (\Sigma_i)_{x,r} \setminus (\Sigma_i \setminus E)_{x,r} \cup (E \setminus \Sigma_i)_{x,r},$$

which implies that

$$\mathcal{H}^d \llcorner E_{E_{x,r}} = \mathcal{H}^d \llcorner (\Sigma_i)_{x,r} - \mathcal{H}^d \llcorner (\Sigma_i \setminus E)_{x,r} + \mathcal{H}^d \llcorner (E \setminus \Sigma_i)_{x,r}.$$

We have three addenda and we can study separately their limiting behaviour as $r \to 0$.

- We have already shown in lemma 6.2.7 that $\mathcal{H}^d \llcorner (\Sigma_i)_{x,r} \overset{*}{\rightharpoonup} \mathcal{H}^d \llcorner \mathrm{Tan}_x \Sigma_i$ locally in the sense of measures for all $x \in \Sigma_i$.
- We claim that for $\mathcal{H}^d$-a.e. $x \in E_i$ there holds that $\mathcal{H}^d \llcorner (\Sigma_i \setminus E)_{x,r} \overset{*}{\rightharpoonup} 0$ locally in the sense of measure. Fix $R > 0$ and $x \in E_i$ s.t. $\Theta^*_d(\Sigma_i \setminus E, x)$ exists and it is 0. By theorem 3.2.3, we know that the required properties occur for $\mathcal{H}^d$-a.e. $x \notin \Sigma_i \setminus E$, that is for $\mathcal{H}^d$-a.e. $x \in E_i$. We have that

  $$\lim_{r\to 0} \mathcal{H}^d \llcorner (\Sigma_i \setminus E)_{x,r}(B_R) = \lim_{r\to 0} \frac{\mathcal{H}^d((\Sigma_i \setminus E) \cap B(x, rR))}{r^d} = \alpha_d R^d \Theta^*_d(\Sigma_i \setminus E, x) = 0.$$

  We deduce that $\mathcal{H}^d \llcorner (\Sigma_i \setminus E)_{x,r} \to 0$ strongly in any ball for $\mathcal{H}^d$-a.e. $x \in E_i$; in particular, the convergence is locally in the sense of measures for $\mathcal{H}^d$-a.e. $x \in E_i$.
- Similarly, it can be checked that $\mathcal{H}^d \llcorner (E \setminus \Sigma_i)_{x,r} \to 0$ as $r \to 0$ locally in the sense of measures for $\mathcal{H}^d$-a.e. $x \in E_i$ (recall that $(E \setminus \Sigma_i) \cap E_i = \emptyset$).

$\square$

Corollary 6.2.9. *If $E \subseteq \mathbb{R}^n$ is d-rectifiable and $\mathcal{H}^d$ locally finite, then for $\mathcal{H}^d$-a.e. $x \in E$ we have that $\Theta_d(E, x)$ exists and equals 1.*

Proof. Take $x \in E$ s.t. E admits an approximate tangent plane at x. Notice that

$$\mathcal{H}^d \llcorner \mathrm{Tan}^a_x E(\partial B_1) = 0;$$

then, by the weak-* convergence, it follows that

$$1 = \lim_{r\to 0} \mathcal{H}^d E_{x,r}(B_1) = \lim_{r\to 0} \frac{\mathcal{H}^d(E \cap B(x, r))}{\alpha_d r^d} = \Theta_d(E, x).$$

$\square$

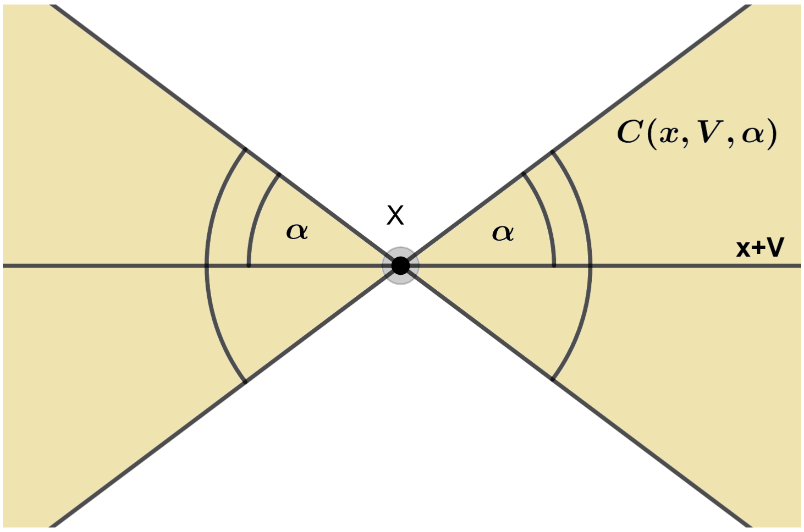

Figure 6.1: The cone $C(x, V, \alpha)$ is highlighted in yellow.

The notation established below will be very useful in the following of the chapter.

Definition 6.2.10 (Cone)**.** Given $x \in \mathbb{R}^n$, $V \in \mathrm{Gr}(n, d)$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$, we set

$$C(x, V, \alpha) := x + \{y \in \mathbb{R}^n \mid \mathrm{dist}(x, V) \leq |y| \sin \alpha\}.$$

We can refine the result given in 6.2.9.

Corollary 6.2.11. *Fix $\alpha \in \left(0, \frac{\pi}{2}\right)$; take $E \subseteq \mathbb{R}^n$ a d-rectifiable set which is $\mathcal{H}^d$ locally finite. Let $x \in E$ s.t. E admits an approximate tangent plane at x (see 6.2). Then, for $\mathcal{H}^d$-a.e. $x \in E$ the followings hold true:*

$$\lim_{r \to 0} \frac{\mathcal{H}^d(E \cap B(x, r) \cap C(x, Tan^a_x E, \alpha))}{\alpha_d r^d} = 1;$$

$$\lim_{r \to 0} \frac{\mathcal{H}^d(E \cap B(x, r) \cap C(x, Tan^a_x E, \alpha)^c)}{r^d} = 0.$$

In particular, these facts holds for $\mathcal{H}^d$-a.e. $x \in E$.

Proof. As for the first statement, using the scaling and translation properties of the Hausdorff measure, we obtain that

$$\frac{1}{r^d}\mathcal{H}^d(E \cap B(x, r) \cap C(x, \mathrm{Tan}^a_x E\alpha)) = \mathcal{H}^d(E_{x,r} \cap B_1 \cap C(0, \mathrm{Tan}^a_x E, \alpha))$$

$$= \mathcal{H}^d \llcorner E_{x,r}(B_1 \cap C(0, \mathrm{Tan}^a_x E, \alpha))$$

Notice that $\partial(B_1 \cap C(0, \mathrm{Tan}^a_x E, \alpha)) \cap \mathrm{Tan}^a_x E$ is the union of the origin and the intersection of a d-dimensional plane with a $(n-1)$-dimensional sphere, that is a $(d-1)$-dimensional sphere. In particular, it is an $\mathcal{H}^d$ null set. Having said that

$$\mathcal{H}^d \llcorner \mathrm{Tan}^a_x E \; (\partial(B_1 \cap C(0, \mathrm{Tan}^a_x E, \alpha))) = 0,$$

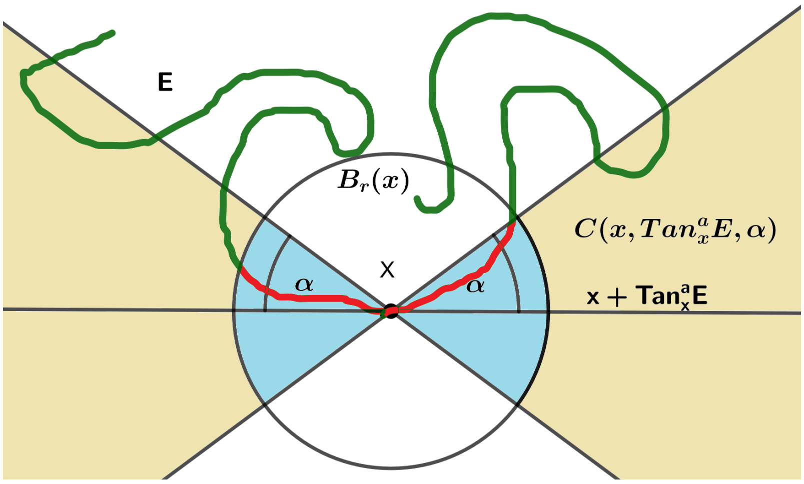

Figure 6.2: We are interested in computing the d-dimensional Hausdorff measure, scaled by the factor $\alpha_d r^d$ of the red part of the set E.

by the weak-* convergence, we deduce that

$$\begin{aligned}\lim_{r\to 0}\frac{1}{r^d}\mathcal{H}^d(E\cap B(x,r)\cap C(x,\mathrm{Tan}^a_x E\alpha)) &= \lim_{r\to 0}\mathcal{H}^d \llcorner E_{x,r}(B_1\cap C(0,\mathrm{Tan}^a_x E,\alpha))\\ &= \mathcal{H}^d\llcorner \mathrm{Tan}^a_x E\ (B_1\cap C(0,\mathrm{Tan}^a_x E,\alpha))\\ &= \mathcal{H}^d(B_1\cap \mathrm{Tan}^a_x E)\\ &= \alpha_d.\end{aligned}$$

The second statement can be proved with a similar argument. □

Remark 6.2.12. An heuristic consequence of corollary 6.2.11 is that cusps and boundary points can occur only for $\mathcal{H}^d$ null sets of points in a d-rectifiable set. In other words, these behaviour are not generic. For instance, assume that E is a 1-rectifiable set in $\mathbb{R}^2$ which has a cusp at the point x (see figure 6.3). Then, one can compute that

$$\mathcal{H}^1\llcorner E_{x,r}\to 2\mathcal{H}^1\llcorner S$$

weakly-* as r goes to 0, where $\mathrm{Tan}_x(E)$ is that we expect to be the tangent plane to E at x and S is half of the translated plane.

An example of boundary point is given in figure 6.4. In this case, we have that the weak-* limit of the scaled measure is $\mathcal{H}^1\llcorner S$, where S is as before.

Remark 6.2.13. We provide an example of a 1-rectifiable set E in $\mathbb{R}^2$ which is $\mathcal{H}^1$ finite and s.t. for $\mathcal{H}^1$-a.e. $x\in E$ there holds

$$\mathcal{H}^1(E\cap B(x,r)\cap C(x,\mathrm{Tan}^a_x E,\alpha)^c)>0\quad \forall r>0\ \forall\alpha\in\left(0,\frac{\pi}{2}\right).$$

Let $(x_n)_n$ be a dense sequence in $\mathbb{R}^2$, $(r_n)_n$ be a sequence of radii s.t. $\sum_n r_n<+\infty$ and $(e_n)_n$ be a sequence of unit vectors in $\mathbb{R}^2$. For all n we set I_n to be the segment joining the points $x_n-r_ne_n$ and $x_n+r_ne_n$. Then

$$E:=\bigcup_n I_n$$

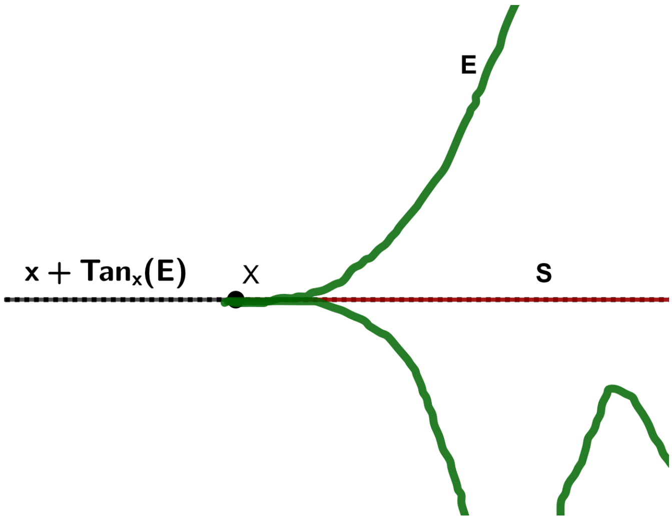

Figure 6.3: In red the support of the weak-* limit of the scaled $\mathcal{H}^d \llcorner E$ measure in a cusp point.

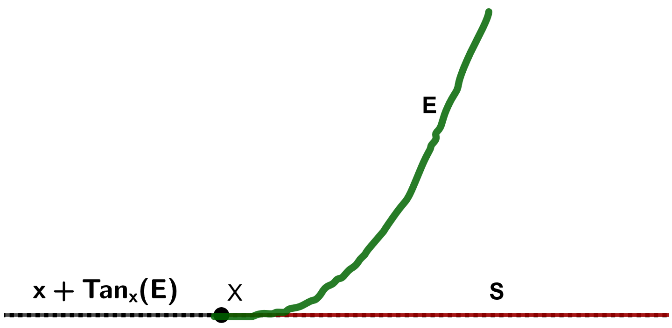

Figure 6.4: In red the support of the weak-* limit of the scaled $\mathcal{H}^d \llcorner E$ measure in a boundary point.

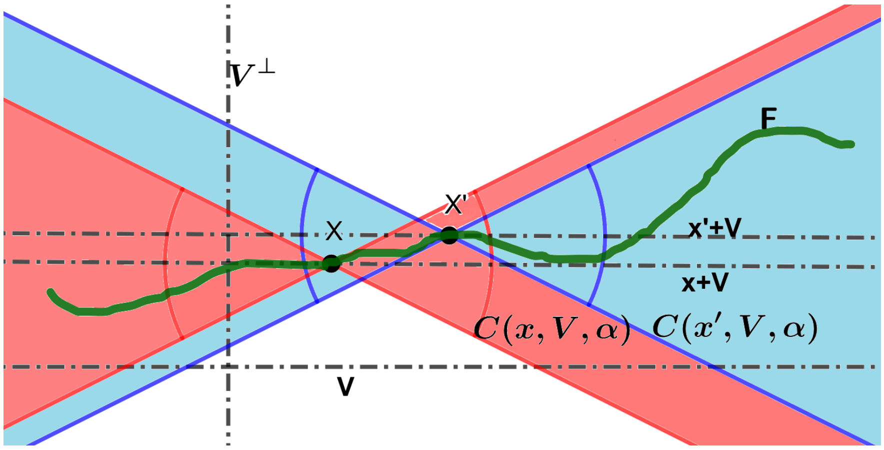

Figure 6.5: The cones $C(x,V,\alpha)$ and $C(x',V,\alpha)$ are respectively in red and in blue.

is 1-rectifiable and $\mathcal{H}^1$ finite. Fix $n \in \mathbb{N}$; for $\mathcal{H}^1$-a.e. $x \in I_n$, there holds that $\mathrm{Tan}^w_x E = \mathrm{Span}(e_n)$. Notice also that $\mathrm{supp}(\mathcal{H}^1 \llcorner E) = \mathbb{R}^2$. Since $B(x,r) \cap C(x, \mathrm{Tan}^a_x E, \alpha)^c$ has non empty interior part for all $r > 0$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$, we deduce that

$$\mathcal{H}^1(E \cap B(x,r) \cap C(x, \mathrm{Tan}^a_x E, \alpha)) = \mathcal{H}^1 \llcorner E(B(x,r) \cap C(x, \mathrm{Tan}^a_x E, \alpha)) > 0.$$

However, arguing as in the proof of corollary 6.2.11, it can be checked that for all $n \in \mathbb{N}$ for $\mathcal{H}^1$-a.e. $x \in I_n$ there holds

$$\lim_{r \to 0} \frac{\mathcal{H}^1 \llcorner E(B(x,r)) \setminus I_n}{r} = 0.$$

6.3 Rectifiability criteria

We introduce some useful rectifiability criteria. We will also state (without proof) some famous results. The first rectifiability criterion is based on the following lemma.

Lemma 6.3.1. *Let F be a set in $\mathbb{R}^n$ s.t. there exist $V \in Gr(n,d)$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ with the following property: for all $x \in F$ there holds $F \subseteq C(x,V,\alpha)$. Then F is included on the graph of a Lipschitz map $g : V \to V^\perp$ s.t. $Lip(g) \leq \tan(\alpha)$ (see figure 6.5).*

Proof. Notice that F is the graph of a map $g : \pi_V(F) \to V^\perp$, where $\pi_v : \mathbb{R}^n \to V$ is the orthogonal projection with respect to V. In deed, the assumptions guarantee that for all $x \in F$ we have $F \cap \pi_V^{-1}(x) = \{x\}$; this argument provides the existence of such a map g. Moreover, g is Lipschitz: in fact, given $x, x' \in F$ there holds that $x' \in C(x,F,\alpha)$. Hence, we have

$$|x - x'| \leq \tan(\alpha) \left| \pi_V(x) - \pi_V(x') \right|,$$

that is g is $\tan(\alpha)$-Lipschitz. At this point, Kirszbraun theorem (see 5.1.6) provides a $\tan(\alpha)$-Lipschitz extension of g to the vector space V with values in $V^\perp$. □

Proposition 6.3.2. *Let d be an integer. Let $E \subseteq \mathbb{R}^n$ be a Borel set s.t. for all $x \in E$ there exist $V(x) \in Gr(n,d)$, $\alpha(x) \in \left(0, \frac{\pi}{2}\right)$ and a radius $r(x) > 0$ s.t.*

$$E \cap B(x, r(x)) \subseteq C(x, V(x), \alpha(x)).$$

Then E is contained in a countable union of Lipschitz graphs. In particular, if the assumptions hold for $\mathcal{H}^d$-a.e. $x \in E$, then E is rectifiable.

Proof. We want to write E as countable union of subsets that fulfil the assumptions of lemma 6.3.1. For all $\alpha \in \left(0, \frac{\pi}{2}\right)$, for all $V \in \mathrm{Gr}(n,d)$, for all $r > 0$, define

$$E_{\alpha,V,r} \coloneqq \{x \in E \mid E \cap B(x,r) \subseteq C(x,V,\alpha)\}.$$

Take a set $Q \subseteq \mathbb{R}^n$ s.t. $\mathrm{diam}(Q) \leq \frac{r}{2}$; then, $E_{\alpha,V,r} \cap Q$ satisfies the assumption of lemma 6.3.1. In deed, for all $x \in E_{\alpha,V,r} \cap Q$ there holds $E_{\alpha,V,r} \subseteq E$ and $Q \subseteq B(x,r)$; hence, we have that

$$E_{\alpha,V,r} \cap Q \subseteq E \cap B(x,r) \subseteq C(x,V,\alpha).$$

So, we infer that $E_{\alpha,V,r} \cap Q$ is included in the graph of a Lipschitz map from V with values in $V^\perp$. At this point, for all N define $\mathcal{G}_N$ to be a finite family of d-dimensional planes in $\mathbb{R}^n$ s.t. every $V \in \mathrm{Gr}(n,d)$ is contained in $C\left(0, \overline{V}, \frac{1}{2N}\right)$ for some $\overline{V} \in \mathcal{G}_N$; for all $r > 0$ let $\mathcal{F}_r$ be a countable decomposition of $\mathbb{R}^n$ in cubes of diameter $\frac{r}{2}$. Then, we can write

$$E = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{G}_n} \bigcup_{m\in\mathbb{N}} \bigcup_{Q\in\mathcal{F}_{\frac{1}{m}}} E_{\frac{\pi}{2}-\frac{1}{n},V,\frac{1}{m}} \cap Q.$$

Since the union is countable, we deduce that E is contained in the union of Lipschitz graphs. □

We provide another criterion of rectifiability, which is based on the following definition and lemma.

Definition 6.3.3 (Approximate tangent cone)**.** Let $E \subseteq \mathbb{R}^n$ be a Borel set; given $x \in E$, $V \in \mathrm{Gr}(n,d)$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$, we say that the cone $C(x,V,\alpha)$ is approximately tangent to E at x if the following holds:

$$\lim_{r\to 0} \frac{\mathcal{H}^d(E \cap B(x,r) \cap C(x,V,\alpha)^c)}{r^d} = 0.$$

Lemma 6.3.4. *Let $E \subseteq \mathbb{R}^n$ be a Borel set; let d be a positive integer. Fix $\alpha \in \left(0, \frac{\pi}{2}\right)$, $V \in Gr(n,d)$, $\delta > 0$ and $r_0 > 0$. Set*

$$\alpha' \coloneqq \frac{1}{2}\left(\frac{\pi}{2} + \alpha\right), \qquad \delta' \coloneqq \delta \left(\frac{\sin(\alpha' - \alpha)}{3}\right)^d.$$

Define E_{α,V,δ,r_0} to be the set of all $x \in E$ s.t.

- $\mathcal{H}^d(E \cap B(x,r)) \geq \delta r^d$ *for all* $r \leq r_0$*;*
- $\mathcal{H}^d(E \cap B(x,r) \cap C(x,V,\alpha)^c) \leq \delta' r^d$ *for all* $r \leq r_0$*.*

Assume that $F \subseteq E_{\alpha,V,\delta,r_0}$ and $diam(F) \leq \frac{r_0}{2}$. Then F is contained in the graph of a Lipschitz map $g : V \to V^\perp$ s.t. $Lip(g) \leq \tan(\alpha')$.

Proof. It suffices to show that for all $x, x' \in F$, then $x' \in C(x,V,\alpha')$. Then, we conclude as in the proof of lemma 6.3.1. Fix $x, x' \in F$ and assume by contradiction that $x' \notin C(x,V,\alpha')$. The geometric construction is resumed in figure 6.6. Denote

$$r \coloneqq 2\,|x' - x|\,.$$

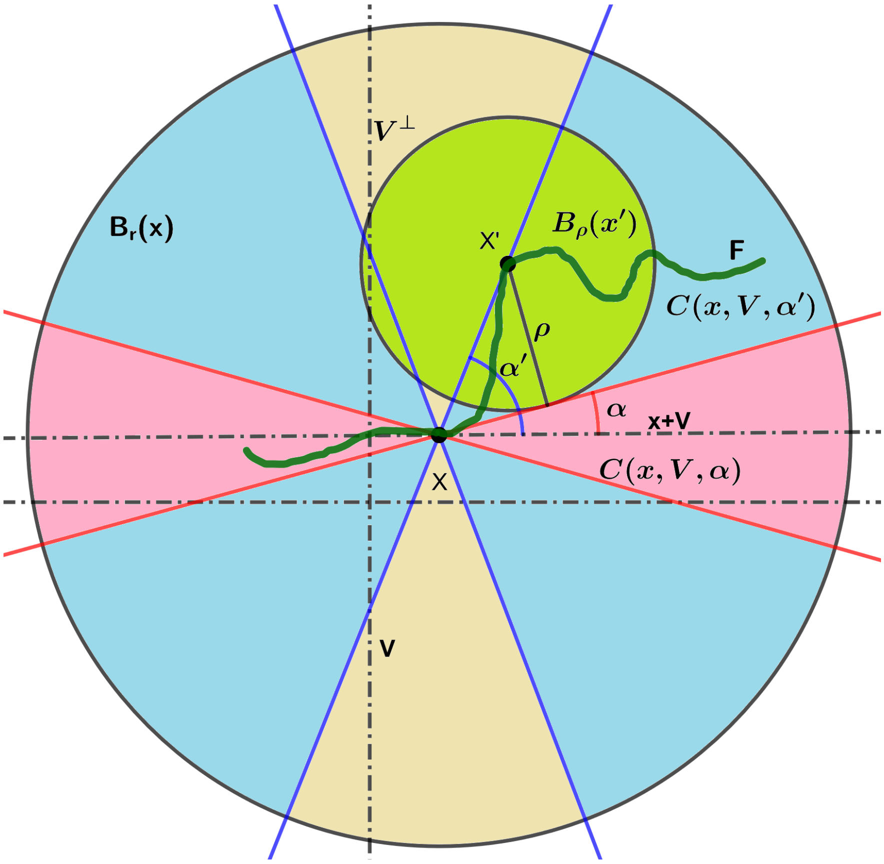

Figure 6.6: The geometric construction of lemma 6.3.4.

Set ρ to be the distance between x' and $C(x, V, \alpha)$, that is

$$\rho := |x' - x| \sin(\alpha' - \alpha) = \frac{r}{2} \sin(\alpha' - \alpha).$$

Since $\operatorname{diam}(F) \leq \frac{r_0}{2}$, then $r \leq r_0$; moreover, we have that $\rho < \frac{r}{2}$. Notice that

$$B(x', \rho) \subseteq B\left(x, \frac{r}{2} + \rho\right) \subseteq B(x, r), \quad B(x', \rho) \cap C(x, V, \alpha) = \emptyset.$$

To resume, there holds

$$B(x', \rho) \subseteq B(x, r) \setminus C(x, V, \alpha).$$

So, by the assumptions, since $r \leq r_0$ we have that

$$\begin{aligned}
\delta' r^d &\geq \mathcal{H}^d(E \cap B_r(x) \cap C(x, V, \alpha)^c) \\
&\geq \mathcal{H}^d(E \cap B_\rho(x')) \\
&\geq \delta \rho^d \\
&= \delta \left(\frac{\sin(\alpha' - \alpha)}{2}\right)^d r^d.
\end{aligned}$$

In conclusion, we have that

$$\delta \left(\frac{\sin(\alpha' - \alpha)}{3}\right)^d = \delta' \geq \delta \left(\frac{\sin(\alpha' - \alpha)}{2}\right)^d,$$

which is a contradiction. $\square$

Theorem 6.3.5. *Let $E \subseteq \mathbb{R}^n$ be a Borel set; let d be a positive integer. Assume that at every $x \in E$ there holds:*

- *E admits an approximately tangent cone $C(x, V(x), \alpha(x))$, where $V(x) \in Gr(n, d)$ and $\alpha(x) \in \left(0, \frac{\pi}{2}\right)$;*
- *$\Theta_{*d}(E, x) > 0$.*

Then E is contained in a countable union of d-dimensional Lipschitz graphs. In particular, if the assumptions holds for $\mathcal{H}^d$-a.e. $x \in E$, then E is d-rectifiable.

Proof. We write E as countable union of subsets that satisfies the assumptions of lemma 6.3.1. More precisely, for all $r_0 > 0$, let $\mathcal{F}_{r_0}$ be a countable family of cubes that cover $\mathbb{R}^n$ with diameter less than or equal to $\frac{r_0}{2}$. For all $m \in \mathbb{N}$, let $\mathcal{G}_m$ be a finite family of planes in $\mathrm{Gr}(n, d)$ s.t. for all $V' \in \mathrm{Gr}(n, d)$ there exists $V \in \mathcal{G}_m$ s.t. $V' \subseteq C\left(0, V, \frac{1}{m}\right)$. For all $V \in \mathrm{Gr}(n, d)$, for all $\alpha \in \left(0, \frac{\pi}{2}\right)$, for all $r, \delta > 0$, define E_{α,V,δ,r_0} to be the set of all $x \in E$ s.t.

- $\mathcal{H}^d(E \cap B(x, r)) \geq \delta r^d$ for all $r \leq r_0$;
- $\mathcal{H}^d(E \cap B(x, r) \cap C(x, V, \alpha)^c) \leq \delta' r^d$ for all $r \leq r_0$.

If we show that

$$E = \bigcup_{m\in\mathbb{N}} \bigcup_{m'\in\mathbb{N}} \bigcup_{V\in\mathcal{G}_m} \bigcup_{Q\in\mathcal{F}_{\frac{1}{m}}} E_{\frac{\pi}{2}-\frac{1}{m},V,\frac{1}{m},\frac{1}{m'}} \cap Q,$$

then we conclude immediately by lemma 6.3.4 and the fact that the union is countable.

Take $x \in E$; we show that there exist $m, m' \in \mathbb{N}$ and $V \in \mathcal{G}_m$ s.t.

$$x \in E_{\frac{\pi}{2}-\frac{1}{m},V,\frac{1}{m},\frac{1}{m'}} \cap Q_x,$$

where Q_x is a cube in $\mathcal{F}_{\frac{1}{m'}}$ that contains x. Choose $m \in \mathbb{N}$ s.t.

$$\Theta_{*d}(E, x) > \frac{1}{m}, \quad \alpha(x) \leq \frac{\pi}{2} - \frac{2}{m}.$$

In particular, there are $m' \in \mathbb{N}$ and $V \in \mathcal{G}_m$ s.t.

$$V(x) \subseteq C\left(0, V, \frac{1}{m}\right),$$

$$\mathcal{H}^d(E \cap B(x, r)) \geq \frac{r^d}{m} \quad \forall r \leq \frac{1}{m'}.$$

We deduce that

$$C(x, V(x), \alpha(x)) \subseteq C\left(x, V, \frac{\pi}{2} - \frac{1}{m}\right).$$

By definition of approximate tangent cone, we deduce that

$$\lim_{r\to 0} \frac{\mathcal{H}^d(E \cap B(x, r) \cap C\left(x, V, \frac{\pi}{2} - \frac{1}{m}\right)}{r^d} = 0.$$

In particular, up to replace m' with a larger one, we can assume that

$$\mathcal{H}^d\left(E \cap B(x, r) \cap C\left(x, V, \frac{\pi}{2} - \frac{1}{m}\right)^c\right) \leq \delta' r^d,$$

where δ' is defined as in lemma 6.3.4, by $\delta = \frac{1}{m}$. This concludes the proof. ☐

Remark 6.3.6. For all $n, d \in \mathbb{N}$ there exists a compact set $K \subseteq \mathbb{R}^n$ s.t. $\mathcal{H}^d(K) > 0$ but $\Theta_{*d}(K, x) = 0$ for $\mathcal{H}^d$-a.e. $x \in K$ and K is not d-rectifiable. This means that the assumption on the lower d-dimensional density in theorem 6.3.5 cannot be removed for free.

The following theorem holds true; we omit the proof, which is not hard.

Theorem 6.3.7. *Assume that $E \subseteq \mathbb{R}^n$ is Borel, $\mathcal{H}^d$ locally finite and for $\mathcal{H}^d$-a.e. $x \in E$ there exists $V(x) \in Gr(n, d)$ s.t. $C(x, V(x), \alpha)$ is an approximate tangent cone to E at x for all $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then, E is d-rectifiable.*

It is also possible to characterize d-rectifiable sets in terms of projections. It can be checked that, if E is d-rectifiable in $\mathbb{R}^n$, then $\mathcal{H}^d(\pi_V(E)) > 0$ for a.e. $V \in \mathrm{Gr}(n, d)$ with respect to the Haar measure on the Grassmannians (where π_V is the orthogonal projection to V). The implication can be reversed in some sense; however, it is very hard to prove.

Theorem 6.3.8 (Besicovitch-Federer)**.** *If $E \subseteq \mathbb{R}^n$ is $\mathcal{H}^d$ finite and d-purely unrectifiable, then $\mathcal{H}^d(\pi_V(E)) = 0$ for a.e. $V \in Gr(n, d)$.*

In the general setting of metric spaces, it is possible to characterize 1-dimensional sets in the following sense. We omit the proof.

Theorem 6.3.9. *Let $\mathbb{X}$ be a metric space; let $K \subseteq \mathbb{X}$ be compact, connected and s.t. $\mathcal{H}^1(K) < +\infty$. Then, K can be covered by a single path $\gamma : [0, 1] \to \mathbb{X}$ which is Lipschitz continuous. In particular, K is 1-rectifiable.*

Remark 6.3.10. In the framework of theorem 6.3.9, tangent lines are tangent to K in the classical sense, not only in the weak sense.

It is possible to characterize rectifiability in terms of density. We have already shown that d-rectifiable sets in $\mathbb{R}^n$ which are $\mathcal{H}^d$ locally finite have d-dimensional density 1 at $\mathcal{H}^d$-a.e. point in E (see 6.2.9). Conversely, the following statement holds true. It is very hard to prove.

Theorem 6.3.11 (Besicovitch, Marstrand-Mattila, Preiss)**.** *If E is $\mathcal{H}^d$ locally finite and $\Theta_d(E, x)$ exists, it is finite and positive for $\mathcal{H}^d$-a.e. $x \in E$, then d is a natural number and E is d-rectifiable.*

6.4 Rademacher theorem and area formula for rectifiable sets

In this final section, we only sketch some very useful and general tools.

Theorem 6.4.1 (Rademacher)**.** *Let $E \subseteq \mathbb{R}^m$ be a d-rectifiable set; let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function. Then, for $\mathcal{H}^d$-a.e. $x \in E$ f is tangentially differentiable at x, that is there exists a linear map $T : T_x^w E \to \mathbb{R}^n$ s.t.*

$$f(x + h) = f(x) + T(h) + o(|h|) \quad \forall h \in T_x^w E, \quad \text{as } |h| \to 0.$$

In particular, T depends only on the restriction of f to E and it is unique. It is denoted by $d^{Tan} f_x$.

Remark 6.4.2. In the framework of theorem 6.4.1, for $\mathcal{H}^d$-a.e. $x \in E$ we can define the tangential jacobian of f at x, that is

$$J_{\text{Tan}} f(x) \coloneqq \left|\det(d^{\text{Tan}} f_x)\right| .$$

Thus, the area formula holds true.

Theorem 6.4.3 (Area formula - 5). *Let $E \subseteq \mathbb{R}^m$ be a d-rectifiable set; let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function. Then, there holds*

$$\int_{f(E)} \#(f^{-1}(y) \cap E) \; d\mathcal{H}^d(y) = \int_E J_{Tan} f(x) \; d \; \mathcal{H}^d(x)$$

and the integrals are well defined.

Chapter 7

The Plateau's problem and the Theory of Currents

The aim of this chapter is to introduce the Plateau's problem, briefly describe some of the possible approaches to the solution and give an overview on the theory of Currents, which can be used to solve this problem. We will define currents to be the dual of differential forms; so, we start with a review of basic multilinear algebra. One of the most complete presentation of the theory of Currents can be found in [2].

7.1 The Plateau's problem

The Plateau's problem can be stated as follows: given a curve Γ in the $\mathbb{R}^3$, we want to find a surface Σ with minimal area that spans Γ. More precisely, we want to prove the existence of such a minimal surface. The same problem can be stated in general dimension and codimension.

The first difficulty is that we need to answer these questions.

- What is a "surface" Σ?
- What is the "area of a surface"?
- What is the meaning of "Σ spans Γ"?

The different answers give rise to many approaches toward the solutions of the Plateau's problem. For instance, if we restrict to regular object, the answers are clear in any dimension and codimension:

- a surface Σ is a d-dimensional submanifold in $\mathbb{R}^n$ of class C^1;
- the area of Σ is the d-dimensional volume, namely $\mathcal{H}^d(\Sigma)$;
- if the boundary of Σ is Γ (that is a $(d-1)$-submanifold), then we say that Σ spans Γ.

To prove the existence, the strategy relies on the so-called Direct Method: we need compactness and lower semicontinuity results. However, the class of regular objects is not the right framework to establish these results; then, the Direct Method does not apply and it is difficult to prove existence; furthermore, the surface with a given boundary that minimizes the area may be not regular. Moreover, there are also modelling reasons to consider more general objects.

We need to find

- a compactification of the collection of the surfaces,
- a lower semicontinuous extension of the area functional,
- an extension of the notion of boundary, which is stable.

These ingredients are enough to prove the existence of the solution of the Plateau's problem among these generalized objects. The theory of Currents provides these tools, giving a possible approach to this problem. Then, one would like to prove the regularity of the minimizers; there is a huge theory in this direction, which is very hard and not fully established. However, the one based on Currents is not the only possible approach to the Plateau's problem. Here, we briefly describe some of the others.

We remark that usually the term "minimal surface" means a surface with vanishing mean curvature, which is the Euler-Lagrange equation associated to the area functional. However, in the following, we will always refer to "minimal surface" as a minimizer of the area functional (in a given class of competitors).

7.1.1 A purely set-theoretic approach

We start describing a simplified version of the Plateau's problem, the so-called Steiner's problem. Given a finite set Γ in $\mathbb{R}^n$, we want to find a connected compact set Σ with minimal length (namely, $\mathcal{H}^1$ measure) that contains Γ. This problem can be solved via Direct Method. One consider the collection

$$\mathcal{X} := \{\Sigma \subseteq \mathbb{R}^n \mid \Gamma \subseteq \Sigma,\ \Sigma \text{ is connected and compact }\}.$$

We can endow the space $\mathcal{X}$ with the Hausdorff distance; we mention the following result.

Theorem 7.1.1 (Golab's theorem)**.** *$\mathcal{H}^1$ is lower semicontinuous on the space $\mathcal{X}$.*

By mean of the Golab's theorem, one can prove the existence of minimizers in this framework. We notice that in the Golab's theorem we need to deal with connected sets. In deed, for any $n \in \mathbb{N}$, one can consider the set $E_n \subseteq [0,1]$ defined as

$$E_n := \left\{ \frac{k}{n} \,\middle|\, k \in \{0, \dots, n\} \right\}.$$

It is immediate to check that the sequence $\{E_n\}_{n\in\mathbb{N}}$ converges to the segment $[0,1]$ with respect to the Hausdorff distance. However, $\mathcal{H}^1(E_n) = 0$ for any $n \in \mathbb{N}$ and $\mathcal{H}^1([0,1]) = 1$; so, the lower semicontinuity with respect to the Hausdorff distance fails. We also remark that the Golab's theorem has no equivalent in higher dimension; then, this approach cannot immediately extended to general dimension and codimension.

An attempt of extension of this purely set-theoretic approach can be the following. Given a curve Γ, we consider the collection of compact sets Σ that contain Γ and such that Γ can be retracted to a point in Σ. Under some reasonable assumptions, this class is compact with respect to the Hausdorff distance; however, the lower semicontinuity of the area functional fails. For instance, if Γ is a circle in $\mathbb{R}^3$, the minimizer of the area functional is the disk that spans Γ. However, by adding tiny tentacles to the disk, one can build a sequence of sets with "almost" the same area of the disk that converges in Hausdorff distance to wild closed set with arbitrarily large 2-dimensional area.

To conclude, we mention the fact that this purely set-theoretic approach was followed in 1960s by Reifenberg. Variants of this approach were proposed after 2000 by Harrison and De Lellis, Ghiraldin, Maggi.

7.1.2 A parametric approach

Fix a model surface D in $\mathbb{R}^d$, $\gamma : \partial D \to \mathbb{R}^n$ and define $\Gamma := \gamma(\partial D)$. Given a map $\Phi : D \to \mathbb{R}^n$, we denote by $\Sigma := \Phi(D)$. In this framework, we say that Σ spans Γ if the boundary condition $\Phi = \gamma$ on ∂D holds. If Φ is injective and regular, the area formula (see 5.2.14) yields that

$$\mathcal{H}^d(\Sigma) = \int_D J\Phi(s)\ ds;$$

if Φ is not regular, multiplicities have to be taken into account. However, one can try to minimize the integral functional

$$F(\Phi) := \int_D J\Phi(s)\ ds$$

on some Sobolev space, for instance $W^{1,2}_\gamma(D, \mathbb{R}^n)$ (the space of $W^{1,2}$ functions from D to $\mathbb{R}^n$ with trace γ on ∂D). We also mention the fact that the lagrangian of the functional F is policonvex; therefore, F is weakly lower semicontinuous on $W^{1,2}_\gamma(D, \mathbb{R}^n)$. However, since the functional F is invariant under reparameterization, then it lacks of coercivity. More precisely, by the change of variable formula, it is immediate to see that

$$F(\Phi \circ \sigma) = F(\Phi)$$

for any $\sigma : D \to D$ diffeomorphism. Assume that there is a minimizer. Then, one can reparameterize this map to construct a sequence of maps in $W^{1,2}_\gamma(D, \mathbb{R}^n)$ with the same energy that weakly converges to maps that has anything to do with the minimizer (for instance a constant map). In other words, the problem of this approach is that we cannot prove in general the compactness of minimizing sequences. However, in dimension 1 and 2 there are shortcuts.

The 1-dimensional case

In dimension 1, it is natural to assume that the reference domain D is the segment $[0, 1]$. Then, we have to minimize the functional

$$F(\Phi) = \int_0^1 |\Phi'(s)|\ ds$$

among the maps $\Phi \in W^{1,2}((0, 1), \mathbb{R}^n)$ with a prescribed boundary datum. Since this problem is trivial, one can add a constraint on the ambient space; for instance, we one can restrict to the Φ that take values in a given submanifold M in $\mathbb{R}^n$. However, since F lacks of coercivity, we introduce the functional

$$E(\Phi) := \int_0^1 |\Phi'(s)|^2\ ds$$

on the space $W^{1,2}_\gamma(D, M)$. The following result holds true; the proof is very easy.

Proposition 7.1.2. *Let Φ in $W^{1,2}_\gamma(D,M)$ be a minimizer for E; then Φ is also a minimizer for F in $W^{1,2}_\gamma(D,M)$ and $|\Phi'|$ is constant.*

Proof. Given $\varphi \in W^{1,2}(D,\mathbb{R}^n)$, there exists a reparameterization $\tilde{\varphi} := \varphi \circ \sigma$ with constant speed (namely $|\varphi'|$ is constant). Here, $\sigma : [0,1] \to [0,1]$ is a diffeomorphism of class C^1 that sends 0 in 0 and 1 in 1. Then, we have that

$$E(\varphi) = E(\tilde{\varphi}).$$

Moreover, by the Hölder's inequality, we deduce that

$$E(\varphi) = \int_0^1 |\varphi'(s)|^2 \ ds \geq \left(\int_0^1 |\varphi'(s)| \ ds \right)^2 = F(\Phi)^2,$$

with equality that holds if and only if $|\varphi'|$ is constant. Therefore, given Φ a minimizer for E and φ any other competitor, let $\tilde{\varphi}$ be the repapameterization of φ at constant speed. To be precise, all these maps are in the space $W^{1,2}_\gamma(D,M)$. Therefore, we have that

$$F(\varphi)^2 = F(\tilde{\varphi})^2 = E(\tilde{\varphi}) \geq E(\Phi) \geq F(\Phi)^2.$$

Hence, Φ is a minimizer for F in the desired space. Moreover, if we take $\varphi = \Phi$, we deduce that $E(\Phi) = F(\Phi)^2$; so, we infer that $|\Phi|'$ is constant. □

This result allows us to minimize the functional E, which has good lower semicontinuity and compactness property. In some sense, we are choosing a specific parameterization for each map.

The 2-dimensional case

In dimension 2, it is natural to assume that the reference domain D is the closed unit disk. If we restrict to maps from the disk to $\mathbb{R}^3$, a similar trick works. Indeed, given a map $\varphi \in W^{1,2}_\gamma(D,\mathbb{R}^3)$, we have that

$$F(\varphi) = \int_D J\Phi(s) \ ds = \int_D \left| \frac{\partial \varphi(s)}{\partial s_1} \times \frac{\partial \varphi(s)}{\partial s_2} \right| \ ds.$$

For such maps, we can define the functional

$$E(\varphi) := \int_D \frac{1}{2} |\nabla \varphi(s)|^2 \ ds.$$

The minimization of the functional E on the space $W^{1,2}_\gamma(D,\mathbb{R}^3)$ relies on the basic theory of Sobolev spaces; as before, we can show that the minimizers for E in the space $W^{1,2}_\gamma(D,\mathbb{R}^3)$ are also minimizer for F in the same space and some additional property holds.

Proposition 7.1.3. *Let Φ be a minimizer for E in the space $W^{1,2}_\gamma(D,\mathbb{R}^3)$. Then, Φ minimizes F in the same space and the differential of Φ is a conformal map at almost every point, namely the partial derivatives of Φ are orthogonal almost everywhere.*

Proof. Given any map $\varphi \in W^{1,2}_{\gamma}(D, \mathbb{R}^3)$, there holds that

$$\begin{aligned} E(\varphi) &= \int_D \frac{1}{2}\left(\left|\frac{\partial\varphi(s)}{\partial s_1}\right|^2 + \left|\frac{\partial\varphi(s)}{\partial s_2}\right|^2\right) ds \\ &\geq \int_D \left|\frac{\partial\varphi(s)}{\partial s_1}\right|\left|\frac{\partial\varphi(s)}{\partial s_2}\right| ds \\ &\geq \int_D \left|\frac{\partial\varphi(s)}{\partial s_1} \times \frac{\partial\varphi(s)}{\partial s_2}\right| ds \\ &= F(\Phi). \end{aligned}$$

Moreover, the equality holds if and only if the two partial derivatives are orthogonal at almost every $s \in D$.

By the Lichtenstein's theorem, we deduce that for any map $\varphi : D \to \mathbb{R}^n$ there exists a reparameterization $\tilde{\varphi} := \varphi \circ \sigma$ such that $\tilde{\varphi}$ is conformal; then, we obtain that $E(\tilde{\varphi}) = F(\tilde{\varphi})$. Here $\sigma : D \to D$ is not necessary the identity on ∂D, but it restricts to a bijection of ∂D. Then, the proof can be concluded as in the case of proposition 7.1.2. □

This parametric approach was developed by Douglas and Radó in the 1930s; it works only in dimension 2 because of lack of conformal parameterization in higher dimension.

7.1.3 A measure theoretic/distributional approach

This approach is based on the definition of generalized surfaces with good compactness and lower semicontinuity properties. This construction is based on the measure theory and it reminds the definition of Sobolev functions (or distributions). The first step in this direction was made by De Giorgi in late 1950s, with the introduction of Sets of Finite Perimeter. This notion generalizes that of oriented surface in $\mathbb{R}^n$ of codimension 1. In late 1960s, Federer and Fleming introduced the theory of Integral Currents: this notion generalizes that of oriented surface in $\mathbb{R}^n$ of arbitrary codimension. We will describe in further details this constructions and show the existence of the solution of the Plateau's problem in this framework.

7.1.4 Some regularity issues

As mentioned before, the regularity theory for minimal surfaces is a big issue. However, there are famous cases in which the Plateau's problem has no solution in the class of C^1 surfaces. We briefly describe some of these examples.

Example 7.1.4. In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, we consider the set

$$\Gamma := (S^1 \times \{0\}) \cup (\{0\} \times S^1).$$

Heuristically, the solution of the Plateau's problem is the set

$$\Sigma := (D^2 \times \{0\}) \cup (\{0\} \times D^2),$$

where D^2 is the unit closed disk in $\mathbb{R}^2$. More precisely, one can show the following statement: if $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a minimizing sequence of oriented surfaces of class C^1 in $\mathbb{R}^4$, namely $\partial\Sigma_n = \Gamma$ for any $n \in \mathbb{N}$ and

$$\mathcal{H}^2(\Sigma_n) \to \inf\{\mathcal{H}^2(\tilde{\Sigma}) \mid \tilde{\Sigma} \text{ is a } C^1 \text{ oriented surface and } \partial\tilde{\Sigma} = \Gamma\},$$

then $\Sigma_n \to \Sigma$ in Hausdorff distance (up to subsequences). However, Σ is not regular at the point $(0,0) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Example 7.1.5. In $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$, we consider the curve

$$\Gamma := \{(w^3, w^2) \mid w \in S^1 \in \mathbb{C}\}.$$

Notice that Γ is a smooth curve with no self-intersections. Not surprisingly, the solution of the Plateau's problem with boundary datum Γ is given by

$$\Sigma := \{(w^3, w^2) \mid w \in D^2 \in \mathbb{C}\},$$

where D^2 is the closed unit disk in $\mathbb{C}$. As before, one can check that any minimizing sequence of oriented surfaces of class C^1 with boundary Γ has a subsequence that converges to Σ with respect to the Hausdorff distance. Since Σ is not regular at the origin, we deduce that the Plateau's problem has no solution in the class of C^1 surfaces.

Example 7.1.6. As in the example 7.1.5, we can consider $\mathbb{R}^{2m} = \mathbb{C}^m$; then, we take a regular domain D in C^k, an open neighbourhood U of D and an holomorphic map $f : U \to C^m$. We define

$$\Gamma := \{f(s) \mid s \in \partial D\}.$$

If f is injective on ∂D and $\nabla f(s)$ has complex rank k at every $s \in \partial D$, then Γ is a regular surface of dimension $2k-1$ in $\mathbb{R}^{2m} = \mathbb{C}^m$. By means of Kälher's forms and Wirtinger's inequality, it is possible to prove that the solution of the Plateau's problem (in the same sense explained in 7.1.4) is given by the surface

$$\Sigma := \{f(s) \mid s \in D\}.$$

However, Σ is not singular at all points $f(s)$ such that the complex rank of $\nabla f(s)$ strictly less that k.

Example 7.1.7. In $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ define the set

$$\Gamma := \{(x, y) \mid |x| = |y| = 1\} = S^{m-1} \times S^{m-1}.$$

Notice that Γ is an analytic surface of dimension $2m-2$ in $\mathbb{R}^m$. If $m \geq 4$, one can prove that the solution of the Plateau's problem is given by

$$\Sigma := \{(x, y) \mid |x| = |y| \leq 1\},$$

with the same meaning of 7.1.4. This fact was proved by Bombieri, De Giorgi, Giusti; a simpler and more recent proof is due to De Phlippis, Paolini. The set Σ is singular at $(0,0)$ at is known as Simon's cone.

As for positive results of regularity, the followings hold true:

- in codimension 1 the singular set of the solution of the Plateau's problem (with any regular boundary datum) has codimension at most 7 in the surface (proved by several authors in the 1960-1970s);
- if the ambient space is $\mathbb{R}^n$ and the codimension is between 2 and $n-2$ (included), then the singular set has codimension at most 2 in the surface (proved by several authors in 1970-2010s).

7.2 Basics of multilinear algebra

7.2.1 The space of k-covectors

In the following, let $\mathbb{V}$ be a real vector space and $\mathbb{V}^*$ its dual. We also denote S_n the group of the permutation of $\{1, \dots, n\}$; given $\sigma \in S_n$, we denote $\mathrm{sgn}(\sigma)$ the sign of the permutation σ.

Definition 7.2.1 (k-covector)**.** A k-linear alternating form (or k-covector) on $\mathbb{V}$ is a multilinear function $\alpha : \mathbb{V}^k \to \mathbb{R}$ with the following property: for all $v_1, \dots, v_k \in \mathbb{V}$ for all permutation σ, there holds

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\alpha(v_1, \dots, v_k).$$

The space of k-covector is denoted by $\Lambda^k(\mathbb{V})$. If $k = 0$, we set $\Lambda^0(\mathbb{V}) := \mathbb{R}$ (viewed as constant functions).

Remark 7.2.2. There follow a list of immediate remarks.

- By definition, $\Lambda^1(\mathbb{V}) = \mathbb{V}^*$.
- $\Lambda^k(\mathbb{V})$ is a linear space.
- In definition 7.2.1, it is equivalent to say that α change sign when swapping v_i and v_j for $i \neq j$.
- Given $\alpha \in \Lambda^k(\mathbb{V})$, if $v_1, \dots, v_k$ are linearly dependent, then $\alpha(v_1, \dots, v_k) = 0$.
- If $k > \dim(\mathbb{V})$, then $\Lambda^k(\mathbb{V}) = \{0\}$.
- If $k = \dim(\mathbb{V})$, then $\dim(\Lambda^k(\mathbb{V})) = 1$; in particular, $\Lambda^k(\mathbb{V})$ is the line spanned by the covector defined by the determinant of a $k \times k$ matrix.

Definition 7.2.3 (Exterior product)**.** Let $\alpha \in \Lambda^h(\mathbb{V})$ and $\beta \in \Lambda^k(\mathbb{V})$ be covectors. We define $\alpha \wedge \beta$ to be the element of Λ^{h+k} given by

$$\alpha \wedge \beta(v_1, \dots, v_{h+k}) = \sum_{\sigma \in S_{h+k}} \mathrm{sgn}(\sigma)\alpha(v_{\sigma(1)}, \dots, v_{\sigma(h)})\beta(v_{\sigma(h+1)}, \dots, v_{\sigma(h+k)})).$$

If $h = 0$, then $\alpha \in \mathbb{R}$ and we define

$$\alpha \wedge \beta = \alpha \cdot \beta \in \Lambda^k(\mathbb{V}).$$

Remark 7.2.4. $\wedge$ is linear in each factor, associative and anticommutative, that is, if $\alpha \in \Lambda^h(\mathbb{V})$ and $\beta \in \Lambda^k(\mathbb{V})$, then

$$\alpha \wedge \beta = (-1)^{hk}\beta \wedge \alpha.$$

In particular, if h is odd, then $\alpha \wedge \alpha = 0$.

From now on, suppose that $\mathbb{V}$ is a finite dimensional vector space. Set $n = \dim(\mathbb{V})$; let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{V}$ and let $\{e_1^*, \dots, e_n^*\}$ be the corresponding dual basis of $\mathbb{V}^*$; more explicitly, we have that $e_i^*(e_j) = \delta_{i,j}$. In other words, e_i^* is the linear functional that compute the i-th coordinate of a vector v with respect to the basis $\{e_1, \dots, e_n\}$. We introduce the following notation for multi-indices:

$$\underline{i} := (i_1, \dots, i_k) \in \{1, \dots, n\}^k;$$

$$I_{n,k} := \{\underline{i} \mid 1 \leq i_1 < \dots < i_k \leq n\}.$$

For all $\underline{i} \in I_{n,k}$, we set

$$e^*_{\underline{i}} := e^*_{i_1} \wedge \dots \wedge e^*_{i_k}.$$

We denote

$$e_{\underline{i}} := (e_{i_1}, \dots, e_{i_k}) \in \mathbb{V}^k;$$

given $\underline{i}, \underline{j} \in I_{n,k}$ we set

$$\delta_{\underline{i},\underline{j}} := \prod_{h=1}^{k} \delta_{i_h, j_h}.$$

Given M an $n \times k$ matrix, we denote as $M_{\underline{i}}$ the $k \times k$ minor given by the rows $i_1, \dots, i_k$.

We want to show that every k-covector can be written as linear combination of $\{e_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$. The proof of this fact relies on the following lemmas.

Lemma 7.2.5. *Take $v_1, \dots, v_k \in \mathbb{V}$; let M be the $n \times k$ matrix whose j-th column is given by the coordinates of v_j with respect to the basis $\{e_1, \dots, e_n\}$. Then, for all $\underline{i} \in I_{n,k}$ there holds that*

$$e^*_{\underline{i}}(v_1, \dots, v_k) = \det(M_{\underline{i}}).$$

In particular, for all $\underline{j} \in I_{n,k}$ there holds

$$e^*_{\underline{i}}(e_{\underline{j}}) = \delta_{\underline{i},\underline{j}}.$$

Proof. The proof can be fixed by induction on k and it is based on the expansion of the determinant. □

Lemma 7.2.6. *Let α be a k-covector s.t. $\alpha(e^*_{\underline{i}}) = 0$ for all $\underline{i} \in I_{n,k}$; then $\alpha = 0$.*

Proof. The proof is an immediate consequence of the fact that α is multilinear and every $v \in \mathbb{V}^k$ can be written as linear combination of $\{e_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$. □

We are now in the position of proving the following result.

Theorem 7.2.7. *Let $0 < k \leq n$ be a natural number. The set $\{e^*_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda^k(\mathbb{V})$. In particular, for all $\alpha \in \Lambda^k(\mathbb{V})$ there holds*

$$\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha_{\underline{i}} e^*_{\underline{i}},$$

where $\alpha_{\underline{i}} = \alpha(e_{\underline{i}})$. Moreover, we have that

$$\dim(\Lambda^k(\mathbb{V})) = \#I_{n,k} = \binom{n}{k}.$$

Proof. **Step 1**: $\{e^*_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ is a linearly independent set of k-covectors. In deed, take a vanishing linear combination

$$\sum_{\underline{i} \in I_{n,k}} \alpha_{\underline{i}} e^*_{\underline{i}} = 0.$$

Evaluating at $e_{\underline{j}}$, where $\underline{j} \in I_{n,k}$, and applying lemma 7.2.5, we obtain that

$$\sum_{\underline{i} \in I_{n,k}} \alpha_{\underline{i}} \delta_{\underline{i},\underline{j}} = 0,$$

that is $\alpha_{\underline{j}} = 0$.

Step 2: We claim that for all $\alpha \in \Lambda^k(\mathbb{V})$ there holds

$$\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha(e_{\underline{i}}) e^*_{\underline{i}}.$$

Consider

$$\beta \coloneqq \alpha - \sum_{\underline{i} \in I_{n,k}} \alpha(e_{\underline{i}}) e^*_{\underline{i}}.$$

Notice that $\beta \in \Lambda^k(\mathbb{V})$ and $\beta(e_{\underline{j}}) = 0$ for all $\underline{j} \in I_{n,k}$. We apply lemma 7.2.6 to conclude that $\beta = 0$. □

The special case of $\mathbb{R}^n$

If $\mathbb{V} = \mathbb{R}^n$, let $\{e_1, \ldots, e_n\}$ be the canonical basis. We denote as $\{dx_1, \ldots, dx_n\}$ the dual basis, which agrees with the notation for differential in the following sense. We call $(x_1, \ldots, x_n)$ the coordinate functions; notice that the differential of x_i agrees with e^*_i as linear maps on $\mathbb{R}^n$; so, the notation $e^*_i = dx_i$ is consistent.

Theorem 7.2.8 (Cauchy-Binet formula)**.** *Given A, B $n \times k$ matrix, with $1 \leq k \leq n$, then*

$$\det(A^T \cdot B) = \sum_{\underline{i} \in I_{n,k}} \det(A_{\underline{i}}) \cdot \det(B_{\underline{i}}).$$

In particular, we have

$$\det(A^T \cdot A) = \sum_{\underline{i} \in I_{n,k}} (\det(A_{\underline{i}}))^2.$$

Proof. Fix A an $n \times k$ matrix; given $v_1, \ldots, v_k \in \mathbb{R}^n$, we define

$$\alpha(v_1, \ldots, v_k) \coloneqq \det(A^T \cdot V),$$

where V is the $n \times k$ matrix whose j-th column is the vector v_j. It is easy to check that α is multilinear and alternating; hence, by theorem 7.2.7, we can write

$$\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha(e_{\underline{i}}) e^*_{\underline{i}}.$$

For every $\underline{i} \in I_{n,k}$, set $E_{\underline{i}}$ to be the $n \times k$ matrix whose j-th column is e_{i_j}. Given V an $n \times k$ matrix, let v_j be the j-column of V. Then, by the computation above and lemma

7.2.5, we deduce that

$$\begin{aligned}\det(A^T \cdot V) &= \alpha(v_1, \dots, v_k)\\ &\sum_{\underline{i} \in I_{n,k}} \alpha(e_{\underline{i}}) e^*_{\underline{i}}(v_1, \dots, v_k)\\ &= \sum_{\underline{i} \in I_{n,k}} \det(A^t \cdot E_{\underline{i}}) \det(V_{\underline{i}})\\ &= \sum_{\underline{i} \in I_{n,k}} \det(A_{\underline{i}}) \det(V_{\underline{i}}).\end{aligned}$$

Then, the proof is concluded. □

7.2.2 Simple k-vectors

Definition 7.2.9 (Simple k-vector)**.** Let k be a positive integer. Given $(v_1, \dots, v_k)$, $(\tilde{v}_1, \dots, \tilde{v}_k) \in \mathbb{V}^k$, we say that they are equivalent if

$$\alpha(v_1, \dots, v_k) = \alpha(\tilde{v}_1, \dots, \tilde{v}_k) \quad \forall \alpha \in \Lambda^k(\mathbb{V})$$

and we write $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k)$. Since $\sim$ is an equivalence relation on $\mathbb{V}^k$, we define the quotient set to be the collection of simple k-vectors. We use the notation $[v_1, \dots, v_k]$ for equivalence classes (later, those will be denoted as $v_1 \wedge \cdots \wedge v_k$).

Remark 7.2.10. If $\mathbb{V}$ is endowed with a topology, then $\Lambda^k(\mathbb{V})$ inherits the quotient topology. However, $\Lambda^k(\mathbb{V})$ is not a vector space (in general); this is the reason to introduce general k-vectors.

The following is the fundamental result about simple k-vectors. It has an interesting geometric meaning.

Proposition 7.2.11. *1. $(v_1, \dots, v_k) \sim (0, \dots, 0)$ if and only if $v_1, \dots, v_k$ are linearly dependent.*

2. If $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k) \nsim (0, \dots, 0)$, then

$$\mathrm{Span}(v_1, \dots, v_k) = \mathrm{Span}(\tilde{v}_1, \dots, \tilde{v}_k)$$

and the change of basis matrix M has determinant 1 (recall that M is s.t. $\tilde{v}_i = \sum_j M_{i,j} v_j$ for all i).

3. Conversely, if $(v_1, \dots, v_k)$ and $(\tilde{v}_1, \dots, \tilde{v}_k)$ span the same subspace W and the change of basis matrix M has determinant 1, then $(v_1, \dots, v_k)$ is equivalent to $(\tilde{v}_1, \dots, \tilde{v}_k)$.

Proof. **Step 1**: If $v_1, \dots, v_k$ are linearly dependent, for all $\alpha \in \Lambda^k(\mathbb{V})$ there holds

$$\alpha(v_1, \dots, v_k) = 0;$$

hence, $(v_1, \dots, v_k) \sim (0, \dots, 0)$. Conversely, assume that $v_1, \dots, v_k$ are linearly independent; we claim that there exists $\overline{\alpha} \in \Lambda^k(\mathbb{V})$ s.t. $\overline{\alpha}(v_1, \dots, v_k) \neq 0$. This is enough to conclude that $(v_1, \dots, v_k) \nsim (0, \dots, 0)$. In deed, complete $\{v_1, \dots, v_k\}$ to a basis $\{v_1, \dots, v_n\}$ of $\mathbb{V}$; let $\{v_1^*, \dots, v_n^*\}$ be the dual basis. Set

$$\overline{\alpha} \coloneqq v_1^* \wedge \dots \wedge v_k^*.$$

If we apply lemma 7.2.5, we obtain that

$$\overline{\alpha}(v_1, \dots, v_k) = \det(Id) = 1.$$

Step 2: Take $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k) \nsim 0$ and assume by contradiction that $v_1, \dots, v_k$ and $\tilde{v}_1, \dots, \tilde{v}_k$ do not span the same space. Then, in the construction of $v_{k+1}, \dots, v_n$ above, we can assume that $v_{k+1} = \tilde{v}_{j_0}$ for some $j_0 \in \{1, \dots, k\}$ ($\tilde{v}_1, \dots, \tilde{v}_k$ are linearly independent; in particular, $\tilde{v}_i \neq 0$ for all i); for simplicity, suppose that $j_0 = 1$. Let $\overline{\alpha}$ be the k-covector defined above. Let Q be the $n \times k$ matrix whose j-th column is given by the coefficients of $\tilde{v}_1$ with respect to the basis $\{v_1, \dots, v_n\}$. By lemma 7.2.5, there holds

$$\overline{\alpha}(\tilde{v}_1, \dots, \tilde{v}_k) = \det(Q_{(1,\dots,k)}) = 0,$$

since the first column of M is $(0, \dots, 0)$. Recall that $\overline{\alpha}(v_1, \dots, v_k) = 1 \neq 0$; so, we find a contradiction.

At this point, let M be the change of basis matrix from $(v_1, \dots, v_k)$ to $(\tilde{v}_1, \dots, \tilde{v}_k)$ (it makes sense, because they space the same subspace). Let $\overline{\alpha}$ be as above; if we complete $\{v_1, \dots, v_k\}$ to a basis $\{v_1, \dots, v_n\}$ of $\mathbb{V}$ and denote as Q the $n \times k$ matrix whose j-column has the component of $\tilde{v}_j$ with respect to the basis $\{v_1, \dots, v_n\}$, we conclude that Q is of the following type:

$$Q = \left(\begin{array}{c} M \\ \hline 0 \end{array} \right).$$

By lemma 7.2.5, we have that

$$1 = \alpha(v_1, \dots, v_k) = \overline{\alpha}(\tilde{v}_1, \dots, \tilde{v}_k) = \det(Q_{(1,\dots,k)}) = \det(M).$$

Step 3: Conversely, assume that $(v_1, \dots, v_k)$ and $(\tilde{v}_1, \dots, \tilde{v}_k)$ span the same subspace W and the change of basis matrix M has determinant 1. Let $\overline{\alpha} \in \Lambda^k(\mathbb{V})$ be as above. Then, by lemma 7.2.5, we have

$$\overline{\alpha}(v_1, \dots, v_k) = 1 = \det(M) = \overline{\alpha}(\tilde{v}_1, \dots, \tilde{v}_k).$$

Pick $\beta \in \Lambda^k(\mathbb{V})$; notice that the restriction of β to W^k is a multiple to the restriction of α to W^k (in deed, $\Lambda^k(W)$ is a linear space of dimension 1). It follows that $\beta(v_1, \dots, v_k) = \beta(\tilde{v}_1, \dots, \tilde{v}_k)$. □

Orientation of a vector space vs simple k-vectors

From now on, we assume that $\mathbb{V}$ is endowed with a scalar product, so that we can define the Hausdorff measure.

Definition 7.2.12. Given $v_1, \dots, v_k \in \mathbb{V}$, let $R(v_1, \dots, v_k)$ be the rectangle spanned by $v_1, \dots, v_k$, that is

$$R(v_1, \dots, v_k) \coloneqq \left\{ \sum_{j=1}^k \lambda_j v_j \;\middle|\; \lambda_j \in [0,1] \right\}.$$

Remark 7.2.13. Take $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k) \nsim (0, \dots, 0)$; set

$$W \coloneqq \mathrm{Span}(v_1, \dots, v_k) = \mathrm{Span}(\tilde{v}_1, \dots, \tilde{v}_k)$$

and $T : W \to W$ the linear map s.t. $T(v_j) = T(\tilde{v}_j)$ for all j. Notice that the matrix associated to T with respect to the basis $\{v_1, \dots, v_k\}$ is the usual change of basis matrix between $\{v_1, \dots, v_k\}$ and $\{\tilde{v}_1, \dots, \tilde{v}_k\}$, that we denote as M; we have shown in 7.2.11 that $\det(M) = 1$. Then, by the area formula stated in 5.2.14, which can be applied because W endowed with the scalar product is isometric to $\mathbb{R}^k$ with the euclidean scalar product and then the Hausdorff measure is preserved, there holds

$$\begin{aligned}\mathcal{H}^k(R(\tilde{v}_1, \dots, \tilde{v}_k)) &= \mathcal{H}^k(T(R(v_1, \dots, v_k))) \\ &= |\det(M)|\, \mathcal{H}^k(R(v_1, \dots, v_k)) \\ &= JT \cdot \mathcal{H}^k(R(v_1, \dots, v_k)) \\ &= \mathcal{H}^k(R(v_1, \dots, v_k)).\end{aligned}$$

Definition 7.2.14 (Norm of k-vector)**.** We define the norm of a k-vector $[v_1, \dots, v_k]$ as

$$|[v_1, \dots, v_k]| \coloneqq \mathcal{H}^k(R(v_1, \dots, v_k)),$$

where the definition is well posed as explained in 7.2.13.

The reason to call the quantity defined in 7.2.14 "norm" will be clear later.

Definition 7.2.15 (Orientation of a vector space)**.** An orientation of $\mathbb{V}$ is an equivalence class of basis with respect to the equivalence relation $\approx$ s.t. $(v_1, \dots, v_n) \approx (\tilde{v}_1, \dots, \tilde{v}_n)$ if the change of basis matrix has positive determinant. We say that a basis of $\mathbb{V}$ is oriented (or positive) if it belongs to the chosen class of equivalence.

Remark 7.2.16. Notice that $\sim$ is finer that $\approx$ (see 7.2.9 and 7.2.15).

Definition 7.2.17 (Oriented Grassmannian)**.** We denote as $\mathrm{Gr}_{or}(\mathbb{V}, k)$ the Grassmanniann of the k-dimensional oriented subspace of $\mathbb{V}$.

Proposition 7.2.18. *Consider the map Ψ which associates to a simple unitary k-vector $[v_1, \dots, v_k]$ the k-dimensional subspace* $\mathrm{Span}(v_1, \dots, v_k)$ *oriented by $(v_1, \dots, v_k)$. This map is well defined and it is a bijection.*

Proof. The map Ψ is well posed, because of proposition 7.2.11, namely it is independent of the representative chosen in a class of equivalence.

We show that Ψ is surjective. Pick $W \in \mathrm{Gr}_{or}(\mathbb{V}, k)$ and an oriented basis of W; up to normalization, we can suppose that $\mathcal{H}^k(R(v_1, \dots, v_k)) = 1$ (notice that, given $\lambda > 0$, then $\{\lambda v_1, v_2, \dots, v_n\}$ is a basis of W equivalent to $\{v_1, \dots, v_n\}$). Hence, W is the image of $[v_1, \dots, v_k]$ through the map Ψ.

We prove that Ψ is injective. Assume that

$$W \coloneqq \mathrm{Span}(v_1, \dots, v_k) = \mathrm{Span}(\tilde{v}_1, \dots, \tilde{v}_k) \coloneqq \tilde{W}$$

and the change of basis matrix M has positive determinant, that is $W = \tilde{W}$ as oriented subspaces. Assume also that $[v_1, \dots, v_k]$ and $[\tilde{v}_1, \dots, \tilde{v}_k]$ are unitary. If we show that $\det(M) = 1$, we conclude that $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k)$. As explained in 7.2.13, we can compute

$$1 = \mathcal{H}^k(R(\tilde{v}_1, \dots, \tilde{v}_k)) = |\det(M)|\, \mathcal{H}^k(R(v_1, \dots, v_k)) = \det(M).$$

□

7.2.3 Differential forms on open sets of $\mathbb{R}^n$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

Definition 7.2.19 (k-form)**.** A k-form ω on Ω is a section of the k-alternating tensor bundle of Ω. We set $\Lambda^k(\Omega)$ to be the set of k-forms in Ω.

Since we are in $\mathbb{R}^n$, the definition 7.2.19 (which holds for abstract manifolds) can be restated in a much more concrete way. We will always use the following facts.

Remark 7.2.20. A k-form in $\mathbb{R}^n$ is nothing but a map $\omega : \Omega \to \Lambda^k(\mathbb{R}^n)$; in other words, $\omega(x)$ is a k-alternating multilinear map from $\mathbb{R}^n$ to $\mathbb{R}$ for all $x \in \Omega$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and let $\{dx_1, \dots, dx_n\}$ be the dual basis. For all $\underline{i} \in I_{n,k}$, we denote

$$dx_{\underline{i}} := dx_{i_1} \wedge \cdots \wedge dx_{i_k};$$

we have shown in 7.2.7 that $\{dx_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda^k(\mathbb{R}^n)$. Hence, we can write ω in coordinates as

$$\omega = \sum_{\underline{i} \in I_{n,k}} \omega_{\underline{i}} dx_{\underline{i}},$$

where $\omega_{\underline{i}} : \Omega \to \mathbb{R}$ is the function s.t. $\omega_{\underline{i}}(x)$ denotes the component of $\omega(x)$ with respect to $dx_{\underline{i}}$. In 7.2.7 we have also shown that

$$\omega_{\underline{i}}(x) = \omega(x)(e_{\underline{i}}).$$

We say that Ω is C^h-regular if all the coefficients $\omega_{\underline{i}}$ are of class C^h. We define the support of ω as the closure of the set in which ω vanishes.

Definition 7.2.21 (Exterior derivative)**.** Let ω be a k-form in Ω of class C^1. For all $j \in \{1, \dots, n\}$ we set

$$\frac{\partial \omega}{\partial x_j}(x) := \sum_{\underline{i} \in I_{n,k}} \frac{\partial \omega_{\underline{i}}}{\partial x_j}(x)\ dx_{\underline{i}}.$$

Then, we define the exterior derivative of ω as the $(k+1)$-form given by

$$d\omega(x) := \sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}(x).$$

Remark 7.2.22. Given $\omega \in \Lambda^k(\Omega)$, we can write more explicitly the exterior derivative $d\omega$. Recall that $\omega_{\underline{i}}$ is a C^1 function on Ω, which can be interpreted as a 0-form in Ω. Definition 7.2.21 makes sense also for 0-forms; thus, we have that

$$d\omega_{\underline{i}}(x) = \sum_{j=1}^{n} \frac{\partial \omega_{\underline{i}}}{\partial x_j}(x)\ dx_j;$$

in other words, $d\omega_{\underline{i}}(x)$ is the differential of $\omega_{\underline{i}}$ at x (they agree as linear maps from $\mathbb{R}^n$ to $\mathbb{R}$). Hence, we have

$$\begin{aligned} d\omega(x) &= \sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}(x) \\ &= \sum_{j=1}^{n} \sum_{\underline{i} \in I_{n,k}} \frac{\partial \omega_{\underline{i}}(x)}{\partial x_j}(x)\ dx_j \wedge dx_{\underline{i}} \\ &= \sum_{j=1}^{n} d\omega_{\underline{i}}(x) \wedge dx_{\underline{i}}. \end{aligned}$$

Remark 7.2.23. We remark that the exterior derivative can be also defined in an intrinsic way; this definition would work for abstract manifold. However, this is not relevant in this context.

Proposition 7.2.24. *Let $k \geq 2$ and ω a k-form with coefficient of class C^2. Then $d^2\omega = 0$.*

Proof. By definition of 7.2.21 and the fact that $\wedge$ is anticommutative, we deduce that

$$\begin{aligned} d^2\omega &= d\left(\sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}\right) \\ &= \sum_{1\leq l<j\leq n} dx_l \wedge dx_j \wedge \frac{\partial}{\partial x_l}\left(\frac{\partial \omega}{\partial x_j}\right) + dx_j \wedge dx_l \wedge \frac{\partial}{\partial x_j}\left(\frac{\partial \omega}{\partial x_l}\right) \\ &= 0, \end{aligned}$$

since $dx_j \wedge dx_l = -dx_l \wedge dx_j$ and $\frac{\partial}{\partial x_j}\left(\frac{\partial \omega}{\partial x_l}\right) = \frac{\partial}{\partial x_l}\left(\frac{\partial \omega}{\partial x_j}\right)$. □

Orientation of a surface

We introduce a notion of orientation of surface, which can be proved to be equivalent to the classical one (choice of an atlas whose transition maps have differential that preserves the orientation or $\mathbb{R}^d$). Let Σ be a k-dimensional submanifold in $\mathbb{R}^n$ (eventually with boundary) of class C^1.

Remark 7.2.25. Being the quotient space of a vector space, the set of the simple k-vector is naturally endowed with a topology.

Definition 7.2.26 (Orientation of a surface)**.** An orientation of Σ is a continuous map τ defined on Σ with valued in the set of simple k-vectors (see 7.2.25) s.t. $\tau(x)$ is unitary and it spans $\mathrm{Tan}_x\Sigma$ for all $x \in \Sigma$.

The boundary of Σ is a $(k-1)$-submanifold; in Σ is oriented, then $\partial\Sigma$ inherits an orientation from that defined on Σ. In can be checked that the definition below agree with the classical one (outward first).

Definition 7.2.27 (Orientation of the boundary)**.** If Σ is oriented by τ, we orient $\partial\Sigma$ by τ' s.t. for all $x \in \partial\Sigma$ there holds

$$[\eta(x), \tau_1'(x), \ldots, \tau_{k-1}'(x)] = [\tau_1(x), \ldots, \tau_k(x)],$$

where η is the outward unit normal.

Integration of k-forms

Let Σ be an oriented k-submanifold (eventually with boundary) of class C^1 in $\mathbb{R}^n$.

Definition 7.2.28 (Integration of a k-form)**.** Assume that Σ is oriented by τ (see 7.2.26). Let ω be a k-form defined on $\mathbb{R}^n$ with continuous coefficients. Assume that either Σ is compact or ω has compact support and Σ is closed. We define

$$\int_\Sigma \omega := \int_\Sigma <\omega(x), \tau(x)> \; d\mathcal{H}^k(x),$$

where $< \omega(x), \tau(x) >$ denotes the action of $\omega(x) \in \Lambda^k(\mathbb{R}^n)$ on $\tau(x)$, which is a simple k-vector. In other words, $< \omega(x), \tau(x) >$ is the value of $\omega(x)$ applied to $\tau(x)$ (it is well defined, namely it is independent on the representative of $\tau(x)$).

Remark 7.2.29. In the framework of 7.2.28, notice that the integral is well defined. Moreover, let $D \subseteq \mathbb{R}^k$ be an open set and $\Phi : D \to \Sigma$ a parameterization of Σ s.t.

$$\left[\frac{\partial \Phi}{\partial s_1}(s), \dots, \frac{\partial \Phi}{\partial s_k}(s)\right] = [\tau_1(\Phi(s)), \dots, \tau_k(\Phi(x))] \quad \forall s \in D.$$

By proposition 7.2.18, it follows that

$$1 = \mathcal{H}^k\left(R\left(\frac{\partial \Phi}{\partial s_1}(s), \dots, \frac{\partial \Phi}{\partial s_k}(s)\right)\right) = \det(J\Phi(s)).$$

By the area formula (see 5.2.14), we obtain

$$\begin{aligned}\int_\Sigma \omega &= \int_\Sigma < \omega(x), \tau(x) > \ d\mathcal{H}^k(x)\\ &= \int_D < \omega(\Phi(s)), \frac{\partial \Phi}{\partial s_1}(s), \dots, \frac{\partial \Phi}{\partial s_k}(s) > |J\Phi(s)| \ ds\\ &= \int_D < \omega(\Phi(s)), \frac{\partial \Phi}{\partial s_1}(s), \dots, \frac{\partial \Phi}{\partial s_k}(s) > \ ds.\end{aligned}$$

One of the main reason to integrate k-forms is the Stokes' theorem, which can be interpreted as an integration by parts formula.

Theorem 7.2.30 (Stokes)**.** *Let Σ be a compact, oriented, k-dimensional submanifold of $\mathbb{R}^n$ of class C^1; let ω be a $(k-1)$-form of class C^1 in $\mathbb{R}^n$. Then*

$$\int_{\partial\Sigma} \omega = \int_\Sigma d\omega.$$

The same holds true if Σ is oriented, closed and ω is C^1 with compact support.

7.2.4 The space of general k-vectors

Given a finite-dimensional vector space $\mathbb{V}$, we construct the vector space of general k-vectors on $\mathbb{V}$, whose dual is $\Lambda^k(\mathbb{V})$, so that simple k-vectors are naturally embedded in this space. We always rely on the canonical identification $\mathbb{V}^{**} = \mathbb{V}$.

Definition 7.2.31 (General k-vector)**.** The space of k-vectors on $\mathbb{V}$ is defined as

$$\Lambda_k(\mathbb{V}) := \Lambda^k(\mathbb{V}^*).$$

Remark 7.2.32. By definition 7.2.31, we have the followings:

- $\Lambda_0(\mathbb{V}) = \mathbb{R}$;
- $\Lambda_1(\mathbb{V}) = \mathbb{V}^{**} = \mathbb{V}$;
- the wedge product $\wedge : \Lambda_h(\mathbb{V}) \times \Lambda_k(\mathbb{V}) \to \Lambda_{h+k}(\mathbb{V})$ is defined (see 7.2.3); $\wedge$ is bilinear, associative and anticommutative (see 7.2.4).

Moreover, given a basis $\{e_1, \dots, e_n\}$ of $\mathbb{V}$, we write

$$e_{\underline{i}} \coloneqq e_{i_1} \wedge \cdots \wedge e_{i_h} \in \Lambda_k(\mathbb{V}).$$

To be precise, we see e_{i_j} as the element of $\mathbb{V}^{**}$ given by the computation of element in $\mathbb{V}^*$ at the vector $e_{i_j} \in \mathbb{V}$. So, we should write

$$e_{\underline{i}}^{**} = e_{i_1}^{**} \wedge \cdots \wedge e_{i_h}^{**} \in \Lambda_k(\mathbb{V}),$$

but we will avoid this useless notation. We have shown in 7.2.7 that $\{e_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda_k(\mathbb{V})$.

Definition 7.2.33 (Duality pairing)**.** We define the duality pairing of $\Lambda^k(\mathbb{V})$ and $\Lambda_k(\mathbb{V})$ as the bilinear form $< \cdot, \cdot >: \Lambda^k(\mathbb{V}) \times \Lambda_k(\mathbb{V}) \to \mathbb{R}$ by setting

$$< e_{\underline{i}}^*, e_{\underline{j}} > \coloneqq \delta_{\underline{i},\underline{j}}.$$

Remark 7.2.34. In the framework of definition 7.2.33, for all $w \in \Lambda_k(\mathbb{V})$ the quantities $\{< e_{\underline{i}}^*, w > \ \mid \underline{i} \in I_{n,k}\}$ are the coordinates of w with respect to the dual basis associated to $\{e_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$, which turns out to be $\{e_{\underline{i}}^* \mid \underline{i} \in I_{n,k}\}$.

Furthermore, we can easily check that $< \cdot, \cdot >$ gives an isomorphism between $\Lambda_k(\mathbb{V})$ and $(\Lambda^k(\mathbb{V}))^*$. In deed, for all $\alpha \in \Lambda_k(\mathbb{V})$, consider the map $T_\alpha : \Lambda^k(\mathbb{V}) \to \mathbb{R}$ defined by

$$T_\alpha(\beta) \coloneqq < \beta, \alpha > .$$

Then, T_α is an element of $\Lambda^k(\mathbb{V})^*$; in other words, the map $T : \Lambda_k(\mathbb{V}) \to \Lambda^k(\mathbb{V})^*$ is well defined. Moreover, T is injective, that is, if $\alpha \in \Lambda_k(\mathbb{V})$ is s.t. $< \beta, \alpha >= 0$ for all $\beta \in \Lambda^k(\mathbb{V})$, then $\alpha = 0$. In deed, for such α there holds $< e_{\underline{i}^*}, \alpha >= 0$ for all $\underline{i} \in I_{n,k}$. If we write α as linear combination of $e_{\underline{i}}$ and recall the fact that $< e_{\underline{i}}^*, e_{\underline{j}} >= \delta_{\underline{i},\underline{j}}$, we conclude that $\alpha = 0$. Since $\Lambda_k(\mathbb{V})$ and $\Lambda^k(\mathbb{V})^*$ have the same dimension, we conclude that T is a linear isomorphism.

Moreover, since $< \cdot, \cdot >$ is bilinear, $\{e_{\underline{i}}^* \mid \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda^k(\mathbb{V})$ and $\{e_i \mid i \in \{1, \dots, n\}\}$ is a basis of $\mathbb{V}$, it immediately follows that

$$< \alpha, v_1 \wedge \cdots \wedge v_k >= \alpha(v_1, \dots, v_k)$$

for all $\alpha \in \Lambda^k(\mathbb{V})$ for all $v_1, \dots, v_k \in \mathbb{V}$. To be precise, we should write

$$< \alpha, v_1^{**} \wedge \cdots \wedge v_k^{**} >= \alpha(v_1, \dots, v_k),$$

where v_i^{**} and v_i are element of $\mathbb{V}^{**}$ and $\mathbb{V}$ respectively, which corresponds via the canonical isomorphism between $\mathbb{V}^{**}$ and $\mathbb{V}$. We will avoid this notation.

We have also shown that the duality pairing does not depend on the choice of the basis of $\mathbb{V}$. It is interesting to notice that the definition of the duality pairing on $\Lambda^k(\mathbb{V}) \times \Lambda_k(\mathbb{V})$ as

$$< \alpha, v_1 \wedge \cdots \wedge v_k > \coloneqq \alpha(v_1, \dots, v_k)$$

is not well posed (without extending by bilinearity), because there are elements in $\Lambda_k(\mathbb{V})$ which are not of the form $v_1 \wedge \cdots \wedge v_k$. For instance, if the dimension of $\mathbb{V}$ is at least 4 and e_1, e_2, e_3, e_4 are linearly independent, then

$$e_1 \wedge e_2 + e_3 \wedge e_4$$

cannot be written as $v_1 \wedge v_2$, as one can easily check. Furthermore, it is obvious that 1-vectors are all simple. It can also be checked (but it is not so immediate) that $(n-1)$-vectors are all simple. Moreover, there holds that k-vectors are never all simple for $1 < k < n-1$.

Corollary 7.2.35. *Simple k-vectors can be naturally embedded in $\Lambda_k(\mathbb{V})$ simply by identifying $[v_1, \ldots, v_k]$ with $v_1 \wedge \cdots \wedge v_k$. In deed, given $v_1, \ldots, v_k, \tilde{v}_1, \ldots, \tilde{v}_k \in \mathbb{V}$, there holds that $(v_1, \ldots, v_k) \sim (\tilde{v}_1, \ldots, \tilde{v}_k)$ if an only if $v_1 \wedge \cdots \wedge v_k = \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k$.*

Proof. We have that $(v_1, \ldots, v_k) \sim (\tilde{v}_1, \ldots, \tilde{v}_k)$ if an only if

$$\alpha(v_1, \ldots, v_k) = \alpha(\tilde{v}_1, \ldots, \tilde{v}_k) \quad \forall \alpha \in \Lambda^k(\mathbb{V}).$$

This means that

$$< \alpha, v_1 \wedge \cdots \wedge v_k >=< \alpha, \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k >;$$

this can happens if and only if $v_1 \wedge \cdots \wedge v_k = \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k$. □

Remark 7.2.36. By corollary 7.2.35, we identify $[v_1, \ldots, v_k]$ with $v_1 \wedge \cdots \wedge v_k$ and the notation $[v_1, \ldots, v_k]$ disappears.

Assume now that $\mathbb{V}$ is endowed with a scalar product; then we can endow $\Lambda^k(\mathbb{V})$ and $\Lambda_k(\mathbb{V})$ with scalar product s.t. $\{e^*_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ and $\{e_{\underline{i}} \mid \underline{i} \in I_{n,k}\}$ are orthonormal basis (we will only use the norm associated to this products on $\Lambda_k(\mathbb{V})$).

Definition 7.2.37 (Euclidean norm)**.** Given $w \in \Lambda_k(\mathbb{V})$ we have that

$$w = \sum_{\underline{i} \in I_{n,k}} w_{\underline{i}} e_{\underline{i}},$$

where $w_{\underline{i}} =< e^*_{\underline{i}}, w >$. Then, the norm of w associated to the scalar product on $\Lambda_k(\mathbb{V})$ is given by

$$|w| := \sqrt{\sum_{\underline{i} \in I_{n,k}} w^2_{\underline{i}}}.$$

Proposition 7.2.38. *The euclidean norm of a simple k-vector defined in 7.2.37 agrees with the norm of simple k-vectors defined in 7.2.14. More precisely, given $v_1 \wedge \cdots \wedge v_k \in \Lambda_k(\mathbb{V})$ there holds*

$$|v_1 \wedge \cdots \wedge v_k| = \mathcal{H}^k(R(v_1, \ldots, v_k)).$$

Proof. Without loss of generaly, we can assume that $v_1 \ldots, v_k$ are linearly independent. Denote by W the $n \times k$ matrix whose j-th column contains the coordinates of v_j with respect to the basis $\{e_1, \ldots, e_n\}$. Define $T : \mathbb{R}^k \to \mathbb{V}$ the linear map the sends the i-th element of the canonical basis (denoted by $\hat{e}_i$) of $\mathbb{R}^n$ in v_i. Denote by JT the jacobian determinant of T at any point (it is constant, since T is linear). Using the property of the duality pairing, lemma 7.2.5, the Cauchy-Binet formula (see 7.2.8) and the area formula (see 5.2.14, 7.2.13 and 5.2.7), we have that

$$\begin{aligned}
|v_1 \wedge \cdots \wedge v_k| &= \sqrt{\sum_{\underline{i} \in I_{n,k}} (< e^*_{\underline{i}}, v_1 \wedge \cdots \wedge v_k >)^2} \\
&= \sqrt{\sum_{\underline{i} \in I_{n,k}} (e^*_{\underline{i}}(v_1, \dots v_k))^2} \\
&= \sqrt{\sum_{\underline{i} \in I_{n,k}} (\det(W_{\underline{i}}))^2} \\
&= \sqrt{\det(W^T \cdot W)} \\
&= JT \\
&= JT \cdot \mathcal{H}^k(R(\hat{e}_1, \dots, \hat{e}_k)) \\
&= \mathcal{H}^k(T(R(\hat{e}_1, \dots, \hat{e}_k))) \\
&= \mathcal{H}^k(R(v_1, \dots, v_k)).
\end{aligned}$$

□

There are tow natural choices of norms on $\Lambda_k(\mathbb{V})$. The first is the euclidean norm $|\cdot|$ defined in 7.2.37; the second is the mass norm, defined below.

Definition 7.2.39 (Mass norm)**.** The mass norm is the largest norm Φ in $\Lambda_k(\mathbb{V})$ s.t.

$$\Phi(v_1 \wedge \cdots \wedge v_k) = |v_1 \wedge \cdots \wedge v_k| \quad \forall v_1, \dots, v_k \in \mathbb{V}.$$

Remark 7.2.40. Clearly, mass norm and euclidean norm agree on simple k-vectors. Moreover, it can be checked that the mass norm is the convex envelope of the restriction of the euclidean norm to simple k-vectors, that is

$$\Phi(w) = \inf \left\{ \sum_i t_i \, |w_i| \; \middle| \; w = \sum_i t_i w_i \text{ is convex combination of simple } k\text{- vectors} \right\}.$$

Definition 7.2.41 (Comass norm)**.** The dual norm Φ^* induced by the mass norm is called comass norm, that is

$$\Phi^*(\alpha) = \sup\{< \alpha, w > \ | \ \Phi(w) \leq 1\} \quad \forall \alpha \in \Lambda^k(\mathbb{V}).$$

Remark 7.2.42. It can be checked that

$$\begin{aligned}
\Phi^*(\alpha) &= \sup\{< \alpha, v_1 \wedge \cdots \wedge v_k > \ | \ |v_1 \wedge \cdots \wedge v_k| \leq 1\} \\
&= \sup\{\alpha(v_1, \dots, v_k) \mid \mathcal{H}^k(R(v_1, \dots, v_k)) \leq 1\}
\end{aligned}$$

for all $\alpha \in \Lambda^k(\mathbb{V})$.

We point out that there are only few results where it is possible to see the difference between euclidean norm and mass/comass norm. In the following, we will use the notation $|\cdot|$ also for mass and comass norm.

Remark 7.2.43. Given $\alpha \in \Lambda^k(\mathbb{V})$ and $\beta \in \Lambda_k(\mathbb{V})$, if we deal with mass and comass norm, then by definition 7.2.39 and 7.2.41 it follows that

$$|< \alpha, \beta >| \leq |\alpha|\,|\beta|\,.$$

However, the same inequality holds true also when dealing with euclidean norms (see 7.2.37). In deed, we have

$$|< \alpha, \beta >| = \left|\sum_{\underline{i}} \alpha_{\underline{i}}\beta_{\underline{i}}\right| \leq \sqrt{\sum_{\underline{i}} \alpha_{\underline{i}}^2} \cdot \sqrt{\sum_{\underline{i}} \beta_{\underline{i}}^2} = |\alpha|\,|\beta|\,.$$

7.3 Theory of currents (De Rham)

The approach in definition of currents is very similar to that of distributions. Our purpose is to introduce currents as the dual space of differential forms. Then we show how to use currents to solve the Plateau's problem.

7.3.1 Definition

We start with a basic remark.

Remark 7.3.1. Let Σ be a closed, oriented, k-dimensional surface of class C^1 in $\mathbb{R}^n$. Then, we define the following linear functional on the space of continuous and compactly supported forms:

$$T_\Sigma(\omega) := \int_\Sigma \omega.$$

- It is easy to show that T_Σ is uniquely determined by Σ, namely $\Sigma \neq \Sigma'$ implies that $T_\Sigma \neq T_{\Sigma'}$.
- We can rewrite Stokes' theorem (see 7.2.30) as follows:

  $$T_{\partial\Sigma}(\omega) = T_\Sigma(d\omega),$$

  where ω is a $(k-1)$-form on $\mathbb{R}^n$ with coefficients in $C^1_c(\mathbb{R}^n)$.
- There holds that

  $$\mathcal{H}^k(\Sigma) = \sup\{T_\Sigma(\omega) \mid \omega \in C^\infty_c(\mathbb{R}^n),\ |\omega(x)| \leq 1\ \forall x \in \mathbb{R}^n\},$$

  where $|\cdot|$ is either the mass/comass norm or the euclidean norm; the key step is explained in 7.2.43. In deed, given ω as above, we have that

  $$\begin{aligned} T_\sigma(\omega) &= \int_\Sigma \omega \\ &= \int_\Sigma < \omega(x), \tau(x) > \ d\mathcal{H}^k(x) \\ &\leq \int_\Sigma |\omega(x)|\,|\tau(x)|\ d\mathcal{H}^k(x) \\ &\leq \int_\Sigma 1\ d\mathcal{H}^k(x) \\ &= \mathcal{H}^k(\Sigma). \end{aligned}$$

Moreover, the mass norm and the euclidean norm of of $\tau(x)$ are both 1, since $\tau(x)$ is a k-simple unitary vector and then proposition 7.2.38. Notice that the inequality above are optimized by ω s.t.

$$< \omega(x), \tau(x) >= 1 \quad \forall x \in \Sigma;$$

such ω exists and it is continuous. By approximation, the supremum can be achieved with compactly supported smooth k-forms.

In order to define currents as the dual of differential forms, we have to introduce a topology.

Definition 7.3.2. Denote by $\mathcal{D}^k(\mathbb{R}^n)$ the space of k-forms on $\mathbb{R}^n$ smooth, compactly supported. Given $K \subseteq \mathbb{R}^n$ compact, we denote by

$$\mathcal{D}^k(K) := \{\omega \in \mathcal{D}^k(\mathbb{R}^n) \mid \operatorname{supp}(\omega) \subseteq K\}.$$

Remark 7.3.3. Given $K \subseteq \mathbb{R}^n$ compact, it can be checked that $\mathcal{D}^k(K)$ is a Fréchet space with the seminorms

$$\|\omega\|_{C^h(K)} := \sum_{\underline{i} \in I_{n,k}} \|\omega_{\underline{i}}\|_{C^h(K)} = \sum_{\underline{i} \in I_{n,k}} \sum_{|\underline{j}| \leq k} \left\|D^{\underline{j}}(\omega_{\underline{i}})\right\|_{L^\infty(K)}$$

for all $h \in \mathbb{N}$.

Definition 7.3.4. We endow $\mathcal{D}^k(\mathbb{R}^n)$ with the smallest (weakest) topology s.t. the inclusion map $i : \mathcal{D}^k(K) \hookrightarrow \mathcal{D}^k(\mathbb{R}^n)$ is continuous for all $K \subseteq \mathbb{R}^n$, the so called "direct limit" topology.

Remark 7.3.5. The topology defined in 7.3.4 is so weak that every reasonable operation is continuous with respect to this topology.

Definition 7.3.6 (k-current)**.** The space of k-currents in $\mathbb{R}^n$ is defined as the dual space of $\mathcal{D}^k(\mathbb{R}^n)$ with respect to the direct limit topology (see 7.3.4) and it is denoted as $\mathcal{D}_k(\mathbb{R}^n) = (\mathcal{D}^k(\mathbb{R}^n))^*$.

Remark 7.3.7. We just mention the fact that the entire space $\mathbb{R}^n$ can be replaced either with an open set in $\mathbb{R}^n$ or a Riemannian manifold.

Definition 7.3.8 (Boundary of a current)**.** Give $T \in \mathcal{D}_k(\mathbb{R}^n)$, we define the boundary of T as the $(k-1)$-current defined as

$$< \partial T, \omega >:=< T, d\omega > .$$

In other words, ∂ is the adjoint operator of d.

Remark 7.3.9. The functional ∂T defined in 7.3.8 is certainly well defined and linear. To be precise, one should also check that ∂T is continuous with respect to the topology introduced on $\mathcal{D}^{k-1}(\mathbb{R}^n)$.

Remark 7.3.10. The counterpart of the fact that $d^2 = 0$ (see 7.2.24), is that $\partial^2 T = 0$ for all $T \in \mathcal{D}_k(\mathbb{R}^n)$, provided $k \geq 2$.

Definition 7.3.11 (Mass of a current)**.** Given $T \in \mathcal{D}_k(\mathbb{R}^n)$, we define the mass of T as

$$\mathbb{M}(T) := \sup\{< T, \omega > \ | \ \omega \in \mathcal{D}^k(\mathbb{R}^n), \ |\omega(x)| \leq 1 \ \forall x \in \mathbb{R}^n\},$$

where $|\omega(x)|$ is the comass norm of $\omega(x)$.

Remark 7.3.12. With the notation introduced, if Σ is a k-dimensional closed, oriented surface of class C^1 in $\mathbb{R}^n$, we have shown in 7.3.1 that

$$\partial T_\Sigma = T_{\partial\Sigma},$$

$$\mathbb{M}(T_\Sigma) = \mathcal{H}^k(\Sigma).$$

Being a dual space, $\mathcal{D}_k(\mathbb{R}^n)$ can be endowed with the weak-* topology induced by the duality with $\mathcal{D}^k(\mathbb{R}^n)$.

Definition 7.3.13 (Convergence of currents)**.** Given a sequence of k-currents $(T_n)_n$ and a k-current T, we say that $(T_n)_n$ converges to T in the sense of currents if the sequence converges to T with respect to the weak-* topology, that is

$$\lim_{n\to+\infty} < T_n, \omega > = < T, \omega > \quad \forall T \in \mathcal{D}_k(\mathbb{R}^n).$$

Proposition 7.3.14. *Let $(T_n)_n$ be a sequence of k-currents that converges to a k-currents T as in 7.3.13. Then, the followings hold true:*

- *$(\partial T_n)_n$ converges to ∂T in the sense of currents;*
- *$\mathbb{M}(T) \leq \liminf_{n\to+\infty} \mathbb{M}(T_n)$.*

Proof. **Step 1**: Given $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^n)$, recall that $d\omega \in \mathcal{D}^k(\mathbb{R}^n)$; by definition 7.3.13, we can infer

$$\lim_{n\to+\infty} < \partial T_n, \omega > = \lim_{n\to+\infty} < T_n, d\omega > = < T, d\omega > = < \partial T, \omega > .$$

Step 2: Take $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ s.t. $|\omega(x)| \leq 1$ for all $x \in \mathbb{R}^n$ (we use the comass norm). Then, we have

$$\liminf_{n\to+\infty} \mathbb{M}(T_n) \geq \liminf_{n\to+\infty} < T_n, \omega > = < T, \omega > .$$

Taking the supremum with respect to $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ s.t. $|\omega(x)| \leq 1$ for all $x \in \mathbb{R}^n$, we conclude that

$$\liminf_{n\to+\infty} \mathbb{M}(T_n) \geq \mathbb{M}(T).$$

□

7.3.2 Significant subclasses of currents

Currents with finite mass

Definition 7.3.15 (Currents with finite mass)**.** We say that a k-current T has finite mass if $\mathbb{M}(T) < +\infty$.

Remark 7.3.16. Given $T \in \mathcal{D}_k(\mathbb{R}^n)$ with finite mass, by definition of mass, we obtain

$$|< T, \omega >| \leq \mathbb{M}(T) \left\|\omega\right\|_{L^\infty} \quad \forall \omega \in \mathcal{D}^k(\mathbb{R}^n),$$

where $\left\|\omega\right\|_{L^\infty}$ is computed with the comass norm. Hence, T can be extended by density to a linear continuous functional defined on the k-forms $\omega \in C_0(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$. By the Riesz's representation theorem (see 1.1.27), we infer that T can be represented by integration with respect to a measure with values in $\Lambda^k(\mathbb{R}^n)^* = \Lambda_k(\mathbb{R}^n)$. Thus, there exists a positive, Borel, finite measure μ on $\mathbb{R}^n$ and a vector field $\tau : \mathbb{R}^n \to \Lambda_k(\mathbb{R}^n)$ in $L^1(\mathbb{R}^d, \mu)$ unitary (in mass norm) and s.t.

$$< T, \omega >= \int_{\mathbb{R}^n} < \omega(x), \tau(x) > \; d\mu(x).$$

Then, it can be checked that

$$\begin{aligned}
\mathbb{M}(T) &= \sup\{< T, \omega > \; \mid \omega \in \mathcal{D}^k(\mathbb{R}^n), \; \left\|\omega\right\|_{L^\infty} \leq 1\} \\
&= \sup\{< T, \omega > \; \mid \omega \in C_0(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n)), \; \left\|\omega\right\|_{L^\infty} \leq 1\} \\
&= \int_{\mathbb{R}^n} |\tau(x)| \; d\mu(x) \\
&= \mu(\mathbb{R}^n),
\end{aligned}$$

where $|\tau(x)|$ is the mass norm and $\left\|\omega\right\|_{L^\infty}$ is computed with the comass norm. To be precise, one should check the two inequalities, but this can be done as in 7.3.1. Thus, we write $T = \tau \cdot \mu$.

Remark 7.3.17. If Σ is a compact, oriented, k-dimensional surface in $\mathbb{R}^n$ of class C^1, then the current T_Σ defined in 7.3.1 can be represented as

$$T_\Sigma = \tau_\Sigma \cdot \mathcal{H}^k \llcorner \Sigma,$$

where τ_Σ is the orientation of Σ. Moreover, we have that

$$\mathbb{M}(T_\Sigma) = \mathcal{H}^k(\Sigma) < +\infty.$$

Currents with finite mass have good compactness properties.

Proposition 7.3.18. *Let $(T_n)_n$ be a sequence of k-currents with finite mass s.t.*

$$\sup_n \mathbb{M}(T_n) < +\infty;$$

then, up to subsequences, $(T_n)_n$ converges to a k-current T in the sense of currents; moreover, there holds

$$\mathbb{M}(T) \leq \liminf_{n \to +\infty} \mathbb{M}(T_n).$$

In particular, T is a k-current of finite mass.

Proof. Since $\mathcal{D}_k(\mathbb{R}^n)$ is the dual space of $\mathcal{D}^k(\mathbb{R}^n)$ and $\mathbb{M}$ is the dual norm, Banach-Alaoglu theorem applies. So, we obtain the compactness result. The lower semicontinuity of the mass has been shown in 7.3.14. □

Normal currents

Definition 7.3.19 (Normal current)**.** If $k \geq 1$, we say that a k-current T is normal if both $T, \partial T$ have finite mass. If $k = 0$, a 0-current is normal if it has finite mass.

Remark 7.3.20. Given a normal k-current T, we can represent $T = \tau \cdot \mu$ and $\partial T = \tau' \cdot \mu'$, with the notation introduced in 7.3.16.

Normal currents have good compactness properties.

Proposition 7.3.21. *Let $(T_n)_n$ be a sequence of normal k-currents s.t.*

$$\sup_n \mathbb{M}(T_n) + \sup_n \mathbb{M}(\partial T_n) < +\infty.$$

Then, up to subsequences, $(T_n)_n$ converges to a k-current T in the sense of currents; moreover, there holds

$$\mathbb{M}(T) \leq \liminf_{n \to +\infty} \mathbb{M}(T_n),$$

$$\mathbb{M}(\partial T) \leq \liminf_{n \to +\infty} \mathbb{M}(\partial T_n).$$

In particular, T is a normal k-current.

Proof. The proof is an immediate consequence of 7.3.18 and 7.3.14. □

Having said that, we can easily solve the Plateau's problem in the class on normal currents. In deed, this is not satisfactory, because this class is definitely too large.

Corollary 7.3.22 (Plateau's problem in normal currents)**.** *Fix T_0 a normal k-current. Then, the minimum problem*

$$\min\{\mathbb{M}(T) \mid T \textit{ is normal and } \partial T = \partial T_0\}$$

admits a solution.

Proof. The functional $\mathbb{M}$ is lower semicontinuous and coercive in the class of normal currents (see 7.3.14 and 7.3.21) with respect to convergence of currents. □

We give some examples of normal (and not) currents.

Example 7.3.23. Let Σ be a compact, oriented, k-dimensional surface of class C^1 in $\mathbb{R}^n$. Then, the k-current T_Σ defined in 7.3.1 is a normal current. In deed, we have

$$T_\Sigma = \tau_\Sigma \cdot \mathcal{H}^k \llcorner \Sigma, \quad \partial T_\Sigma = T_{\partial\Sigma} = \tau'_{\partial\Sigma} \cdot \mathcal{H}^{k-1} \llcorner \partial\Sigma,$$

where τ_Σ is the orientation on Σ and $\tau'_{\partial\Sigma}$ is the orientation of $\partial\Sigma$ as a boundary of the surface Σ oriented by τ. Thus, we have

$$\mathbb{M}(T_\Sigma) = \mathcal{H}^k(\Sigma), \quad \mathbb{M}(\partial T_\Sigma) = \mathcal{H}^{k-1}(\partial\Sigma).$$

Example 7.3.24. In $\mathbb{R}^2$ consider the rectangle $R \coloneqq [-1, 1] \times [0, 1]$, the measure $\mu \coloneqq \mathscr{L}^2 \llcorner R$ and the constant vector field $e_1 \coloneqq (1, 0)$. Then, $T \coloneqq e_1 \cdot \mu$ is a normal 1-current. Define $I^\pm \coloneqq \{\pm 1\} \times [0, 1]$ (see figure 7.1); we claim that

$$\partial T = \mathcal{H}^1 \llcorner I^+ - \mathcal{H}^1 \llcorner I^- = \tau' \cdot \mu',$$

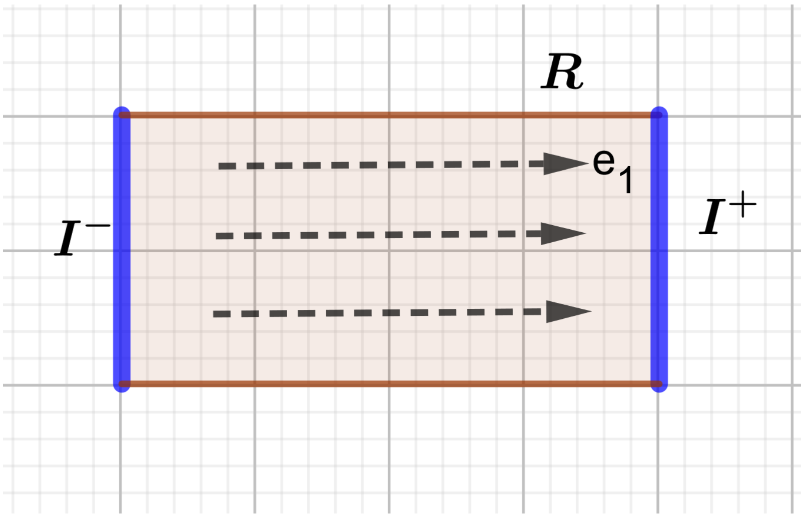

Figure 7.1: The 1-current associated to the rectangle with vector field e_1.

where $\mu' = \mathcal{H}^1 \llcorner (I^+ \cup I^-)$ and

$$\tau'(x) := \begin{cases} 1 & x \in I^+, \\ -1 & x \in I^-; \end{cases}$$

thus, we have that $\mathbb{M}(\partial T) = 2$. In deed, this can be proved by a straightforward computation. Take $\Phi \in \mathcal{D}^0(\mathbb{R}^2)$, that is $\Phi \in C_c^\infty(\mathbb{R}^2)$; by definition, we have

$$\begin{aligned} < \partial T, \Phi > &=< T, d\Phi > \\ &= \int_{\mathbb{R}^2} < d\Phi(x), e_1 > \ d\mathscr{L}^2 \llcorner R \\ &= \int_R < d\Phi(x), e_1 > \ d\mathscr{L}^2(x) \\ &= \int_R \frac{\partial \Phi}{\partial x_1}(x_1, x_2) \ dx_1 dx_2 \\ &= \int_0^1 \left(\int_{-1}^1 \frac{\partial \Phi}{\partial x_1}(x_1, x_2) \ dx_1 \right) dx_2 \\ &= \int_0^1 [\Phi(1, x_2) - \Phi(-1, x_2)] \ dx_2 \\ &= \int_{\mathbb{R}^2} < \tau', \Phi > \ d\mu'. \end{aligned}$$

The computation above can be resumed as follows: given $t \in [0,1]$, set $I_t := [-1,1] \times \{t\}$ oriented by e_1. Let $T_t := T_{I_t}$ be the 1-current associated to I_t. By Fubini's theorem, we can check that

$$T = \int_0^1 T_t \ dt, \quad \partial T = \int_0^1 \partial T_t \ dt,$$

that is

$$< T, \omega >= \int_0^1 < T_t, \omega > \ dt, \quad < \partial T, \omega >= \int_0^1 < \partial T_t, \omega > \ dt$$

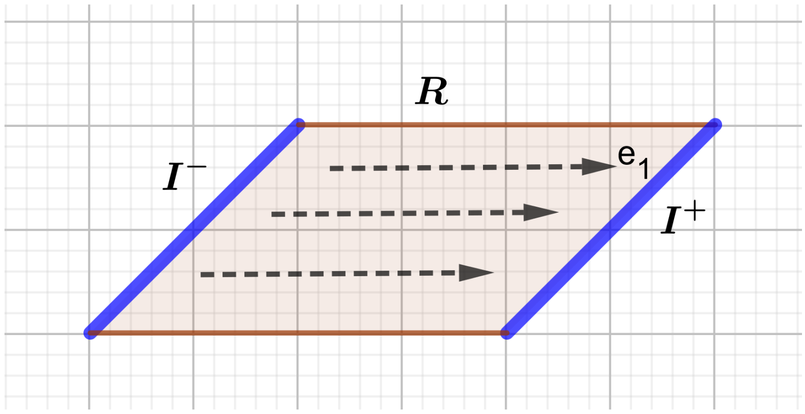

Figure 7.2: The 1-current associated to the parallelogram with vector field e_1.

for all admissible test forms ω. However, it is immediate to see that

$$\partial T_t = \delta_{(1,t)} - \delta_{(-1,t)}.$$

Then, we conclude that

$$\partial T = \int_0^1 [\delta_{(1,t)} - \delta(-1,t)]\, dt = \mathcal{H}^1 \llcorner I^+ - \mathcal{H}^1 \llcorner I^-.$$

In deed, to check the details, one should repeat word by word the computation above. To resume, in some sense, the idea is to "slice" the current along parallel sections and then summing them up by integration.

Example 7.3.25. Let R be the rectangle in $\mathbb{R}^2$ of vertices $(-1,0),(1,0),(0,1),(2,1)$; set $\mu := \mathscr{L}^2 \llcorner R$ and $e_1 := (1,0)$. Define $T := e_1 \cdot \mu$ (see figure 7.2). As in 7.3.24, we claim that T is a normal current and

$$\partial T = \mathcal{H}^1 \llcorner I^+ - \mathcal{H}^1 \llcorner I^- = \tau' \llcorner \mu',$$

where

$$I^\pm := \{(\pm 1 + t, t) \mid t \in [0,1]\},$$
$$\mu' := \mathcal{H}^1 \llcorner (I^+ \cup I^-),$$
$$\tau'(x) := \begin{cases} 1 & x \in I^+, \\ -1 & x \in I^-. \end{cases}$$

The computation is completely similar to that shown in 7.3.24.

Example 7.3.26. Let T be the 1-current on $\mathbb{R}$ given by $T := e_1 \cdot \delta_0$, where e_1 is the standard orientation of $\mathbb{R}$, namely $e_1 = 1$ (see figure 7.3). Then ∂T is not represented by a measure; in other words, T is not a normal current. In deed, given $\Phi \in \mathcal{D}^0(\mathbb{R})$, that is $\Phi \in C_c^\infty(\mathbb{R})$, there holds

$$< \partial T, \Phi > = < T, d\Phi > = \int_{\mathbb{R}} < d\Phi(x), e > \; d\delta_0 = \Phi'(0).$$

It follows that

$$\mathbb{M}(\partial T) = \sup\{\Phi'(0) \mid |\Phi| \leq 1,\ \Phi \in C_c^\infty(\mathbb{R})\} = +\infty.$$

Then, ∂T is not bounded with respect to the C^0 norm; equivalently, ∂T cannot be represented by a measure.

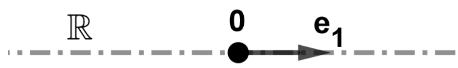

Figure 7.3: The 1-current associated to the point 0 in $\mathbb{R}$ with vector field e_1.

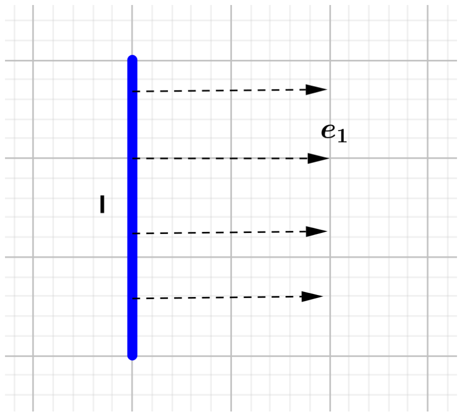

Figure 7.4: The 1 current in $\mathbb{R}^2$ associated to a vertical segment with horizontal vector field.

Example 7.3.27. In $\mathbb{R}^2$ consider the 1-currents $T_1 := e_1 \cdot \delta_0$ and $T_2 := e_1 \cdot \mathcal{H}^1 \llcorner I$, where $e_1 := (1,0)$ and $I := \{0\} \times I$ (see 7.4). Arguing as in 7.3.26, it is immediate to show that $\mathbb{M}(\partial T_i) = +\infty$ for $i = 1, 2$; hence, T_1, T_2 are not normal currents. In deed, given $\Phi \in C_c^\infty(\mathbb{R}^2)$, we have that

$$< \partial T_1, \Phi >=< T_1, d\Phi >= \frac{\partial \Phi}{\partial x_1}(0);$$

then, we deduce that

$$\mathbb{M}(\partial T_1) = \sup \left\{ \frac{\partial \Phi}{\partial x_1}(0) \;\middle|\; |\Phi| \leq 1,\ \Phi \in C_c^\infty(\mathbb{R}^2) \right\} = +\infty.$$

Similarly, we have that

$$< \partial T_2, \Phi >=< T_2, d\Phi >= \int_I < d\Phi, e_1 > \ d\mathcal{H}^1 = \int_0^1 \frac{\partial \Phi}{\partial x_1}(0,t)\ dt.$$

Hence, we have that

$$\mathbb{M}(\partial T_2) = \sup \left\{ \int_0^1 \frac{\partial \Phi}{\partial x_1}(0,t)\ dt \;\middle|\; |\Phi| \leq 1,\ \Phi \in C_c^\infty(\mathbb{R}^2) \right\} = +\infty.$$

The previous example is a particular instance of the following fact (that can be fixed): given the 1-current $T := \tau \cdot \mathcal{H}^1 \llcorner I$, where I is the support of a C^1 curve in $\mathbb{R}^n$ and τ is a continuous vector field on I, if $\mathbb{M}(\partial T) < +\infty$, then τ is tangent to I. This suggests that normal currents are geometrically relevant, while currents with finite mass are not.

Rectifiable currents

Definition 7.3.28 (Orientation of a rectifiable set)**.** Let E be a k-dimensional rectifiable set in $\mathbb{R}^n$; τ is an orientation of E if $\tau : E \to \Lambda_k(\mathbb{R}^d)$ is s.t. $\tau(x)$ is an orientation of $\mathrm{Tan}^w_x E$ for $\mathcal{H}^k$-a.e. $x \in E$; in other words, $\tau(x) = \tau_1(x) \wedge \cdots \wedge \tau_k(x)$ is simple, $|\tau(x)| = 1$ and

$$\mathrm{Span}(\tau(x)) \coloneqq \mathrm{Span}(\tau_1(x), \ldots, \tau_k(x)) = \mathrm{Tan}^w_x E$$

for $\mathcal{H}^k$-a.e. $x \in E$.

Definition 7.3.29 (Rectifiable current)**.** Given E, τ as above, let $m \in L^1(\mathbb{R}^n, \mathcal{H}^k \llcorner E)$ be "multiplicity ". We set $T \coloneqq [E, \tau, m]$ be the k-current defined as follows:

$$< T, \omega > \coloneqq \int_E < \omega(x), \tau(x) > m(x) \; d\mathcal{H}^k(x) \quad \forall \omega \in \mathcal{D}^k(\mathbb{R}^n),$$

or equivalently $T = \tau m \cdot \mathcal{H}^k \llcorner E$. We say that T is a k-rectifiable current. Moreover, if m takes values in $\mathbb{Z}$, we say that T has integral multiplicity.

Remark 7.3.30*.* A 0-rectifiable current with integral multiplicity is nothing else that

$$T = \sum_{i=1}^{N} m_i \delta_{x_i}$$

for some finite integers m_i and some points $x_i \in \mathbb{R}^n$. In deed, a function $m \in L^1(\mathcal{H}^0 \llcorner E)$ with values in $\mathbb{Z}$ is a function that attains a finite number of values in a finite number of points and it vanishes elsewhere.

Remark 7.3.31*.* If $T \coloneqq [E, \tau, m]$ is a k-rectifiable current, then

$$\mathbb{M}(T) = \|m\|_{L^1(\mathcal{H}^k \llcorner E)} .$$

In deed, using the mass and comass norm, we have that

$$\begin{aligned}
\mathbb{M}(T) &= \sup \left\{ \int_E < \omega(x), \tau(x) > m(x) \; d\mathcal{H}^k(x) \;\middle|\; |\omega| \leq 1, \; \omega \in \mathcal{D}^k(\mathbb{R}^n) \right\} \\
&\leq \sup \left\{ \int_E |< \omega(x), \tau(x) >| \, |m(x)| \; d\mathcal{H}^k(x) \;\middle|\; |\omega| \leq 1, \; \omega \in \mathcal{D}^k(\mathbb{R}^n) \right\} \\
&\leq \sup \left\{ \int_E |\omega(x)| \, |\tau(x)| \, |m(x)| \; d\mathcal{H}^k(x) \;\middle|\; |\omega| \leq 1, \; \omega \in \mathcal{D}^k(\mathbb{R}^n) \right\} \\
&\leq \int_E |m(x)| \; d\mathcal{H}^k(x).
\end{aligned}$$

Recall that the mass norm of the simple k-vectors agree with their euclidean norm; hence, $\tau(x)$ has unitary mass norm $\mathcal{H}^k$-a.e. in E. The reverse inequality can be proved by approximation (see 7.3.16).

Remark 7.3.32*.* Given T a k-rectifiable current, then E, τ, m in the representation of T (see 7.3.29) are not uniquely determined: for instance, one can switch τ, m with $-\tau, -m$. It is possible to show that, if we require in addition that $m \geq 0$ for $\mathcal{H}^k$-a.e. $x \in E$, then E, τ, m are uniquely determined (up to $\mathcal{H}^k$-null sets).

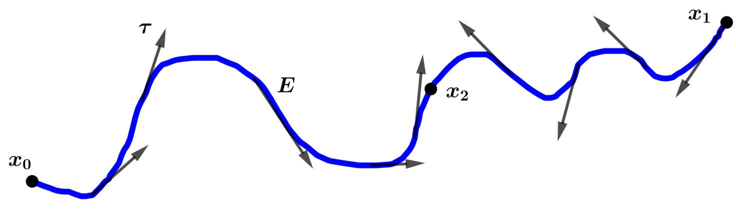

Figure 7.5: The 1-current associated to a regular curve with discontinuous orientation.

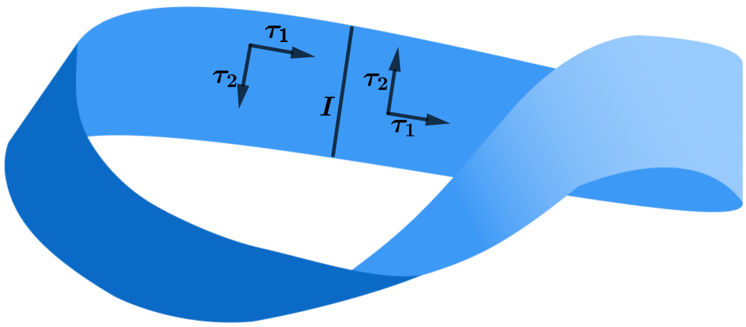

Figure 7.6: The 2-current on the Möebius strip associated to a discontinuous orientation in I.

Example 7.3.33. Given $x_0, x_1 \in \mathbb{R}^n$, let E be a C^1 curve in $\mathbb{R}^n$ joining x_0, x_1. Take $x_2 \in E$, $x_2 \neq x_0, x_1$ and define as orientation τ of E discontinuous in x_2 (see figure 7.5). Then, $T := [E, \tau, 1]$ is an 1-rectifiable current with integral multiplicity. By testing with 0-forms (i.e. smooth functions), it is immediate to compute that

$$\partial T = 2\delta_{x_2} - \delta_{x_0} - \delta_{x_1},$$

up to sign. Similarly, it can be computed the boundary of $T = [E, \tau, m]$ where E is an oriented curve of class C^1 (piecewise).

Example 7.3.34. Let S be the Möebius strip in $\mathbb{R}^3$; we can choose an orientation τ which is discontinuous along a vertical segment I (see figure 7.6); denote $T := [T, S, 1]$ the 2-rectifiable current. Then, it is possible to compute that

$$\partial T = 2\mathcal{H}^2 \llcorner I.$$

Integral currents

Definition 7.3.35 (Integral current)**.** If $k \geq 1$, we say that T is an integral k-current if both T and ∂T are rectifiable currents with integral multiplicity, thus we have

$$T = [E, \tau, m], \quad \partial T = [E', \tau', m'].$$

If $k = 0$, T is a 0-integral current if it is rectifiable with integral multiplicity, that is

$$T = \sum_{i=1}^{N} m_i \delta_{x_i}$$

for some finite $m_i \in \mathbb{Z}$ and $x_i \in \mathbb{R}^n$ (see 7.3.30).

Example 7.3.36. Let Σ be a k-dimensional compact oriented surface in $\mathbb{R}^n$ of class C^1; then, $\mathcal{H}^k(S) < +\infty$ and $\mathcal{H}^{k-1}(\partial\Sigma) < +\infty$. Assume that Σ is oriented by τ_Σ. Then, the k-current T_Σ defined in 7.3.1 is rectifiable with integral multiplicity; in deed, there holds $T_\Sigma = [\Sigma, \tau_\Sigma, 1]$. Moreover, ∂T_Σ is still rectifiable with integral multiplicity. In deed, we have

$$\partial T_\Sigma = T_{\partial\Sigma} = [\partial\Sigma, \tau_{\partial\Sigma}, 1],$$

where $\tau_{\partial\Sigma}$ is the orientation of $\partial\Sigma$ induced by the orientation τ_Σ of Σ.

7.3.3 Federer-Fleming theorem

We state without proof one of the fundamental results in the theory of integral currents.

Theorem 7.3.37 (Federer-Fleming compactness)**.** *Let $(T_n)_n$ be a sequence of integral k-currents in $\mathbb{R}^n$, with $0 \leq k \leq n$. Assume that*

$$\sup_n \mathbb{M}(T_n) + \sup_n \mathbb{M}(\partial T_n) < +\infty.$$

Then, up to subsequences, $(T_n)_n$ converges in the sense of currents to an integral k-current T.

As a bonus, we then get the solution of the Plateau's problem in the class integral currents. In deed, the assumptions in Federer-Fleming theorem are the natural ones for application to the Plateau's problem.

Corollary 7.3.38. *Let T_0 be an integral k-current in $\mathbb{R}^n$; then, the minimum problem*

$$\min\{\mathbb{M}(T) \mid T \textit{ is integral, } \partial T = \partial T_0\}$$

has a solution.

Proof. The functional $\mathbb{M}(T)$ is lower semicontinuous with respect to the weak-* convergence (as shown in 7.3.14); by Federer-Fleming theorem (see 7.3.37), we infer that the class of integral currents with given boundary is closed under weak-* convergence. Moreover, the functional $\mathbb{M}$ is coercive in the defined class. □

Remark 7.3.39. The proof of Federer-Fleming's theorem is very hard. Under the assumption

$$\sup_n \mathbb{M}(T_n) + \sup_n \mathbb{M}(\partial T_n) < +\infty,$$

we already know that $(T_n)_n$ converges up to subsequences to a normal current T: this is the soft statement proved in 7.3.21, which relies on the compactness of bounded sets with respect to the weak-* topology. The hard part in Federer-Fleming theorem is to show that T is an integral current. In fact, the Federer-Fleming compactness theorem is also called Federer-Fleming closure theorem. We also mention the fact that there are no counterparts of theorem 7.3.37 for rectifiable sets or rectifiable measure; so, Federer-Fleming theorem is something very specific of currents.

The following examples show that the assumptions in Federer-Fleming theorem are all needed.

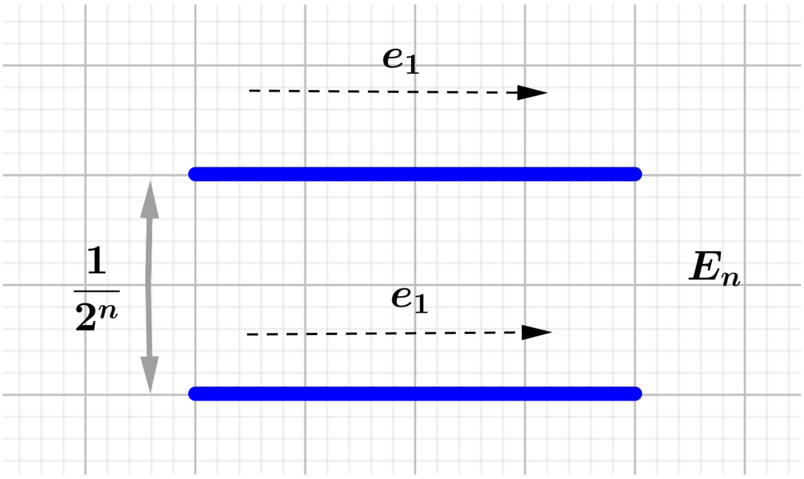

Figure 7.7: A sequence of integral 1-current with multiplicity 1 that converges to an integral 1-current with multiplicity 2.

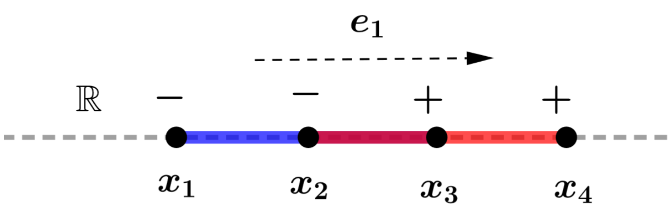

Figure 7.8: A solution of the Plauteau's problem with multiplicity different that 1.

Example 7.3.40. Theorem 7.3.37 does not hold if we work in the class of integral currents with multiplicity 1. In deed, consider $e_1 := (1,0)$ in $\mathbb{R}^2$ and for all $n \in \mathbb{N}$ the segments

$$E_n := [0,1] \times (\{0, 2^{-n}\}).$$

Define the integral 1-current

$$T_n := [E_n, e_1, 1],$$

as pictured in figure 7.7. It is immediate to see that

$$\partial T_n = \delta_{(1,2^{-n})} + \delta_{(1,0)} - \delta_{(0,2^{-n})} - \delta_{(0,0)}.$$

So, we have that

$$\mathbb{M}(T_n) = 2, \quad \mathbb{M}(\partial T_n) = 4 \ \forall n \in \mathbb{N}.$$

$(T_n)_n$ is a family of integral 1-currents with multiplicity 1; moreover, a simple computation shows that $(T_n)_n$ converges in the sense of currents to the integral 1-current

$$T := [[0,1] \times \{0\}, e_1, 2].$$

Example 7.3.41. It can happens that solutions of the Plateau's problem may have multiplicity different from 1. For instance, consider the following cases.

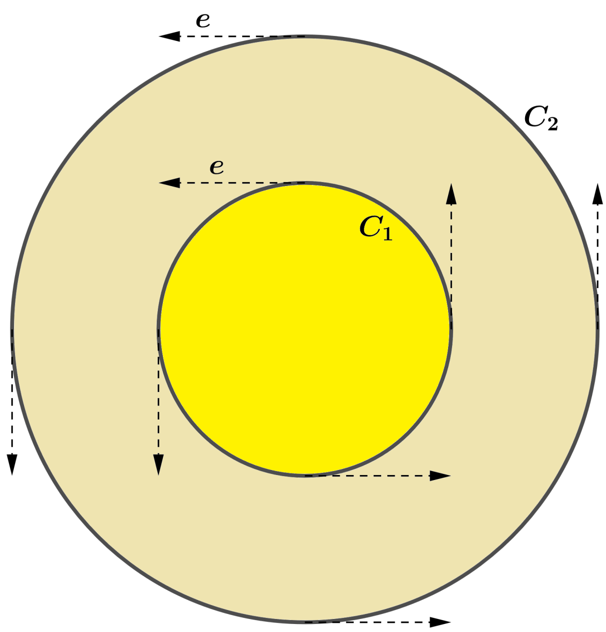

Figure 7.9: A solution of the Plauteau's problem with multiplicity different that 1.

- It the line, let $x_i = (i, 0)$ for $i = 0, 1, 2, 3$. Let T_0 be an integral 1-current with boundary
$$\partial T_0 = \delta_{x_4} + \delta_{x_3} - \delta_{x_2} - \delta_{x_1}.$$
Heuristically, the solution of the Plateau's problem with boundary datum ∂T_0 should be the integral 1-current
$$T := [[0, 3] \times \{0\}, e_1, m],$$
where $e_1 := (1, 0)$ and m is the multiplicity s.t.
$$m(x, 0) = \begin{cases} 1 & \text{if } x \in [0, 1] \cup [2, 3], \\ 2 & \text{if } x \in [1, 2]. \end{cases}$$
The situation is pictured in figure 7.8; this heuristic argument can be made precise.

- In the plane, consider the circumferences C_1, C_2 of center 0 and radii $1, 2$, respectively; assume that C_1, C_2 are oriented counterclockwise. Let T_0 be an integral 1-current in $\mathbb{R}^2$ s.t.
$$\partial T_0 = [C_1 \cup C_2, \tau^+, 1],$$
where τ^+ denotes the counterclockwise orientation of both C_1, C_2. Heuristically, the solution of the Plateau's problem with boundary datum T_0 should be integral 1-current
$$T := [D, e, m],$$
where D is the disc of radius 2 centered at the origin in the plane, e is the standard orientation of the plane and m is the multiplicity s.t.
$$m(x) = \begin{cases} 2 & \text{if } |x| \leq 1, \\ 1 & \text{if } 1 < |x| \leq 2. \end{cases}$$

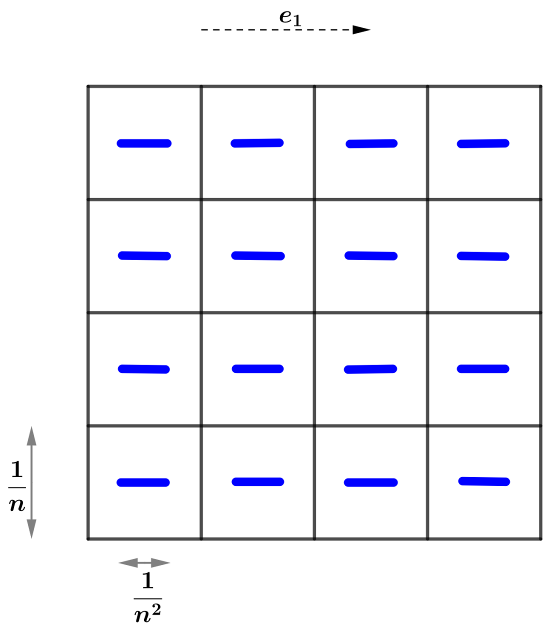

Figure 7.10: The set E_n, highlighted in blue, gives a sequence of integral 1-currents that converges to a non-rectifiable 1-current.

The situation is pictured in figure 7.9; as in the previous case, this heuristic argument can be made precise.

Example 7.3.42. In the plane, consider the square $Q := [0,1]^2$; for all $n \in \mathbb{N}$ divide Q in n^2 squares of side-length $\frac{1}{n}$; in each of these small squares, pick an horizontal segment on length $\frac{1}{n^2}$; set E_n to be the union of these n^2 segments. Let T_n be the integral 1-current in $\mathbb{R}^2$ given by

$$T_n := [E_n, e_1, 1],$$

where $e_1 := (1,0)$. The situation is resumed in figure 7.10. It is immediate to compute that

$$\mathbb{M}(T_n) = 1, \quad \mathbb{M}(\partial T_n) = 2n^2 \ \forall n \in \mathbb{N}.$$

Moreover, a straightforward computation shows that $(T_n)_n$ converges in the sense of currents to the 1-current $T := e_1 \cdot \mathscr{L}^2 \llcorner Q$. However, T is not a rectifiable 1-current. This example shows that the assumption

$$\sup_n \mathbb{M}(\partial T_n) < +\infty$$

in theorem 7.3.37 is really needed.

Example 7.3.43. In the plane, consider the square $Q := [0,1]^2$; for all $n \in \mathbb{N}$ divide Q in n horizontal stripes of thickness $\frac{1}{n}$ and take an horizontal segment of length 1 in each of these stripes. Call E_n the union of these n segments. For all $n \in \mathbb{N}$, let T_n be the integral current defined by

$$T_n := \left[E_n, e_1, \frac{1}{n}\right],$$

where $e_1 := (1,0)$, as pictured in figure 7.11. A simple computation shows that

$$\mathbb{M}(T_n) = 1, \quad \mathbb{M}(\partial T_n) = 2 \ \forall n \in \mathbb{N}.$$

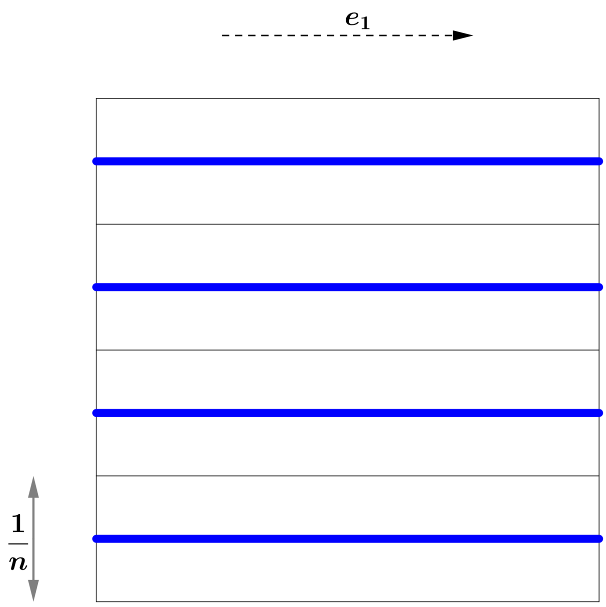

Figure 7.11: The set E_n, highlighted in blue, gives a sequence of 1-currents with non-integral multiplicity that converges to a non-rectifiable 1-current.

Moreover, as in the example 7.3.42, it is possible to show that $(T_n)_n$ converges in the sense of currents to the 1-current $T := e_1 \cdot \mathscr{L}^2 \llcorner Q$, which is not a rectifiable 1-current. Hence, the assumption of integral multiplicity in theorem 7.3.37 is really needed.

Example 7.3.44. In the plane, consider the square $Q := [0,1]^2$; for all $n \in \mathbb{N}$ divide Q in n^2 squares of side-length $\frac{1}{n}$; in each of this small squares, pick a circle $D_{i,n}$ of radius $\frac{1}{2n^2}$; set E_n to be the union of the n^2 circumferences given by $\partial D_{i,n}$ for $i = 1, \dots, n^2$. Let T_n be the integral 1-current in $\mathbb{R}^2$ given by

$$T_n := [E_n, \tau, 1],$$

where τ is the unitary vector field that denotes the fact that each circumferences is oriented counterclockwise (see figure 7.12). It is immediate to compute that

$$\mathbb{M}(T_n) = \pi, \quad \mathbb{M}(\partial T_n) = 0 \ \forall n \in \mathbb{N}.$$

In deed, by Stokes' theorem, there holds that $\partial T_n = 0$ for all $n \in \mathbb{N}$. In particular, Federer-Fleming theorem applies. However, we can show with a straightforward computation that $(T_n)_n$ converges to 0 in the sense of currents. In deed, given $\omega \in \mathcal{D}^1(\mathbb{R}^2)$,

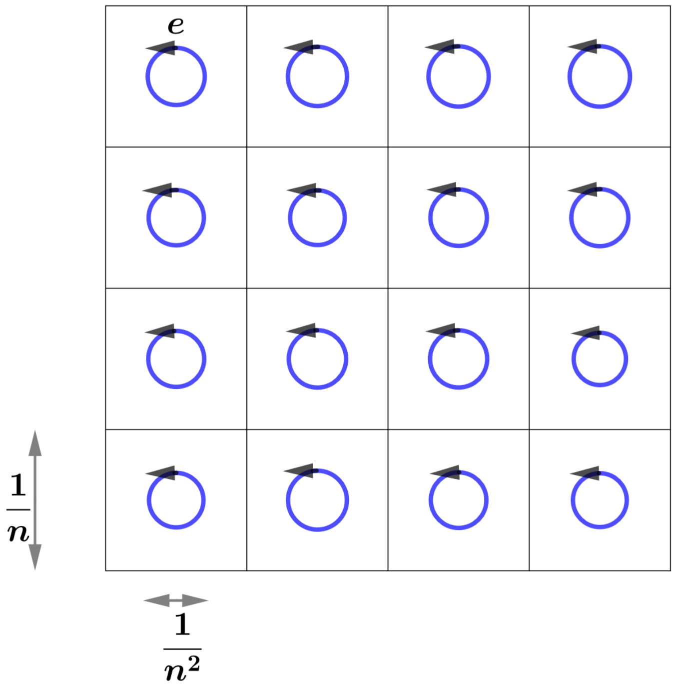

Figure 7.12: The set E_n, highlighted in blue, gives a sequence of integral 1-currents that converges to the current that vanishes identically.

we can apply Stokes' theorem (see 7.2.30) to obtain

$$\begin{aligned}
|< T_n, \omega >| &= \left| \sum_{i=1}^{n^2} \int_{\partial D_{i,n}} < \omega(x), \tau(x) > \; d\mathcal{H}^1(x) \right| \\
&= \left| \sum_{i=1}^{n^2} \int_{\partial D_{i,n}} \omega \right| \\
&= \\
&= \left| \sum_{i=1}^{n^2} \int_{D_{i,n}} d\omega \right| \\
&\leq \sum_{i=1}^{n^2} \int_{D_{i,n}} |d\omega| \, d\mathcal{H}^2 \\
&\leq \|\omega\|_{C_1} \, n^2 \mathcal{H}^2 \left(B_{\frac{1}{2n^2}} \right) \\
&= \|\omega\|_{C_1} \, \frac{\pi}{4n^2}.
\end{aligned}$$

Taking the limit as $n \to +\infty$, we conclude that $(T_n)_n$ converges to 0 in the sense of currents.

Here, we state without proof another fundamental result in the theory of Currents.

Theorem 7.3.45 (Boundary rectifiability). *Let T be a rectifiable current with integral multiplicity s.t. $\mathbb{M}(\partial T)$ is finite. Then, ∂T is rectifiable with integral multiplicity. In particular, T is an integral current.*

Remark 7.3.46. The proof of theorem 7.3.45 is very hard. Moreover, the assumption $\mathbb{M}(\partial T) < +\infty$ is not trivial, in the sense that there exists a rectifiable current T with integral multiplicity s.t. $\mathbb{M}(\partial T) = +\infty$. For instance, consider in $\mathbb{R}$ the 1-current defined by

$$T := [E, e, 1],$$

where e is the standard orientation or $\mathbb{R}$ (namely $e = 1$) and

$$E := \bigcup_{n \in \mathbb{N}} \left[\frac{1}{2 \cdot 4^n}, \frac{1}{4^n} \right].$$

Then, T is 1-rectifiable, but $\mathbb{M}(\partial T) = +\infty$.

Having in mind boundary rectifiability theorem (see 7.3.45), Federer-Fleming theorem (see 7.3.37) can be restated as follows.

Theorem 7.3.47. *Let $(T_n)_n$ be a sequence of rectifiable k-currents in $\mathbb{R}^n$, with $0 \leq k \leq n$. Assume that*

$$\sup_n \mathbb{M}(T_n) + \sup_n \mathbb{M}(\partial T_n) < +\infty.$$

Then, up to subsequences, $(T_n)_n$ converges to an integral k-current T in the sense of currents.

A significant improvement of the Federer-Fleming theorem is the following result, which is very hard to prove.

Theorem 7.3.48 (Ambrosio-Kirchneim, Jerrard)**.** *Let (T_n) be a sequence of rectifiable k-currents in $\mathbb{R}^n$ with multiplicities (m_n). Assume that*

$$\inf_n |m_n| \geq \delta > 0, \quad \sup_n \mathbb{M}(T_n) + \mathbb{M}(\partial T_n) < +\infty.$$

Then, up to subsequences, $(T_n)_n$ converges in the sense of currents to a rectifiable k-current T with multiplicity m s.t. $|m| \geq \delta$.

7.3.4 Approximation of currents

Definition 7.3.49 (k-polyhedral current)**.** A k-polyhedral current (or chain) in $\mathbb{R}^n$ is a current of the form

$$T := \sum_{i=1}^{N} [S_i, \tau_i, m_i],$$

where

- S_i is a k-dimensional simplex in $\mathbb{R}^n$,
- τ_i is a constant orientation of S_i,
- m_i is a constant multiplicity in $\mathbb{R}$ or in $\mathbb{Z}$.

If the multiplicities take values in $\mathbb{Z}$ or $\mathbb{R}$ we say that T is a real or integral polyhedral chain.

The following statement holds true; the proof is hard.

Theorem 7.3.50. *If T is an integral current in $\mathbb{R}^d$, then there exists a sequence $(T_n)_n$ of integral polyhedral chains s.t.*

- $T_n \overset{*}{\rightharpoonup} T$ *and* $\partial T_n \overset{*}{\rightharpoonup} \partial T$ *in the sense of currents,*
- $\mathbb{M}(T_n) \to \mathbb{M}(T)$ *and* $\mathbb{M}(\partial T_n) \to \mathbb{M}(\partial T)$.

If T is only a normal current, the same holds with $(T_n)_n$ a sequence of real polyhedral chains.

Remark 7.3.51. In the theorem 7.3.50 (when dealing with normal currents) it is crucial that $\mathbb{M}$ is defined with mass and comass norm for k-vectors and k-covectors (non with the euclidean norm). For example, if $T = \tau \cdot \mathscr{L}^n \llcorner [0,1]^n$ and τ is a constant, non-simple k-vector ($2 \leq k \leq n-2$), to construct τ by hands, we need to write $\tau = \sum \lambda_i \tau_i$, where τ_i are simple and $|\tau| \approx \sum \lambda_i |\tau_i|$.

Remark 7.3.52. In this theory, the notion of mass generalizes the volume with multiplicity of polyhedral chains, which is not the volume.

Remark 7.3.53. The approximation result 7.3.50 can be improved in many ways: for instance, if $\partial T = 0$, we can additionally require that $\partial T_n = 0$ for all n. In general, we can also require that T_n is cobordant to T, that is $T_n - T$ is the boundary of a current U.

Bibliography

[1] Falconer, K. J. *The Geometry of Fractal Sets*. Cambridge Tracts in Mathematics. Cambridge University Press, 1985.

[2] Federer, H. *Geometric Measure Theory*. Classics in Mathematics. Springer Berlin Heidelberg, 2014.

[3] HUTCHINSON, J. E. Fractals and self similarity. *Indiana University Mathematics Journal 30*, 5 (1981), 713–747.

[4] Maggi, F. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012.

[5] Mattila, P. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[6] Rudin, W. *Real and Complex Analysis*. Mathematics series. McGraw-Hill, 1987.

[7] Weil, A. *L'intégration dans les groupes topologiques et ses applications*. Actualités scientifiques et industrielles. Edwards brothers, Incorporated, 1942.