Viewpoints on stability and forking

A micro-course for the intrigued

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Wwu Münster
Shorth Model Theory Huddle 2

17th June 2021

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- . . . and if you feel like it, please put/leave your camera on. It feels less like talking to a screen :)

Plan of the talk

Why stability?

Structure from scarcity

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In fact, it is enough to check for $n = 1$. But it is not enough to check on one A of size λ, even if it is a model.

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There are so few types $\Leftrightarrow$ there is a good reason for it $\Leftrightarrow$ there are *many* such.

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A list of stable theories

These theories/structures are stable: (a structure is stable iff its theory is)

- Any number of infinitely cross-cutting equivalence relations.

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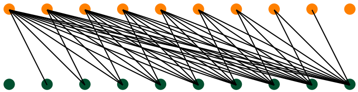

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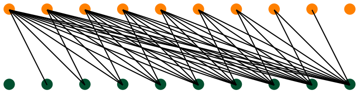

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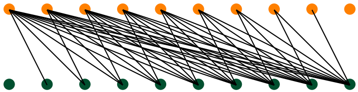

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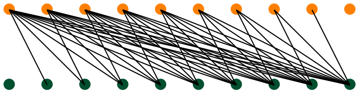

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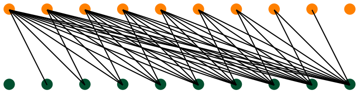

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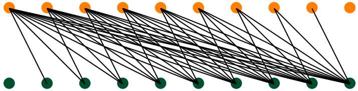

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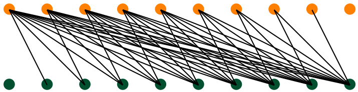

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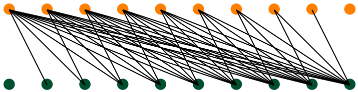

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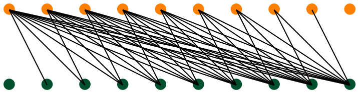

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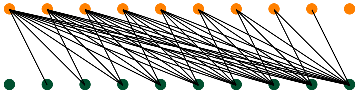

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Instability is usually easier to prove than stability. By the second-last point (more on this later).

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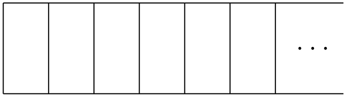

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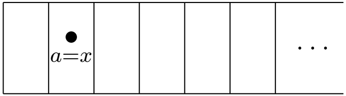

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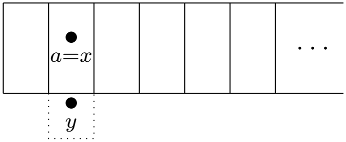

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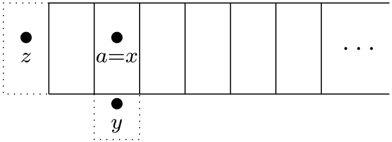

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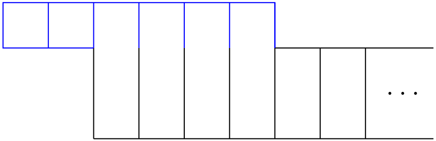

Enter $B \supseteq M$.

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T is complete with q.e. Fix $M \vDash T$. There are three kinds of types in $S_1(M)$:

- Realised: $r_a(x) \equiv \{x = a\}$.
- New point in old equivalence class:
  $q_a(y) \equiv \{E(y, a)\} \cup \{y \neq d \mid d \in M\}$
- Point in new equivalence class:
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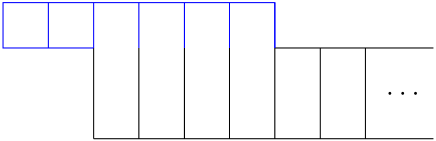

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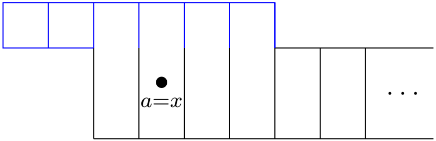

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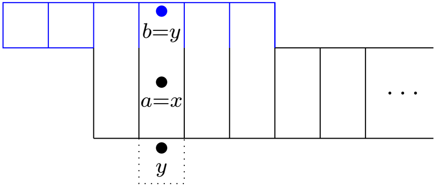

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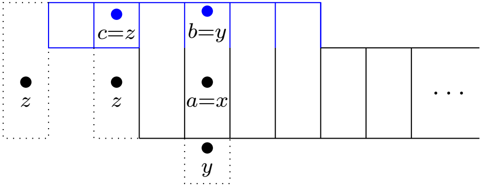

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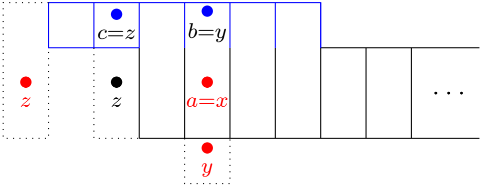

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Note how some choices seem to "preserve the spirit" of the original type.

Algebraically closed fields of characteristic 0

Fix $M \models \mathsf{ACF}_0$, and let $p(x_0, x_1) \in S_2(M)$ say that x_0, x_1 are not in M and algebraically independent over M.

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- p_1, p_2 have "small" sets, e.g. the line $x_0 - x_1 = b$.

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Moreover, if T is stable, the unique heir and coheir of $p \in S_n(M)$ to B coincide.

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Note that if p is definable then it is so over some A of size $|A| \leq |T|$.

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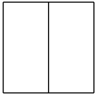

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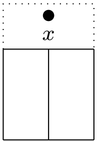

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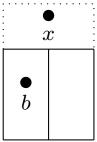

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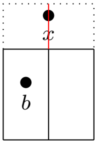

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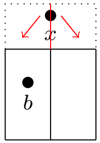

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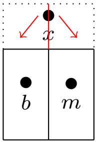

A toy example

[spoiler alert] it is less of a toy than you may expect

Naive idea: “nice extensions” are those which don’t represent more formulas.
It’s not that easy. $T :=$ “E is an equivalence relation with 2 classes, both infinite”.

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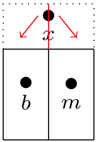

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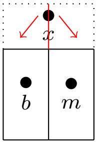

For the unique $p(x) \in S_1(\emptyset)$ in $(\mathbb{Q}, <)$, we still have two extensions to $S_1(\mathbb{Q})$ representing as few formulas as possible: $\mathrm{tp}(-\infty/\mathbb{Q})$ and $\mathrm{tp}(+\infty/\mathbb{Q})$. But the represented formulas are not the same! ($\{x < w\}$ and $\{x > w\}$).

Passing to arbitrary bases, for real

Theorem (of the bound)

Let $A \subseteq M$ and $p(x) \in S_n(A)$.

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Bonus info: for topological reasons, there are either finitely many or at least $2^{\aleph_0}$.

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Then $\sqsubset$ equals nonforking.

The forking ideal

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Definition

A formula $\varphi(x, d)$ *divides over* A iff there is an A-indiscernible sequence $(d^i)_{i<\omega}$ with $d = d^0$ such that $\{\varphi(x, d^i) \mid i < \omega\}$ is inconsistent.

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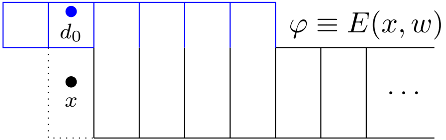

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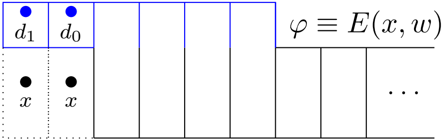

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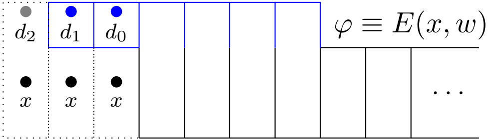

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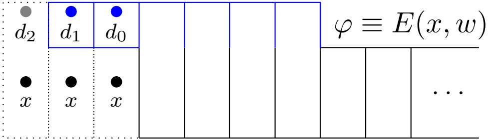

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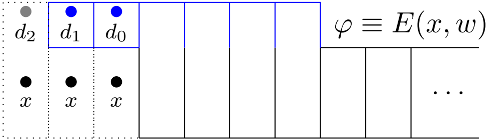

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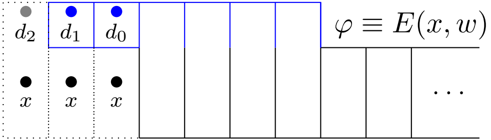

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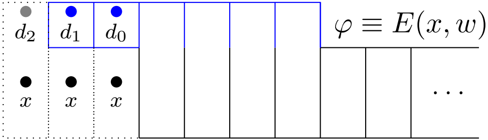

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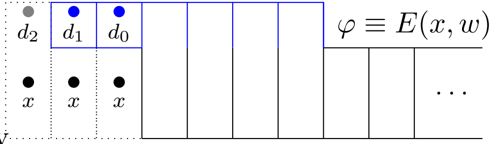

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This idea can be adapted to the general stable case, but one needs a *family* of ranks:
instead of just one rank R, one has a rank R_Δ for every finite family of formulas Δ.
An extension forks iff at least one of the ranks drops. (so this is another way to define forking)

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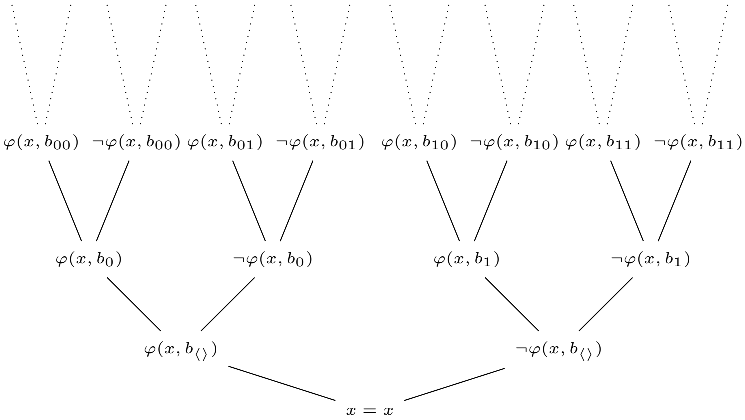

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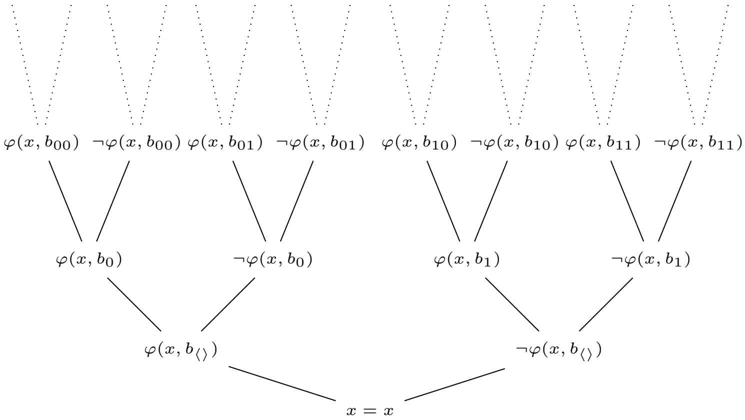

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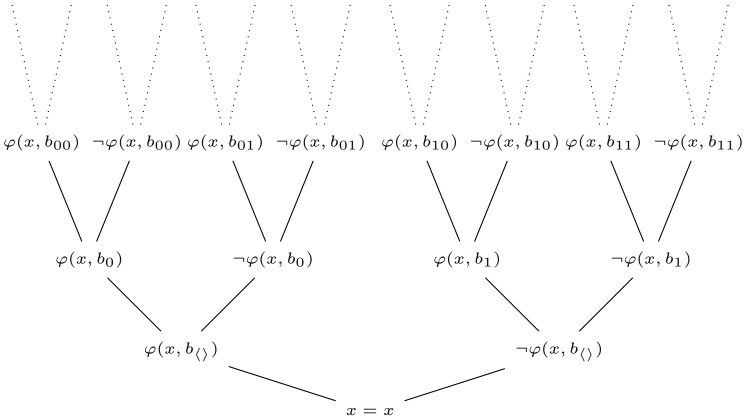

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(which is in turn clearly related to the number of types)

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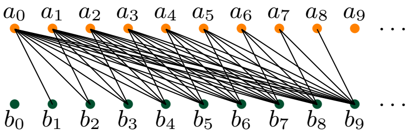

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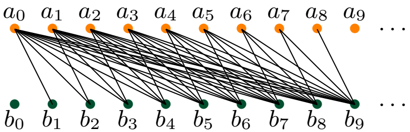

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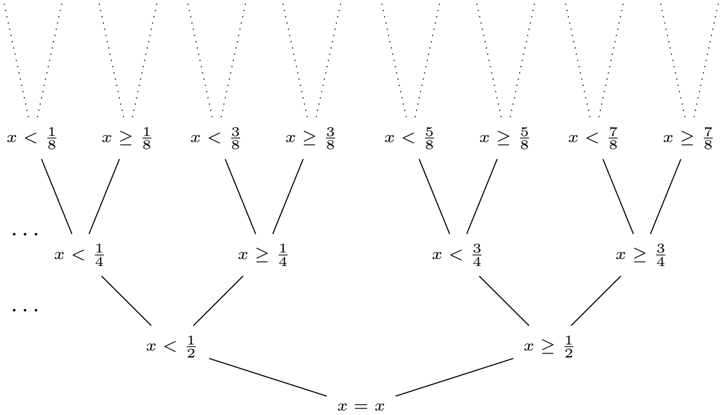

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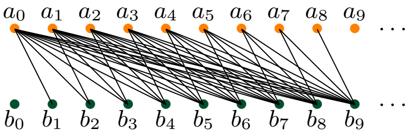

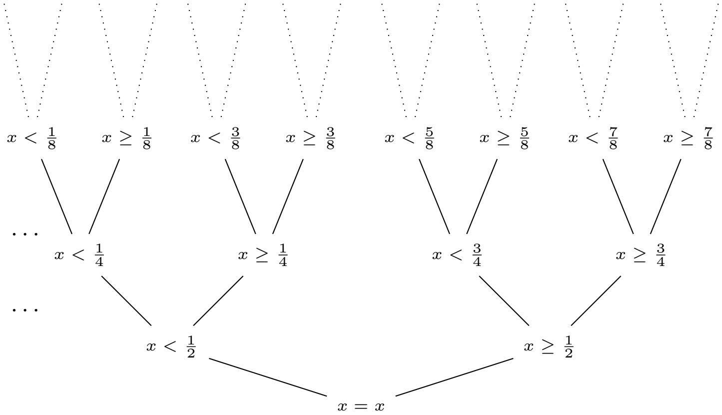

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These definitions are *local*: we may talk of stable/unstable *formulas* (and types) in arbitrary theories (T is stable iff it has no unstable formulas).

(also, this is the reason why T is stable iff every indiscernible *sequence* is an indiscernible *set*)

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- In planar graphs, $a \underset{C}{\downarrow} b$ iff every path from a to b goes through $\operatorname{acl}(C)$.

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Theorem (Hrushovski)

Mordell–Lang for function fields. (a finiteness result in algebraic geometry)

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- *Rosy* theories (includes simple and o-minimal): theories with an independence notion with certain nice properties.

Beyond stability

Many interesting theories are unstable. And a lot of recent model-theoretic research concerns generalising methods from stable theories to other classes.

A quick list (with no presumption of exhaustivity):

- *Simple* theories: nonforking independence (defined via dividing) still behaves well. Something is lost, e.g. stationarity over models. Examples: the Random Graph, pseudofinite fields, algebraically closed fields with generic automorphism.
- NIP theories: a generalisation orthogonal to simplicity (stable=NIP+simple). Good behaviour of measures on spaces of types. Examples: all o-minimal theories (e.g. the exponential real field $\mathbb{R}$), all ordered abelian groups, algebraically closed valued fields, transseries, dense meet-trees.
- Plenty more classes: NSOP_1, NTP_2,... See Conant's `forkinganddividing.com`.
- *Rosy* theories (includes simple and o-minimal): theories with an independence notion with certain nice properties.
- *Continuous structures*: stability (and more) can be generalised in the setting of *continuous logic*. For example, Hilbert spaces are stable.

Where to read more?

"Introductions": (all contain way more than just an introduction)

- Baldwin, *Fundamentals of Stability Theory*.
- Buechler, *Essential Stability Theory*.
- Lascar, *Stability in Model Theory*.
- Pillay, *An Introduction to Stability Theory*.
- Poizat, *A Course in Model Theory*.
- Tent–Ziegler, *A Course in Model Theory*.

Applications and more advanced material: (most also contain an introduction)

- Bouscaren et al, *Model Theory and Algebraic Geometry*.
- Marker et al, *Model Theory of Fields*.
- Pillay, *Geometric Stability Theory*.
- Poizat, *Stable Groups*.
- Shelah, *Classification Theory*.
- Wagner, *Stable Groups*.

Beyond stability:

- Casanovas, *Simple Theories and Hyperimaginaries*.
- Kim, *Simplicity Theory*.
- Simon, *A guide to NIP Theories*.
- Wagner, *Simple Theories*.

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Thanks for listening!