Logica Matematica 2023/2024
Partial lecture notes

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Contents

Readme

What is this? This document contains **a part** of the lecture notes for a course in mathematical logic, held in the spring of 2024 at the Università di Pisa by Marcello Mamino and myself. More specifically, it contains the notes relative to the lectures given by me.

What is this not?

Sound. These are lecture notes that never underwent a proper editorial process, hence they are likely to contain mistakes, typos, etc. If you spot any, please write me at the email address at page vi.

Complete. These notes **do not contain the whole material covered in the course**. For the rest, see the official page of the latter: (`https://ciovil.li/logica23/`).

Original work. Clearly nothing in these notes is an original result of the author. The exposition also owes debts to multiple sources, notably course notes from previous iterations of this course by Alessandro Berarducci [Ber17].

Are the contents of a chapter exactly what was covered in the lecture held on the day in the chapter title? No. They are that modulo having different[1] errors, different explanatory paragraphs, more exercises, etc.

Are these notes going to change much? Besides fixing possible mistakes pointed out by readers, there is no current plan to modify this document.

I have a previous version of these notes. How do I know what changed? If the `pdf`s are not too different, you may be lucky running `diffpdf`. Otherwise, fetch the source files (see below) and use your favourite program. I recommend `latexdiff` or `ediff` (a function in `emacs`).

Perché questi appunti sono in inglese? Perché aumenta il numero di persone che possono leggerli, e perché in questo modo posso riciclare materiale che ho già scritto nella parte introduttiva di [Men22]. Probabilmente hai già dovuto studiare su materiale in inglese per altri corsi, quindi spero che questa decisione non sia troppo traumatica. Se lo è, come scritto sopra ci sono delle dispense [Ber17] in italiano di un corso tenuto qualche anno fa. I contenuti però

[1] I would really like to say "fewer", but Murphy's law has no mercy.

non sono esattamente gli stessi, e in particolare il sistema di deduzione naturale è leggermente diverso. Comunque, di solito, a un certo punto del corso di laurea si arriva a un momento in cui il materiale in italiano semplicemente non esiste; se per te questo momento non è ancora arrivato, leggere questi appunti potrebbe essere un buon riscaldamento.

Ma poi non so come si chiamano i teoremi in italiano! Cercherò di aggiungere note con il nome italiano quando questo può essere difficile da indovinare. Se me ne dimentico, o la traduzione è quella ovvia, oppure la mia mail è a pagina vi.

Why do pages move left and right? Because you are looking at a digital copy, but this version is made to be printed. If you really don't like this, just recompile with `twoside` replaced by `oneside` (see below for the source files).

Acknowledgements Thanks to Marcello Mamino for help in the planning of these notes, and for feedback on drafts. To Vincenzo Mantova, for a discussion leading to Example 3.1.1 and the sentence above it. I am also grateful to Pietro Lorenzo Bianchi, Massimo Gasparini, Nora Líf Masi, Mirko Taibi, and a number of other students taking the course, for pointing out mistakes in previous versions of, as well as various questions and comments that resulted in improvements to, this document. If you recognise a contribution of yours (even if very implicit), and would like your name to appear here, please let me know at the email address at page vi.

Info You can contact me at `R.Mennuni@posteo.net`. This version has been compiled on the 5th June 2024. To get the source code click on the leftmost paper clip. The bibliography source file is in the rightmost one.

Rosario Mennuni

Chapter 1

19/03

1.1 Of things to come

In the previous lectures you have seen the formal definitions of terms, formulas, etc, in first-order logic. You have also seen the definitions of first-order structure and of Tarski semantics, that is, what it means for a structure M together with a valuation v of the variables to satisfy a formula φ, in symbols, $M \vDash \{v\}_M \varphi$.

The next step is to formalise the notion of *proof*. More precisely, we want a *proof system*, that is, a system in which one can code proofs as (something that can ultimately be coded with) strings. There are many proof systems around, each with its own advantages and disadvantages. Some of these have "simple compilers" at the price of "having to write complicated code". In other words, some proof systems (that we are not going to see) have desirable technical properties, which allow to prove easily certain theorems *about the system*, but the drawback is that actually writing[1] proofs *in the system* is more difficult.

The one we are about to look at is called *Natural Deduction*, which we are going to abbreviate as ND. It is called that way because it is designed in such a way as to make it easy (relative to some other proof systems) to code an argument expressed in natural language into a proof "accepted by the system", so to speak. The nuts and bolts of the system are going to be *rules*, which will typically look like this:

$$\frac{\text{(assume that) from } T_0 \text{ you prove } \varphi_0 \qquad \text{(and that) from } T_1 \text{ you prove } \varphi_1}{\text{(then) from } T_2 \text{ you prove } \varphi_2}$$

We will use the standard symbol for "proves", namely $\vdash$. Hence, things like the above will be written more compactly as follows:

$$\frac{T_0 \vdash \varphi_0 \qquad T_1 \vdash \varphi_1}{T_2 \vdash \varphi_2} \tag{1.1}$$

So a rule has two parts: a finite set[2] of *assumptions* (in the example above, $T_0 \vdash \varphi_0$ and $T_1 \vdash \varphi_1$) and a single *conclusion* (in the example above, $T_2 \vdash \varphi_2$).

[1] ..., and even convincing yourself that you would in principle be able to write...

[2] Well, really a *labelled* set. More on this later.

The rules are going to code the most basic steps in proofs, stuff like "if T proves φ, and also proves ψ, then it proves $\varphi \wedge \psi$". Some rules will have two assumptions, as in (1.1) above, some will have more, and some will have less. In particular, a couple of rules will have zero: they will correspond to the very most trivial statements from which a proof starts[3]. Stuff like:

$$\frac{\text{(assume nothing)}}{\text{(then) from } \varphi \text{ you prove } \varphi}$$

We can then say that a proof of φ from T is a finite "something"[4] that starts by a bunch of rules with zero assumptions, uses their conclusions as assumptions for other rules, which in turn give conclusions that are used as assumptions for other rules, etc., and ends with a single rule whose conclusion is $T \vdash \varphi$.

All of this can be taken as a(n inductive) definition of the relation $T \vdash \varphi$, that can be though of as "there is a proof of φ from T". Perhaps we should write $\vdash_{\mathrm{ND}}$ in order to make it clear that this is provability in Natural Deduction, as opposed to provability in a different system, but at least for now we will stick to simply writing $\vdash$ in order to keep the notation lighter. Just remember that "$\vdash$" is relative to a fixed system, while this is not true for "$\vDash$", whose very definition only depends on formulas, structures, and valuations.

After defining a system, it is common courtesy to also show that it actually works. What does it mean for a proof system to work? Or, in other words, what is a proof? A proof of φ from T may be thought of as a certificate that whenever T holds, then φ must also hold. This is written as $T \vDash \varphi$, which is formally defined as: whenever M is an L-structure and v a valuation of the variables, if $M \vDash \{v\}_M T$ then $M \vDash \{v\}_M \varphi$.

Therefore, in order to be able to claim that our system *works*, at the very least we have to make sure that system is *sound*, i.e. that whatever φ it proves from T is actually a consequence of T; in symbols, that $T \vdash \varphi \implies T \vDash \varphi$. This is the content of the *Soundness Theorem*[5].

Now, having a sound system is easy: just ~~buy a portable bluetooth spea~~ take a system that cannot prove anything. Of course, if our goal is to formally represent actual proofs like the ones you can find in a math book, then such a system is entirely useless; it would be more interesting if we could prove soundness for a system that actually allows us to produce enough proofs. In particular: does Natural Deduction allow us to "certify" with a proof *all* consequences of a theory T? In the *Completeness Theorem*[6] we will show that the answer is positive; more precisely, that $T \vDash \varphi \implies T \vdash \varphi$. In other words, we will prove that, if every model of T must be a model of φ, then there is formal proof of φ from T in Natural Deduction. Note that, by definition of soundness, this is the best, in terms of amount of things you can prove, that one can ask from a sound system.

Why bother doing all this? One reason is practical: $\vdash$ is usually more convenient to use when we want to show that some φ follows from T, while $\vDash$ tends to be easier to use to show that φ does *not* follow from T. Another

[3]Or "ends", depending on taste.

[4]For the impatient: for "something" in $\{\text{labelled tree}, \text{sequence}\}$. Both points of view are useful, and one can go from one to the other. The impatient reader of this footnote may also want to observe that both (finite) labelled trees and (finite) sequences can be coded as strings.

[5]In italiano, *Teorema di Correttezza.*

[6]In italiano, *Teorema di Completezza.*

technical reason, which will come handy later on in the course when you will see the Incompleteness Theorems, is that the definition of $\models$, say for a fixed language L, quantifies on *all L-structures*, which form a proper class, while the definition of $\vdash$ (that we have not seen yet) only quantifies about the way smaller collection of *proofs*.[7]

WARNING To some people, the next few sections may seem obvious and boring. To be fair, to some extent they are. The catch is that, if the nitty-gritty details of certain things are clear to you *now*, you have good chances of having an easier time further down the road.

1.2 Natural Deduction rules

Natural Deduction operates on all formulas of first-order logic with equality. The rules will therefore need to specify how to deal with all of $\top, \bot, \wedge, \vee, \rightarrow, \neg, \forall, \exists, =$. Most of these symbols are going to require more than one rule in order to function properly: for example, we need a rule to be able to deduce something of the form $\varphi \wedge \psi$ —a so-called *introduction rule*— but we also need rules to make use of $\varphi \wedge \psi$ further down our proofs, that is, we also need what are known as *elimination rules*.

Most of the rules below will be of one of these two forms, with three exceptions: a rule called Reductio ad Absurdum, corresponding to arguments by contradiction; a rule called *weakening*, saying that $\vdash$ is monotone in its left argument (that is, that assuming more does not decrease the number of provable things), and an *axiom* rule, saying that each formula proves itself.

Now let us recall some notation, and then look at the full list of rules.

Notation 1.2.1. We use the following notational conventions.

1. Fix a first-order vocabulary L.
2. s, t range over L-terms.
3. φ, ψ, θ range over first-order L-formulas.
4. T, T' range over L-theories.
5. Commas denote unions and formulas denote their singletons whenever appropriate; for instance, T, φ denotes $T \cup \{\varphi\}$.
6. The set of free variables of t is denoted by $\mathrm{fv}(t)$; similarly for sets of free variables of φ and of T.
7. $\varphi[t/x_k]$ is only defined when t is a term that can legally be substituted in place of the variable x_k, in which case it denotes the result of the substitution. In other words, when we write $\varphi[t/x_k]$ we are also saying that no free variable of t will be captured by the substitution.

[7] Again for the impatient: once certain theorems to be proven later in these notes are available, in certain uses of $\models$ one will be able to avoid quantifying over a proper class, by quantifying only over structures of bounded cardinality. But coding infinite structures is still more complicated than coding finite strings.

Definition 1.2.2. The *rules of natural deduction* (ND) are:

$$\text{(Ax)}\ \frac{}{\varphi \vdash \varphi} \qquad \text{(Wk) For } T \subseteq T'\ \frac{T \vdash \varphi}{T' \vdash \varphi}$$

$$(\mathrm{In}_\top)\ \frac{}{\vdash \top} \qquad (\mathrm{El}_\bot)\ \frac{T \vdash \bot}{T \vdash \varphi}$$

$$(\mathrm{In}_\wedge)\ \frac{T \vdash \varphi \qquad T \vdash \psi}{T \vdash \varphi \wedge \psi} \qquad (\mathrm{El}_{\wedge.})\ \frac{T \vdash \varphi \wedge \psi}{T \vdash \varphi} \qquad (\mathrm{El}_{.\wedge})\ \frac{T \vdash \varphi \wedge \psi}{T \vdash \psi}$$

$$(\mathrm{In}_{\vee.})\ \frac{T \vdash \varphi}{T \vdash \varphi \vee \psi} \qquad (\mathrm{In}_{.\vee})\ \frac{T \vdash \psi}{T \vdash \varphi \vee \psi} \qquad (\mathrm{El}_\vee)\ \frac{T \vdash \varphi \vee \psi \qquad T, \varphi \vdash \theta \qquad T, \psi \vdash \theta}{T \vdash \theta}$$

$$(\mathrm{In}_\to)\ \frac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi} \qquad (\mathrm{El}_\to)\ \frac{T \vdash \varphi \to \psi \qquad T \vdash \varphi}{T \vdash \psi}$$

$$(\mathrm{In}_\neg)\ \frac{T, \varphi \vdash \bot}{T \vdash \neg\varphi} \qquad (\mathrm{El}_\neg)\ \frac{T \vdash \neg\varphi \qquad T \vdash \varphi}{T \vdash \psi} \qquad \text{(RaA)}\ \frac{T, \neg\varphi \vdash \bot}{T \vdash \varphi}$$

$$(\mathrm{In}_\forall) \text{ For } x_k \notin \mathrm{fv}(T)\ \frac{T \vdash \varphi}{T \vdash \forall x_k\, \varphi} \qquad (\mathrm{El}_\forall)\ \frac{T \vdash \forall x_k\, \varphi}{T \vdash \varphi[t/x_k]}$$

$$(\mathrm{In}_\exists)\ \frac{T \vdash \varphi[t/x_k]}{T \vdash \exists x_k\, \varphi} \qquad (\mathrm{El}_\exists) \text{ For } x_k \notin \mathrm{fv}(T, \psi)\ \frac{T \vdash \exists x_k\, \varphi \qquad T, \varphi \vdash \psi}{T \vdash \psi}$$

$$(\mathrm{In}_=)\ \frac{}{\vdash t = t} \qquad (\mathrm{El}_=)\ \frac{T \vdash s = t \qquad T \vdash \varphi[s/x_k]}{T \vdash \varphi[t/x_k]}$$

When we speak of an *instance* of a rule, we mean the obvious thing: for instance[8], the following is an instance of Ax:

$$\frac{}{x = x \vdash x = x}$$

We will talk of the *assumption* and of the *conclusion* of a rule: they are respectively what's above and below the large horizontal line. Sometimes I may be sloppy and talk of *assumptions* of a rule to mean an element of its assumption, e.g. by saying that $T, \varphi \vdash \theta$ is an assumption of $\mathrm{El}_\vee$. In fact, I have already done it some pages ago. Note that this ambiguity does not apply to conclusions, because every rule has only one thing of the form $T \vdash \varphi$ below the horizontal rule. By way, a "thing of that form" is called a *sequent*[9]. Let me stress that ~~the assumption is~~ we will treat the assumption as a *set* of sequents: the order of the assumptions does not matter. If we want to be extremely precise, the assumption really is a (finite) *labelled set*: a function from a (finite) set to sequents. This is needed in order to deal with some edge cases (for example, think of instances of $\mathrm{In}_\wedge$ where $\varphi = \psi$). Notationally, we will still treat the assumption A of a rule as if it were a set, and write e.g. $(T \vdash \varphi) \in A$ instead

[8] Pun intended.

[9] Well, to be precise it is a special case of a sequent, but all sequents in these notes (and I suspect also in this course) are going to be of the kind above.

of $(T \vdash \varphi) \in A(\mathrm{Set}(A))$, in order not to overburden our notation. But again, if you want full precision, they are labelled sets.

These rules correspond to very basic steps in proofs. For instance, a step like "...since we proved that $g \cdot g = e$, and g was arbitrary, we deduce that every element of G has order 2" will be formalised by using ($\mathrm{In}_\forall$). The restriction "For $x_k \notin \mathrm{fv}(T)$" in the rule corresponds precisely to the "and g was arbitrary" part of the informal argument. Let us take a closer look.

Remark 1.2.3. The rules $\mathrm{In}_\forall$ and $\mathrm{El}_\exists$ have a restriction on where the variable x_k can appear. This is essential, because without it we would be able to prove silly (read: false) things from true ones. For example, let $L_{\mathrm{ab}} \coloneqq \{+, 0, -\}$ be the language of abelian groups, and let T_{ab} be the L_{ab}-theory of abelian groups. If $\mathrm{In}_\forall$ did not have that restriction, then it would have an instance allowing us to start with $T_{\mathrm{ab}}, x_0 = 0 \vdash x_0 = 0$, which intuitively should be true, and deduce $T_{\mathrm{ab}}, x_0 = 0 \vdash \forall x_0\ (x_0 = 0)$, which intuitively should be false.

Let us look even closer. Recall that we want (and will prove) the equivalence of $T \vdash \varphi$ with $T \vDash \varphi$, and that the latter means that for all M and v we have $M \vDash \{v\}_M T \implies M \vDash \{v\}_M \varphi$. Suppose, to fix ideas, $\mathrm{fv}(T) = \mathrm{fv}(\varphi) = \{x\}$. If T mentions x, then having $M \vDash \{v\}_M T$ subjects x to some special constraints. In other words, with the same M but a different v' (one with $v'(x) \neq v(x)$) it may happen that $M \nvDash \{v'\}_M T$. On the other hand, whether $M \vDash \{v\}_M \forall x\ \varphi$ or not does not depend on the choice of $v(x)$, because the quantifier $\forall$ is "hiding" x from the scope of v. Hence, dropping the restriction from $\mathrm{In}_\forall$ would amount to saying that a property that may hold or not depending on v implies one that ignores v. At this level of generality, this smells bad, and in fact we already saw in Remark 1.2.3 that the reason it smells bad is that it is rotten.

What the rules $\mathrm{In}_\forall$ and $\mathrm{El}_\exists$ are telling us is that we should think of free variables not mentioned by T as universally quantified upstream when they appear on the right of $\vdash$, and as existentially quantified upstream when they appear on its left. Observe that this interacts well with the fact that the rules $\mathrm{In}_\neg$ and RaA tell us that formulas can jump across $\vdash$ at the price of a negation. Another viewpoint is to think about free variables as new constants, cf. Lemma 2.3.4. From this perspective, the fact that having one right of $\vdash$ without having it left of $\vdash$ corresponds to a universal quantification follows from the fact that on this new constant we know nothing specific, hence whatever we prove of it should be true regardless of how it is interpreted. Yet another viewpoint is to think of free variables, even if mentioned by T, as quantified universally *before the sequent*, that is, upstream of $\vdash$. To see why this makes sense, it is useful to think of $\vdash$ as a sort of generalised implication.

Exercise 1.2.4. Come up with an example showing that the restriction on free variables in $\mathrm{El}_\exists$ is necessary. Why do we also need to require $x \notin \mathrm{fv}(\psi)$ there?

1.3 Natural Deduction proofs

We would like to say that "a proof is something obtained by gluing instances of rules in a sensible way". That is, we want a formal definition capturing the fact that a proof is something like in Figure 1.1. As evident from the figure, the

$$(\mathrm{In}_{\wedge})\ \dfrac{(\mathrm{El}_{.\wedge})\ \dfrac{(\mathrm{Ax})\ \dfrac{}{\varphi \wedge \psi \vdash \varphi \wedge \psi}}{\varphi \wedge \psi \vdash \psi} \qquad (\mathrm{El}_{\wedge .})\ \dfrac{(\mathrm{Ax})\ \dfrac{}{\varphi \wedge \psi \vdash \varphi \wedge \psi}}{\varphi \wedge \psi \vdash \varphi}}{\varphi \wedge \psi \vdash \psi \wedge \varphi}$$

Figure 1.1: A proof that conjunction is commutative.

fact that every rule has one conclusion and between 0 and 3 assumptions makes proofs look like a tree.[10] So let us define trees.

Definition 1.3.1. A partial order $(P, \leq)$ is a *tree* iff it is

1. *downward directed*: for every $p, p' \in P$, there is $q \in P$ such that $q \leq p$ and $q \leq p'$; and
2. *lower semilinear*[11]: for every $p \in P$ the restriction of $\leq$ to the set $\{q \in P \mid q \leq p\}$ is linear (i.e. total).

A maximal element of a tree is called a *leaf.* A minimal one is called a *root.* If $p_0 < p_2$ and there are no p_1 with $p_0 < p_1 < p_2$, we call p_2 an *immediate successor* of p_0, and call p_0 the[12] *immediate predecessor* of p_2.

Exercise 1.3.2. Every tree has at most one root.

Exercise 1.3.3. In a nonempty finite tree

1. there always exists a root;
2. every non-root element has (precisely) one immediate predecessor; and
3. every non-leaf has at least one immediate successor.

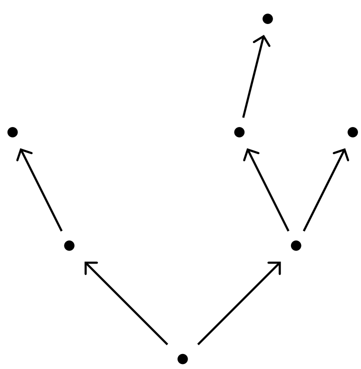

Figure 1.2: A finite tree. An arrow $p \to q$ denotes that p is the immediate predecessor of q.

Now we can define a proof to be a finite tree, labelled with instances of rules in a certain way.

[10] If you are not worrying about diamonds, stop reading this footnote. Otherwise, just observe that to use a lemma twice we can just copy its proof elsewhere in the tree.

[11] Lower semilinear orders are also called *forests.*

[12] See Exercise 1.3.2.

Definition 1.3.4. A *Natural Deduction proof* P of φ from T (or "of $T \vdash \varphi$") is a function from a finite tree $(\mathrm{Tree}(P), \leq_P)$ to the set of instances of rules of ND such that[13]

1. if $p \in \mathrm{Tree}(P)$ is a leaf of P, then $P(p)$ has no assumption; namely, it is an instance of either Ax, of $\mathrm{In}_\top$, or of $\mathrm{In}_=$;

2. if r is the root of $\mathrm{Tree}(P)$, then $P(r)$ has $T \vdash \varphi$ as its conclusion;

3. if $p \in \mathrm{Tree}(P)$ and $q_0, \ldots, q_n$ are its immediate successors, then the assumption of $P(p)$ is precisely the set of[14] conclusions of $P(q_0), \ldots, P(q_n)$.

We write $T \vdash_{\mathrm{ND}} \varphi$ to mean that there exists a natural deduction proof of φ from T. Whenever it is clear that the proof system we are referring to is Natural Deduction, we will drop the ND and just write $T \vdash \varphi$, and say that T *proves* φ, or that φ is *provable* from T.

Example 1.3.5. An easy (but useful) proof is depicted in Figure 1.1.

Remark 1.3.6. Figure 1.1 is, in an extremely pedantic sense, misleading: the labels on the tree are *instances of rules*, and not just sequents, although the picture may seem to suggest the latter. Clearly, it is graphically more convenient to glue a sequent in the assumption of an instance of a rule with the conclusion of the instance above it (since they are the same by definition of proof), so we will keep following this convention.

Example 1.3.7. To avoid forgetting that structures with empty domain are forbidden, Figure 1.3 depicts a proof of $\exists x\; x = x$ from the empty theory.

$$\dfrac{(\mathrm{In}_=)\ \dfrac{}{\vdash x = x}}{\vdash \exists x\; x = x}\ (\mathrm{In}_\exists)$$

Figure 1.3: A proof that the universe is nonempty.

At this point, we should convince ourselves that every informal proof can be formalised in this system. If you have seen a proof assistant before[15] this will probably not be a surprise for you. Otherwise, there are a number of ways to grasp what happens when one tries to formalise a proof.

One way is to ruminate on a proof while staring at Definition 1.3.4. A better way is to take a proof written in natural language and actually writing a labelled tree formalising it, like the ones above (except it is probably going to be quite higher and broader). If you are planning to do this, I would recommend waiting until the end of Section 1.4.

Perhaps a good way to start convincing ourselves that it is possible to formalise anything in this system is to start writing down a proof of a tautology from the empty theory.

[13]Depending on how we deal with trivial cases, point 1 can be seen as the special case of point 3 when there is no q_i.

[14]If you are being precise and translating "set" with "labelled set", here you should have an isomorphism of labelled sets, that is, a bijection of the underlying sets preserving labels. From now on I will stop commenting on such pedantries.

[15]E.g. if you have taken the Lean course last (as of March 2024) semester.

Example 1.3.8. A proof of the Law of Excluded Middle, i.e. $\vdash \varphi \vee \neg\varphi$, is represented in Figure 1.4.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(Ax)}
\UnaryInfC{$\neg\theta \vdash \neg\theta$}
\LeftLabel{(Wk)}
\UnaryInfC{$\neg\theta, \varphi \vdash \neg\theta$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\varphi \vdash \varphi$}
\RightLabel{(In$_{\vee.}$)}
\UnaryInfC{$\varphi \vdash \theta$}
\RightLabel{(Wk)}
\UnaryInfC{$\neg\theta, \varphi \vdash \theta$}
\LeftLabel{(El$_\neg$)}
\BinaryInfC{$\neg\theta, \varphi \vdash \bot$}
\LeftLabel{(In$_\neg$)}
\UnaryInfC{$\neg\theta \vdash \neg\varphi$}
\AxiomC{}
\LeftLabel{(Ax)}
\UnaryInfC{$\neg\theta \vdash \neg\theta$}
\LeftLabel{(Wk)}
\UnaryInfC{$\neg\theta, \neg\varphi \vdash \neg\theta$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\neg\varphi \vdash \neg\varphi$}
\RightLabel{(In$_{.\vee}$)}
\UnaryInfC{$\neg\varphi \vdash \theta$}
\RightLabel{(Wk)}
\UnaryInfC{$\neg\theta, \neg\varphi \vdash \theta$}
\RightLabel{(El$_\neg$)}
\BinaryInfC{$\neg\theta, \neg\varphi \vdash \bot$}
\RightLabel{(RaA)}
\UnaryInfC{$\neg\theta \vdash \varphi$}
\LeftLabel{(El$_\neg$)}
\BinaryInfC{$\neg\theta \vdash \bot$}
\LeftLabel{(RaA)}
\UnaryInfC{$\vdash \theta$}
\end{prooftree}
$$

Figure 1.4: A proof of the Law of Excluded Middle from the empty theory, writing θ in place of $\varphi \vee \neg\varphi$.

You have already seen a proof of the Compactness Theorem using ultrafilters. Therefore, if we take the Completeness Theorem for granted, Theorem 1.3.9 below will follow immediately. Nevertheless, the latter will be used in proving the Completeness Theorem, hence we give a direct proof below.

Theorem 1.3.9 (Syntactic Compactness)**.** If $T \vdash \varphi$, then there is a finite $T_\varphi \subseteq T$ such that $T_\varphi \vdash \varphi$.

Proof. Let P be a proof of $T \vdash \varphi$. We prove the conclusion by structural induction on finite trees; if you prefer, by induction on the height[16] of Tree(P). Let r be the root of Tree(P); in other words, it is the last[17] step of the proof. Let R be the rule of which $P(r)$ is an instance. We proceed by cases, depending on R.

The rule Ax only introduces one formula left of $\vdash$.

If R is one of rules Wk, In$_\top$, El$_\bot$, In$_\wedge$, El$_{\wedge.}$, El$_{.\wedge}$, In$_{\vee.}$, In$_{.\vee}$, El$_\to$, El$_\neg$, In$_\forall$, El$_\forall$, In$_\exists$, In$_=$, El$_=$, then R has the property that there is only one theory appearing to the left of the various $\vdash$ in its assumption, and it is included in[18] the theory left of the $\vdash$ in its conclusion. For each $T_i \vdash \varphi_i$ appearing the assumption of $P(r)$ we may replace, by induction, the subtree above it by a proof of $S_i \vdash \varphi_i$, where S_i is finite. We let T_φ be the union of the S_i and conclude, by applying enough instances of Wk and an instance of R, that $T_\varphi \vdash \varphi$.

The case where R is one of El$_\vee$, In$_\to$, In$_\neg$, RaA, El$_\exists$ is similar, but with a twist. These rules have a conclusion of the form $T \vdash \varphi$ and at least one assumption of the form $T, \psi_i \vdash \varphi_i$. When we apply the inductive assumption to T, ψ_i, we obtain a finite S_i such that $S_i \vdash \varphi_i$. This set may not contain ψ_i, but we simply add it back: namely, we take again as T_φ the union of the S_i, weaken each S_i to T_φ, ψ_i (or to be T_φ, if the i-th sequent in the assumption only had T left of $\vdash$), and conclude by an instance of R. □

[16]The height of a finite tree is the maximal cardinality of a totally ordered subset.
[17]Again, or first, depending on taste.
[18]In fact, except for Wk, the "included in" is actually an "equal to".

1.4 Larger systems

Definition 1.4.1. A rule

$$\frac{T_0 \vdash \varphi_0 \qquad \cdots \qquad T_{n-1} \vdash \varphi_{n-1}}{T \vdash \varphi}$$

is *admissibile* iff whenever, for all $i < n$, we have $T_i \vdash \varphi_i$, then $T \vdash \varphi$.

Note that the case $n = 0$ corresponds to rules with empty assumption, i.e. of the form $\overline{T \vdash \varphi}$. For example, the admissibility of the LEM rule $\overline{\vdash \varphi \vee \neg\varphi}$ follows from Figure 1.4.

If we know that a rule is admissible, then we can add it to ND and obtain a system, say S, with more rules, but such that $T \vdash_{\mathrm{ND}} \varphi \iff T \vdash_{\mathrm{S}} \varphi$; the proof of this is easy: one just replaces an instance of a new rule by a labelled tree obtained from the fact that the new rule is admissible, see e.g. the solution to Exercise 1.4.5 below.

This means having more proofs of the same things, some of which may be easier to write, some of which may be shorter, etc. Below are some exercises asking you to prove the admissibility of some rules that we will use later. Of course you can use any rule that has already been proven admissible, e.g. LEM.

Exercise 1.4.2. Prove that the "proof by cases" rule below is admissible.

$$\frac{T, \psi \vdash \varphi \qquad T, \neg\psi \vdash \varphi}{T \vdash \varphi}$$

Exercise 1.4.3. Prove that the following rule is admissible.

$$\frac{T, \varphi \to \psi \vdash \bot}{T, \varphi \vdash \neg\psi}$$

Exercise 1.4.4. Prove that the following rule is admissible.

$$\frac{T \vdash \exists x\, \varphi \qquad T \vdash \varphi[t/x] \to \psi}{T \vdash \psi}$$

Exercise 1.4.5. Prove that the following rule is admissible.

$$\frac{T, \varphi \vdash \bot \qquad T, \varphi \to \psi \vdash \bot}{T \vdash \bot}$$

Solution. By assumption, there are a proof of $T, \varphi \vdash \bot$ and a proof of $T, \varphi \to \psi \vdash \bot$. Glue them (in the correct place) to the labelled tree in Figure 1.5. □

Exercise 1.4.6. Take a proof from your favourite book/notes/whatever and formalise it in ND, for a suitable choice of L. If needed, expand the system by admissible rules (feel free to prove new ones).

$$\dfrac{(\mathrm{In}_\to)\,\dfrac{(\mathrm{El}_\bot)\,\dfrac{T, \varphi \vdash \bot}{T, \varphi \vdash \psi}}{T \vdash \varphi \to \psi} \qquad \dfrac{T, \varphi \to \psi \vdash \bot}{T \vdash (\varphi \to \psi) \to \bot}\,(\mathrm{In}_\to)}{T \vdash \bot}\,(\mathrm{El}_\to)$$

Figure 1.5: The core part of the solution to Exercise 1.4.5.

Chapter 2

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2.1 A smaller system

If you want to write a formal proof, the system ND is quite nice. The drawback is that, in order to prove statements about this system directly, we need to check that things go through for a lot of rules, cf. the proof of Theorem 1.3.9.

For this reason, we introduce a smaller system, with fewer logical symbols and rules. Although, in this system, writing proofs (in fact, even just coding the statements we want to prove) is more cumbersome, proving properties of the system will be easier. To take advantage of this, we will show that the "small" system is, in a certain precise sense, equivalent to the "large" one (ND), allowing to transfer to the latter some properties that we will prove of the former.

The smaller system operates exclusively on formulas whose only logical symbols are $\bot, \to, \exists, =$. It is obtained from ND by discarding all rules involving $\top, \wedge, \vee, \neg, \forall, =$, discarding[1] also $\mathrm{El}_\bot$, and adding back (RaA) in a modified form, where $\neg\varphi$ is replaced by $\varphi \to \bot$.

Definition 2.1.1. The *rules of* $\mathrm{ND}_{\bot,\to,\exists,=}$ are those in Figure 2.1. The notion of a proof in $\mathrm{ND}_{\bot,\to,\exists,=}$ is given by adapting Definition 1.3.4: namely, we replace ND by $\mathrm{ND}_{\bot,\to,\exists,=}$ and avoid mentioning $\mathrm{In}_\top$. We write $T \vdash_0 \varphi$ to mean that there is such a proof whose root's conclusion is $T \vdash \varphi$.

Again, we will usually drop the subscript. For the sake of precision, let us stipulate that, unless otherwise stated, in this section $\vdash$ will stand for $\vdash_0$; before the end of it, we will see that the difference with $\vdash_{\mathrm{ND}}$ is immaterial.

To (formulate and) prove the equivalence of $\mathrm{ND}_{\bot,\to,\exists,=}$ with ND, the first step is to code the "discarded" logical symbols in terms of the remaining ones.

Definition 2.1.2. Define the following abbreviations[2]

1. $\neg\varphi := \varphi \to \bot$
2. $\top := \neg\bot$.
3. $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$

[1] Because it is redundant, see Proposition 2.1.5.

[2] Each is to be read as the formula obtained by unpacking all previously introduced abbreviations. For instance, the abbreviation for $\varphi \wedge \psi$ unpacks to $(\varphi \to (\psi \to \bot)) \to \bot$

$$\text{(Ax)}\ \frac{}{\varphi \vdash \varphi} \qquad \text{(Wk) For } T \subseteq T'\ \frac{T \vdash \varphi}{T' \vdash \varphi}$$

$$(\text{In}_{\to})\ \frac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi} \qquad (\text{El}_{\to})\ \frac{T \vdash \varphi \to \psi \qquad T \vdash \varphi}{T \vdash \psi}$$

$$\text{(RaA)}\ \frac{T, \varphi \to \bot \vdash \bot}{T \vdash \varphi}$$

$$(\text{In}_{\exists})\ \frac{T \vdash \varphi[t/x_k]}{T \vdash \exists x_k\, \varphi} \qquad (\text{El}_{\exists}) \text{ For } x_k \notin \text{fv}(T, \psi)\ \frac{T \vdash \exists x_k\, \varphi \qquad T, \varphi \vdash \psi}{T \vdash \psi}$$

$$(\text{In}_{=})\ \frac{}{\vdash t = t} \qquad (\text{El}_{=})\ \frac{T \vdash s = t \qquad T \vdash \varphi[s/x_k]}{T \vdash \varphi[t/x_k]}$$

Figure 2.1: The rules of $\text{ND}_{\bot,\to,\exists,=}$.

4. $\varphi \vee \psi \coloneqq \neg(\neg\varphi \wedge \neg\psi)$
5. $\forall x_k\, \varphi \coloneqq \neg\exists x_k\, \neg\varphi$

Applying Definition 2.1.2 inductively allows to translate every first-order formula φ into one mentioning only $\bot, \to, \exists, =$ (as well as the symbols of L), call it $\text{tr}(\varphi)$. We extend translations to theories and rules in the obvious way, namely by translating everything in them.

Proposition 2.1.3. Let φ be a first-order formula, and let ψ be obtained from φ by any number of applications of the abbreviations in Definition 2.1.2. Then ND proves that ψ and φ are *equivalent*, that is, $\psi \vdash_{\text{ND}} \varphi$ and $\varphi \vdash_{\text{ND}} \psi$. In particular, ND proves that φ and $\text{tr}(\varphi)$ are equivalent.

Proof. Exercise. □

Exercise 2.1.4. Show that, for every first-order formulas φ, ψ (possibly involving $\wedge, \vee$, etc) ND proves that φ and ψ are equivalent if and only if $\vdash_{\text{ND}} (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Proposition 2.1.5. The translation of every rule of ND is admissible in the system $\text{ND}_{\bot,\to,\exists,=}$.

Proof. Exercise. □

Corollary 2.1.6. For all T and φ we have $T \vdash_{\text{ND}} \varphi \iff \text{tr}(T) \vdash_0 \text{tr}(\varphi)$.

Corollary 2.1.7. A rule is admissible for ND if and only if its translation is admissible for $\text{ND}_{\bot,\to,\exists,=}$.

Note that, in particular, if you have done any of Exercises 1.4.2 to 1.4.5, then Corollary 2.1.7 implies that the rules in them, after replacing $\neg\psi$ by $\psi \to \bot$, are admissible for $\text{ND}_{\bot,\to,\exists,=}$, even if your proof of admissibility used other logical symbols. If you have not done those exercises yet, you may want to prove the admissibility of those rules in $\text{ND}_{\bot,\to,\exists,=}$ directly.

Corollary 2.1.8. If $T \vdash_0 \varphi$, then there is a finite $T_\varphi \subseteq T$ such that $T_\varphi \vdash_0 \varphi$.

Proof. By Theorem 1.3.9 and Corollary 2.1.6. □

Proposition 2.1.9. Translations work semantically, i.e. for every formula φ, structure M and valuation v, we have $M \vDash \{v\}_M \varphi \iff M \vDash \{v\}_M \operatorname{tr}(\varphi)$.

Proof. Exercise. □

Essentially the combination of Propositions 2.1.3, 2.1.5 and 2.1.9 tells us that we may treat ND as "syntactic sugar" on top of $\mathrm{ND}_{\perp,\to,\exists,=}$. In particular, we will be able to prove the Soundness and Completeness Theorems for $\vdash_0$ and immediately deduce their counterparts for $\vdash_{\mathrm{DN}}$.

Therefore, from now on, we may (and will) freely write $\vdash$ for either $\vdash_{\mathrm{ND}}$ or $\vdash_0$, without risk of ambiguity.

If you had fun solving the exercises above, here is one more tautology to prove.

Exercise 2.1.10. Prove *Peirce's law*: $\vdash ((\varphi \to \theta) \to \varphi) \to \varphi$.

Hint. Peirce's law is a way to express LEM with implication only. You *must* use RaA at some point. □

2.2 Soundness

We start with a small lemma, whose content can be summed up as "valuations commute with substitutions". More in detail, substituting a term for a variable and then evaluating the resulting formula is the same as evaluating the term and then substituting its value. Even more in detail:

Lemma 2.2.1 (Evaluating substitutions)**.** For every structure M, valuation v, and term t that can be legally substituted for x_k in φ,

$$M \vDash \{v\}_M \varphi[t/x_k] \iff M \vDash \{v[(\{v\}_M t)/x_k]\}_M \varphi$$

Proof. You have probably already seen this. If not, it is a good exercise in unravelling definitions. □

Theorem 2.2.2 (Soundness)**.** For every first-order theory T and formula φ, if $T \vdash \varphi$ then $T \vDash \varphi$.

Proof. Let $T \vdash \varphi$, say witnessed by a proof P. As in the proof of Theorem 1.3.9, we work by induction on finite trees and consider the various cases depending on the rule R instantiated in the label of the root of $\operatorname{Tree}(P)$. By Corollary 2.1.6 and Proposition 2.1.9, it suffices to check the rules in Definition 2.1.1. Essentially, we need to check that every rule remains true if we replace $\vdash$ by $\vDash$.

(Ax) If $M \vDash \{v\}_M \varphi$ then obviously $M \vDash \{v\}_M \varphi$, hence this rule is sound.

(Wk) Let $M \vDash \{v\}_M T'$. Because $T \subseteq T$', in particular $M \vDash \{v\}_M T$, hence $M \vDash \{v\}_M \varphi$ by assumption.

($\mathrm{In}_\to$) By definition of $\vDash$, in order to check that $M \vDash \{v\}_M \varphi \to \psi$, we have to show that if $M \vDash \{v\}_M \varphi$ then $M \vDash \{v\}_M \psi$. But if $M \vDash \{v\}_M T$ and $M \vDash \{v\}_M \varphi$ then $M \vDash \{v\}_M T, \varphi$, hence and by assumption $M \vDash \{v\}_M \psi$.

($\mathrm{El}_{\to}$) Let $M \vDash \{v\}_M T$. By assumption, $M \vDash \{v\}_M \varphi$ and $M \vDash \{v\}_M \varphi \to \psi$. Again by definition of $\vDash$ we have $M \vDash \{v\}_M \psi$.

(RaA) Let $M \vDash \{v\}_M T$. Since $M \vDash \{v\}_M \bot$ never holds, by assumption we must have $M \nvDash \{v\}_M \varphi \to \bot$. Unravelling the definition of $\vDash$, this means that $M \vDash \{v\}_M \varphi$ (and $M \nvDash \{v\}_M \bot$).

($\mathrm{In}_{\exists}$) By definition of $\vDash$, having a term t such that $M \vDash \{v\}_M \varphi[t/x_k]$ implies that $M \vDash \{v\}_M \exists x_k\, \varphi$.

($\mathrm{El}_{\exists}$) Let $M \vDash \{v\}_M T$. By assumption $M \vDash \{v\}_M \exists x_k\, \varphi$, hence there is $a \in M$ such that $M \vDash \{v[a/x_k]\}_M \varphi$. Since $x_k \notin \mathrm{fv}(T)$, the restriction of $v[a/x_k]$ and v to $\mathrm{fv}(T)$ is the same, hence $M \vDash \{v[a/x_k]\}_M T, \varphi$, and again by assumption $M \vDash \{v[a/x_k]\}_M \psi$. Because $x_k \notin \mathrm{fv}(\psi)$, the restriction of $v[a/x_k]$ and v to $\mathrm{fv}(\psi)$ is the same, hence $M \vDash \{v\}_M \psi$.

($\mathrm{In}_{=}$) Trivially, $\{v\}_M(t)$ equals $\{v\}_M(t)$, hence $M \vDash \{v\}_M t = t$ always holds.

($\mathrm{El}_{=}$) If $M \vDash \{v\}_M(s = t)$ then s and t are interpreted as the same element, say $\{v\}_M(t) = \{v\}_M(s) = a \in M$. We conclude this case, hence the whole proof, by observing that, using Lemma 2.2.1 twice,

$$\begin{aligned} M \vDash \{v\}_M \varphi[s/x_k] &\iff M \vDash \{v[(\{v\}_M s)/x_k]\}_M \varphi \\ &\iff M \vDash \{v[a/x_k]\}_M \varphi \\ \iff M \vDash \{v[(\{v\}_M t)/x_k]\}_M \varphi &\iff M \vDash \{v\}_M \varphi[t/x_k] \quad \square \end{aligned}$$

Corollary 2.2.3. If $T \vdash \bot$, then T has no model.

Proof. Let $M \vDash T$. If $T \vdash \bot$, the Soundness Theorem implies that $M \vDash \bot$. But this can never be the case by definition of the semantic of $\bot$. ☐

Corollary 2.2.4. For every propositional theory T and formula φ, if $T \vdash \varphi$ then $T \vDash \varphi$.

Proof. Recall that we allow in our language 0-ary relation symbols, i.e. so-called *propositional variables.*[3] As a proof system, take the same one we are using and just look at proofs only involving propositional formulas (if you really want, throw away all rules involving quantifiers or equality). Essentially, this corollary is proven by observing that models of a propositional theory in the first-order sense are the same as models in the propositional sense "plus" a nonempty set of points.

More in detail, let L be a first-order language with these 0-ary relation symbols and nothing else. Recall that a model of a propositional theory T is a *truth value assignment*, that is, a function from propositional variables to $\{\top, \bot\}$, that makes all formulas in T true. If we regard T as a first-order L-theory, then its models are obtained by choosing an arbitrary nonempty set as the domain and interpreting the 0-ary relation symbols according to the truth value assignment above. Conversely, given a model in the first-order sense, we obtain one in the propositional sense by just looking at the interpretation of the 0-ary relation symbols. The conclusion is immediate from Theorem 2.2.2.

[3] Not to be confused with the variables x_k we have been dealing with so far.

If you really don't like 0-ary relations[4] just add a 1-ary relation symbol P_A for each propositional variable A and code A with $\exists x\ P_A(x)$. $\square$

2.3 Reducing to sentences

Definition 2.3.1. A *sentence*[5] is a formula φ such that $\mathrm{fv}(\varphi) = \emptyset$.

From now on, whenever convenient,[6] we want to be able to exclusively work with sentences. This has the advantage of not having to name valuations in order to say whether φ holds or not in M. The fact that we lose no expressive power is justified by the following lemmas.

Lemma 2.3.2. Let T be an L-theory, φ an L-formula, $x_k \in \mathrm{fv}(\varphi)$, and c a constant symbol[7] not in L. Then:

1. $\varphi[c/x_k]$ is well-defined, and
2. if $x_k \notin \mathrm{fv}(T)$, then $T \vdash \varphi$ in L if and only if $T \vdash \varphi[c/x_k]$ in $L \cup \{c\}$.

Proof. Exercise. $\square$

Exercise 2.3.3. What if $x_k \in \mathrm{fv}(T)$?

Lemma 2.3.4. [on Constants[8]] Let T be an L-theory, φ an L-formula, x_k a variable, and c a constant symbol not in L. Then $\varphi[c/x_k]$ is well-defined, and the following are equivalent.

1. $T \vdash \varphi[c/x_k]$ (in $L \cup \{c\}$).
2. $T \vdash \forall x_k\ \varphi$ (in L).
3. $T, \exists x_k\ \neg\varphi \vdash \bot$ (in L).

Proof. Exercise. Hint for $1 \Rightarrow 2$: use Syntactic Compactness to obtain a finite $T_0 \subseteq T$ and a proof of $T_0 \vdash \varphi[c/x_k]$; pick a variable x_ℓ not appearing in this proof. Show that $T_0 \vdash \varphi[x_\ell/x_k]$ in L by structural induction on trees (essentially, check that replacing all instances of c by x_ℓ still gives a proof). Now you should be able to conclude by applying a couple of rules. $\square$

Finally, observe that, even a free variable x_k is mentioned by an L-theory T, we can still think of x_k as a constant $c \notin L$: the point is that an L-structure M together with a valuation v is essentially the same thing as an $L \cup \{c\}$-structure M', with the same domain as M and interpreting the symbols of L in the same way, together with a valuation of the variables that is only defined on the variables different from x_k. Substituting a term t for x_k corresponds to changing the interpretation of c, and the substitution being legal corresponds to the fact that it is not possible to quantify over constants.

[4]...then you are wrong, they make everything more elegant. Also, they are nice people. But if you really hate them...

[5]In italiano, *enunciato*.

[6]For the impatient: the idea is that, at least when giving axioms of a theory, or when saying whether something is true in a structure, we want to use sentences. Formulas with free variables still play a crucial role, but will be though of as *definable sets*. More on this in Section 5.4.

[7]Recall that in our formalism a constant symbol is a 0-ary function symbol.

[8]In italiano: *Lemma delle Costanti*.

Chapter 3

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3.1 Sketching a road map

The purpose of this section is to give a rough idea of how we will prove the Completeness Theorem. There will be some symbols and words, in particular some of the *emphasised* ones, that have not been defined yet, but will shortly be, or that will never be defined at all in the whole course. If this bothers you, or if you prefer to distil the ideas directly from the cold hard proofs, feel free to skip this section altogether. If you do not care about the heuristics but would still like to see the proof strategy before delving into the details, go straight to the list at the end of the section.

We will need to show that the only reason why a theory T, say consisting of sentences, may fail to have a model, is because $T \vdash \perp$. The core idea is akin to that of conjuring a group by presenting it, so we are going to take a close look at group presentations and try to figure out what are the obstacles to carrying out a similar construction for more general structures, and the pitfalls one may encounter by being careless.

Recall that in order to give a *presentation* of a group G one specifies a set S of *generators*, and a set R of *relations* between them. For example, working in the language of (multiplicative) groups $L_{\mathrm{grp}} = \{\cdot, e, {}^{-1}\}$, we can present $\mathbb{Z}^2$ by taking as generators a, b, say, and as relations the singleton of $aba^{-1}b^{-1} = e$. We can then build $\mathbb{Z}^2$ by taking all words in e, a, b, a^{-1}, b^{-1}, and identifying two of them whenever they are mandated to coincide by the axioms of group theory together with the relation $ab = ba$. In a general group presentation, all relations in R may be written in the form [some word] $= e$, and R identified with the set of those words.

The clean group-theoretic way of defining the group $\langle S \mid R\rangle$ presented by S and R is to say that it is:

1. the free group generated by S
2. modulo the *normal closure* of R.

Let us look at this list. In 1, in order to talk about elements of S, we are adding to L_{grp} a constant symbol for each of its elements. We may assume these constants are the elements of S themselves. Then, still in 1, there are two things happening: we are

a. forming *all* words, including things like $eaa^{-1}b$, i.e., considering all terms in the *expansion* of L_{grp} by S, and

b. identifying two words whenever T_{grp} says that they must denote the same element, for example $eaa^{-1}b$ and b.

In point 2 we are taking a further quotient. Intuitively, we want to quotient by R. But setting $aba^{-1}b^{-1} = e$ has consequences, for example that all the products of elements in R, as well as their conjugates, will still be equal to e. In other words, if we want to quotient a group by something, and obtain another group, then this something would better be a normal subgroup. Hence, we take the smallest normal subgroup containing R. From the perspective of a logician the point is that, for example, $T_{\mathrm{grp}}, ab = ba \vdash abba = a^{-1}b^{-1}ab^{-1}abbbba$.

To recap, this construction builds a model of the $L_{\mathrm{grp}}(S)$-theory $T_{S,R} \coloneqq T_{\mathrm{grp}} \cup \{r = e \mid r \in R\}$ by quotienting the set of closed $L_{\mathrm{grp}}(S)$-terms by an equivalence relation determined by $T_{S,R}$: the relation identifying t and t' iff they must be identified because of a combination of the axioms of group theory and the relations in R. That is, t and t' are identified precisely when their equality is written in the *deductive closure* of $T_{S,R}$, which means that $T_{S,R} \vdash t = t'$.

Now, for $T_{S,R}$ the construction is quite easy, for a number of reasons. For example, this theory consists of sentences of the form $\forall x_0 \dots \forall x_n\ t_0(x_0, \dots, x_n) = t_1(x_0, \dots, x_n)$. Note that in particular there are no negations, implications, nor relation symbols, and the only quantifiers are universal and at the start of the sentence. Perhaps unexpectedly, this implies $T_{S,R} \nvdash \bot$.

If you try to think about the reason why, you may realise that said reason is (the contrapositive of) Corollary 2.2.3. Namely, it is very easy to build a model of $T_{S,R}$: it is readily checked that the trivial group $\{e\}$, with the only possible[1] $L_{\mathrm{grp}}(S)$-structure on it, satisfies the axioms of $T_{S,R}$. Essentially, the axioms we wrote can only ensure that the natural map from the free group generated by S has as kernel *some* normal subgroup containing R.

You are now asking yourself a subset of the set of these questions.[2]

1. What if we want to do this kind of construction for a theory in a language with no constant symbols, e.g. ZFC?

2. Why does $\langle S \mid R\rangle$ not collapse onto the trivial group?

3. What if I want to ensure that it does not, and add axioms saying, for example, that the named generators of the group are pairwise distinct?

For the first question, the solution is to add "enough" constant symbols. What does "enough" mean? Well, ZFC proves that there is an empty set. If we want a model, it should have a point for $\emptyset$ in it, so maybe we should add a constant $c_\emptyset$ to our language. As you probably know, models of any reasonable set theory are not made of a single point[citation needed], so we will need more than that.

In fact, whenever $\mathsf{ZFC} \vdash \exists x\ \varphi(x)$, the model we build needs a witness of this. Because we want to build our model as a quotient of the set of terms in a suitable language, perhaps we should place in our language a constant c_φ and write down in our extension of ZFC that c_φ is to be interpreted as a point

[1] Exercise: the fact that there is, up to isomorphism, a unique L-structure with one element is equivalent to L not containing any relation symbol.

[2] Yes, you are. I never said "nonempty".

satisfying $\varphi(x)$. This is the idea behind *Henkin theories*: we will enlarge L to some L', and T to some T', so that the condition above holds.

The answer to the second question is that we are identifying t and t' *only* when $T_{S,R} \vdash t = t'$. But how do we know if this happens? Well, in this particular case, it just does not. In general, this is known as the *Word Problem*, and there is no *effective* way to solve it. I am not sure whether you will see a proof of this fact in this course. But you *will* later on see how to prove that there is no effective way to figure out whether $\mathsf{ZFC} \vdash \varphi$ or not.

So, if we want a general result, we will need some nonconstructive argument. Essentially, by an appeal to Zorn's Lemma, we will further enlarge T' to some T'' such that, if $T'' \nvdash t = t'$, then $T'' \vdash t \neq t'$. While we are at it, we may as well make T'' *deductively closed*, so that we know exactly when two terms are being identified by looking at whether the corresponding formula is an element of T. Similarly, we should know which relations hold. In fact, we want T'' to tell us directly whether a sentence should be true or false in the constructed model. For example, if $T \vdash c_0 = c_1 \vee c_0 = c_2$, then in a model of T'' one of the two disjuncts will be true; we want T'' to already know which one(s) in advance. Note that, in the case of ZFC, this will mean passing to an extension where, e.g., it is decided whether the Continuum Hypothesis is true or not.

Regarding the third question, the problem is that we need to make sure not to create any inconsistencies. For example, suppose that doing the enlargements above results, for some reason (recall that we need some nonconstructive step), in having added, among other things, generators c, d and relations $aba^{-1}b^{-1} = e$, $cb^{-1}a^{-1} = e$, and $da^{-1}b^{-1} = e$. If we try to also add axioms saying that the generators a, b, c, d are pairwise distinct, we quickly produce an inconsistency. Hence, whatever axioms we add to our T in the steps foreshadowed above, we need to make sure that the resulting T'' does not prove $\perp$, for this would preclude the possibility that T'' has a model, leading our construction to certain doom. In particular, T should not prove $\perp$, to being with. Here we are lucky, because this is the hypothesis of the Completeness Theorem, but we have to ensure that this property is preserved by our constructions. For example, it would be foolish to just say "take as T'' the set of all sentences that can be added to T without proving $\perp$", because it can happen that such a set proves $\perp$.

Example 3.1.1. If $T = \{c_0 = c_1 \vee c_0 = c_2\}$, then $T \cup \{c_0 \neq c_1\} \nvdash \perp$ and $T \cup \{c_0 \neq c_2\} \nvdash \perp$, but $T \cup \{c_0 \neq c_1, c_0 \neq c_2\} \vdash \perp$.

While we are on the theme of foreshadowings: you will see later on in the course that, in the case of ZFC, proving this, i.e., that $\mathsf{ZFC} \nvdash \perp$, is not something you can prove in ZFC alone.

Wary of the hurdles and perils awaiting us, we devise the following plan.

(A) Start with a theory T not proving $\perp$.

(B) Add enough constant symbols and formulas to T, resulting in a T' with constants witnessing all statements of the form $\exists x\, \varphi(x)$ that it proves. Be careful not to make T' prove $\perp$ in the process.

(C) Pass to an even larger T'' that proves as many things as possible, still subject to the requirement that it must not prove $\perp$. Also, make sure that (B) is satisfied by T'' as well.

(D) Take the set of terms in the language of T'', identify them whenever T'' says so, and define functions and relations on the quotient according to what is written in T''. Note that this is possible because every point of the quotient is named by a term (in fact, by several).

(E) Check that what we built is (well-defined and) indeed a model of T'', and in particular of T.

3.2 Expansions, reducts, and deductive closures

So far we mostly fixed a language L. As you have seen above, sometimes we need to enlarge and shrink languages, so it is time to introduce the relative terminology.

Definition 3.2.1. If $L \subseteq L'$, we call L' an *expansion* of L, and L a *reduct* or *sublanguage* of L'.

If $L \subseteq L'$ and M is an L'-structure, the *reduct* $M \upharpoonright L$ of M to L is the L-structure with the same domain as M, where every $s \in L$ is interpreted as in M. We call M an *expansion* of $M \upharpoonright L$ to L'.

In other words, the reduct of an L'-structure to $L \subseteq L'$ is obtained by forgetting the interpretations of symbols in $L' \setminus L$.

Let us check that expanding the language by constants[3] does not change the amount of statements in the unexpanded language that we can prove.

Corollary 3.2.2. Let T be an L-theory and φ an L-formula. Let C be a set of constants, none of which is in L, and let $L(C)$ be the expansion of L by the symbols in C. Whether $T \vdash \varphi$ does not depend on whether we regard T, φ as L-theories or as $L(C)$-theories.

Proof. Any proof in L is in particular a proof in $L(C)$, so if $T \vdash \varphi$ in L then $T \vdash \varphi$ in $L(C)$.

Conversely, let P be a proof of $T \vdash \varphi$ in $L(C)$. By Syntactic Compactness (Theorem 1.3.9) we may assume T is finite. In particular, there are only finitely many formulas involving constants in C that appear in the proof P. Now apply the Lemma on Constants and induction to replace the $L(C)$-formulas appearing in the proof by L-formulas. $\square$

Definition 3.2.3. The *deductive closure* of a theory T is $\{\varphi \mid T \vdash \varphi\}$. A theory is *deductively closed* iff it equals its deductive closure.

For many purposes[4], a theory may be identified with its deductive closure. Especially when arguing semantically (roughly, when there are many $\vDash$ and few $\vdash$ around), this is so common that, since we are approaching semantic territory, I am virtually sure that I will at some point forget to say that I am doing this identification. Apologies. I will leave it up to you whether to forgive me for doing so, or to accept the following sloppily phrased assumption.

[3]The same is actually true for any expansion. You will be able to prove this easily by using the Completeness Theorem, but this case will be used in the road leading there.

[4]But not for all of them, cf. Corollary 3.2.2.

Assumption 3.2.4. Henceforth, all theories mentioned are assumed to be deductively closed, unless it is obvious from context that we do not want to assume this.

In order to "undo" this assumption locally, it is common to say things like "T_0 is an axiomatisation of T" to mean "the deductive closure of T_0 is T".

3.3 Consistent and complete theories

At the risk of stating the obvious, let me say that the notation $T \nvdash \varphi$ means that $T \vdash \varphi$ does not hold. In general, this is *not* the same as $T \vdash \neg\varphi$. In one direction, if $T \vdash \bot$ then T proves anything, and in particular $T \vdash \neg\bot$.

In the other direction, for example, the theory of groups does not prove $\forall x\, \forall y\; x \cdot y = y \cdot x$ because, by the Soundness Theorem, if it did then all groups would be abelian. For similar reasons, it does not prove its negation. Theories with the properties that, for every φ, they prove φ or they prove $\neg\varphi$, but they do not prove both, deserve a special name.

Definition 3.3.1. A theory T is

1. *consistent*[5] iff $T \nvdash \bot$;
2. *complete* iff it is consistent and for every sentence φ either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

We can rephrase these two definitions as follows.

Exercise 3.3.2. T is consistent if and only if there is no sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Lemma 3.3.3. If T is consistent, then it is consistent with φ or with $\neg\varphi$, i.e. at least one of $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ is consistent.

Proof. If not, then $T, \varphi \vdash \bot$ and $T, \neg\varphi \vdash \bot$. It follows, e.g. by using Exercise 1.4.2, that $T \vdash \bot$, against the assumptions. □

Note that a general T may be separately consistent both with φ and with $\neg\varphi$: think of the example above with the theory of groups.

Corollary 3.3.4. Let T be a consistent theory. The following are equivalent.

1. (The deductive closure of) T is maximal (with respect to inclusion) among (deductive closures of) consistent theories.
2. If $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.
3. T is complete.

Proof. E(asy e)xercise. □

Example 3.3.5. If M is an L-structure, its *theory* $\mathrm{Th}(M)$ is the set of all L-sentences φ such that $M \vDash \varphi$. Clearly $M \vDash \mathrm{Th}(M)$ hence the latter, having a model, must be consistent (by Corollary 2.2.3). Moreover $\mathrm{Th}(M)$ is always complete because, if $M \nvDash \varphi$, then $M \vDash \neg\varphi$ by definition of Tarski semantics.

[5] In italiano: *coerente*.

Spoiler 3.3.6. In fact Example 3.3.5 is, in an abstract sense, the only example: it will follow from the Completeness Theorem that every complete theory is of the form $\mathrm{Th}(M)$, for some suitable M. For the moment, observe that $M \vDash T$ (say for T a set of sentences) is the same as $\mathrm{Th}(M) \supseteq T$.

Non-Example 3.3.7. Let $L_{\mathrm{ring}} := \{+, 0, -, \cdot, 1\}$ be the language of rings. The L_{ring}-theory of fields is not complete, since it does not prove $1 + 1 = 0$ (by the Soundness Theorem and looking at $\mathbb{Q}$) nor its negation (by the Soundness Theorem and looking at $\mathbb{F}_2$).

Non-Example 3.3.8. The $\{\leq\}$-theory of linear orders is not complete, since it does not decide[6] whether $\exists x\, \forall y\; x \leq y$.

You may object that all we have seen here are some examples of incomplete theories, and an example of a complete theory that is general beyond the point of usefulness. The fact is that proving that a fixed theory is complete requires, at least in interesting cases, nontrivial mathematical work.[7] Later in this course we will see some tools that can be used to prove completeness of a theory, beginning with the Completeness Theorem.[8] More such tools are usually developed in a model theory course, see e.g. [Men22]. At any rate, if you are happy to take these statements on trust:

Example 3.3.9. All of these are complete theories.[9]

1. The theory of dense linear orders with no endpoints, i.e. $\mathrm{Th}(\mathbb{Q}, \leq)$.
2. The theory of nontrivial $\mathbb{K}$-vector spaces, for $\mathbb{K}$ a fixed infinite field, in the language $\{+, 0, -\} \cup \{\lambda \cdot - \mid \lambda \in \mathbb{K}\}$.
3. The theory of infinite-dimensional $\mathbb{F}_q$-vector spaces, for $\mathbb{F}_q$ a finite field, in the language above with $\mathbb{K} = \mathbb{F}_q$.
4. The theory of algebraically closed fields of fixed characteristic, e.g. for characteristic 0 this is $\mathrm{Th}_{L_{\mathrm{ring}}}(\mathbb{C})$.
5. The theory of real closed fields, i.e. the theory of $\mathrm{Th}_{L_{\mathrm{ring}}}(\mathbb{R})$.
6. The theory of atomless[10] boolean algebras, in the language $\{\sqcap, \sqcup, 0, 1, -^{\complement}\}$, or in the language $\{\sqsubseteq\}$, or in L_{ring}.[11]

[6] Of course this means that it does neither prove that sentence nor its negation.

[7] Depending on taste, this may trivially be true by definition of "interesting".

[8] Coincidence? Yes: the "complete" in "Completeness Theorem" is not the same "complete" as in "complete theory"; it refers to the fact that the theorem can be phrased as "Natural Deduction is complete for Tarski semantics". We will (very briefly) come back to this later on.

[9] The interpretations of the symbols is the obvious one suggested by the notation. The axiomatisations are left to the reader. Alternatively, interpret e.g. "the theory of real closed fields" as "the set of sentences true in all real closed fields", that is, take $\mathrm{Th}(K) := \bigcap_{M \in K} \mathrm{Th}(M)$ where K is the class of real closed fields.

[10] An *atom* of a boolean algebra $(B, \sqsubseteq)$ is a $\sqsubseteq$-minimal element of $B \setminus \{0\}$. *Atomless* of course means without atoms. Examples of such a thing: the powerset of the natural numbers modulo the ideal of finite sets, Lebesgue measurable sets modulo those of measure zero, and regular open subsets of $\mathbb{R}$ (or of any other Hausdorff space with no isolated points).

[11] Fun fact: boolean algebras may be identified with the (commutative) rings (with identity) where every element is idempotent, that is, equals its square. Under this identification, ideals of this ring are the same as ideals of the boolean algebra in the sense of duals of filters, where "dual" means "take the complement of every element"; e.g. in the σ-algebra underlying a probability space, the dual of the ideal of measure zero set is the filter of measure one sets. Maximal ideals are the same as duals of ultrafilters.

7. There is a (unique) complete theory T in the language $L_{\text{graph}} \coloneqq \{E\}$ of graphs[12] such that: if, for every p with $0 < p < 1$, we place an edge between pairs of distinct points of $\mathbb{N}$ with probability p, independently, then we obtain a model of T with probability 1. This can be used to show that, in fact, we obtain with probability 1 a unique graph, called the *Random Graph*, or *Rado Graph*.

Exercise 3.3.10. Let $\mathbb{F}_q$ be a finite field, and let $\mathbb{F}_q{-}\mathsf{VS}$ be the theory of nontrivial $\mathbb{F}_q$-vector spaces, axiomatised in the obvious way in the language $\{+, 0, -\} \cup \{\lambda \cdot - \mid \lambda \in \mathbb{F}_q\}$. Prove that $\mathbb{F}_q{-}\mathsf{VS}$ is not complete.

For some other standard examples of theories, see e.g. [Mar02, Section 1.2] or [Hod93, Section 2.2].

Lemma 3.3.11 (Lindenbaum)**.** Every consistent theory is contained in a complete theory.

Proof. By Corollary 3.3.4 and Zorn's Lemma, we only need to check that the union of a chain of consistent theories is consistent. But this is obvious from Syntactic Compactness (Theorem 1.3.9). □

3.4 Henkin theories

Notation 3.4.1. We write $\varphi(x_0, \dots, x_n)$ to mean that φ is a formula with $\mathrm{fv}(\varphi) \subseteq \{x_0, \dots, x_n\}$. For $\varphi(x)$ a formula and c a constant, we write $\varphi(c)$ for $\varphi[c/x]$.[13]

Definition 3.4.2. An L-theory T is *Henkin* iff, for every L-sentence of the form $\exists x\ \varphi(x)$, there is a constant symbol $c \in L$ such that $T \vdash (\exists x\ \varphi(x)) \to \varphi(c)$.

Note that in order for an L-theory to be Henkin, first of all L needs to have some constant symbols. This gives us plenty of examples of theories that are not Henkin, for example all theories in a *relational language*, that is, one only containing relation symbols. But of course just having some constant symbols in the language is not enough.

Example 3.4.3. The L_{ring}-theory of $\mathbb{C}$ (made into an L_{ring}-structure in the natural way) is not Henkin.

Proof. Let $\varphi(x) \coloneqq x = 1 + 1$. Then this theory proves $\exists x\ \varphi(x)$, but it does not prove $\varphi[0/x]$ nor $\varphi[1/x]$. Even if you add constant symbols for all closed terms (e.g. a symbol 2 for $1 + 1$), this theory is still not Henkin: just consider $\psi(x) \coloneqq x \cdot x = 1 + 1$. □

[12] E is a binary relation symbol, and we make a graph into an $\{E\}$-structure by setting $M \vDash E(a, b)$ iff there is an Edge between a and b. The theory of graphs simply says that E is irreflexive and symmetric, that is, graphs are without self-loops, undirected, and with no multiple edges (the latter is not an axiom, it is a built-in consequence of the fact that we are coding edges with a single relation symbol). If you want to allow multiple edges between the same two points, you can use a different language and have some points of the domain of your structure representing vertices, and other points representing edges, perhaps distinguished by a unary predicate.

[13] Note that it is always legal to substitute a constant for a variable.

Spoiler 3.4.4. If you skipped Section 3.1, you may wonder what the point of introducing this notion is. It is that, if a Henkin theory is complete, then it is possible to build a model for it *out of formulas*. Therefore, in order to prove the Completeness Theorem, we will show that any T is contained in such a theory. This will be an inductive construction, the key step being Lemma 3.4.5 below.

Lemma 3.4.5. Let T be a consistent L-theory, φ an L-formula, and c a constant symbol not in L. Then the $L\cup\{c\}$-theory $T\cup\{(\exists x\ \varphi(x)) \to \varphi[c/x]\}$ is consistent.

Proof. Note immediately that $\varphi[c/x]$ is well-defined, i.e. the substitution is legal, because φ is an L-formula and $c \notin L$.

Suppose towards a contradiction that $T, ((\exists x\ \varphi(x)) \to \varphi[c/x]) \vdash \bot$. By Exercise 1.4.3, $T, \exists x\ \varphi(x) \vdash \neg\varphi[c/x]$. By Exercise 1.4.4 (instatiated with $\psi \coloneqq \bot$) we obtain that $T, \exists x\ \varphi(x) \vdash \bot$, and using Exercise 1.4.5 we conclude that T, regarded as a $L\cup\{c\}$-theory, is inconsistent. By Corollary 3.2.2, T is inconsistent as an L-theory as well, against the assumptions. □

Proposition 3.4.6. Let T be a consistent L-theory. There are $L' \supseteq L$ with $|L|' = |L| + \aleph_0$ and $T' \supseteq T$ such that T' is a complete Henkin L'-theory.

Proof. We simply keep enlarging T and L until we get a consistent Henkin theory, and then complete it.

In detail, let $L_0 \coloneqq L$ and $T_0 \coloneqq T$. For every L-formula of the form $\varphi(x)$, let c_φ be a new constant symbol (in particular, it can be legally substituted in every L_0-formula), and let $L_1 \coloneqq \{c_\varphi \mid \varphi(x) \text{ an } L_0\text{-formula}\}$. Define

$$T_1 \coloneqq T_0 \cup \{(\exists x\ \varphi(x)) \to \varphi[c_\varphi/x] \mid \varphi(x) \text{ an } L_0\text{-formula}\}$$

Claim 3.4.7. T_1 is consistent.

Proof of the Claim. By Syntactic Compactness, if $T_1 \vdash \bot$ then there are L-formulas $\varphi_0(x_{k_{i_0}}), \dots, \varphi_{n-1}(x_{k_{i_{n-1}}})$ such that

$$T_0 \cup \{\exists x_{k_{i_j}}\ \varphi_j(x_{k_{i_j}}) \to \varphi[c_{\varphi_j}/x_{k_{i_j}}] \mid j < n\} \vdash \bot$$

By Lemma 3.4.5 and induction on n, we find that $T_0 \vdash \bot$, against the assumptions. □ CLAIM

Now T_1 satisfies the special case of Definition 3.4.2 where φ is an L_0-formula, but is not Henkin, because T_1 is an L_1-theory and $L_1 \supsetneq L_0$, hence we need to take care of formulas involving the newly introduced symbols.

To do this, we simply iterate the argument, by inductively defining T_{n+1} from T_n in the same way as T_1 was defined from T_0. Clearly, by the same argument as in the proof of Claim 3.4.7, every T_n is consistent, and clearly $T_n \subseteq T_{n+1}$, hence by another application of Syntactic Compactness we obtain the consistency of $T_\omega \coloneqq \bigcup_{n<\omega} T_n$, which is a theory in the language $L' \coloneqq \bigcup_{n<\omega} L_n$. Note that the latter satisfies the required cardinality bound. Besides being consistent, the theory T_ω is Henkin, because if $\varphi(x)$ is an L' formula, then there is n such that $\varphi(x)$ is an L_n-formula, and by construction the formula required by Definition 3.4.2 appears in T_{n+1}.

Now apply Lindenbaum's Lemma 3.3.11 to complete T_ω to a complete L'-theory T', and observe that a completion[14] of a Henkin theory is still Henkin. □

[14] A *completion* of T' is a complete theory $T'' \supseteq T'$ the same language.

Usually, when the size of a language is relevant, what really matters in the cardinality of the set of *formulas*, not of the set of *symbols*. An easy counting argument shows that these are different if and only if there are finitely many symbols, in which case there are $\aleph_0$ formulas. Carrying that $+\aleph_0$ around, as in the statement of Proposition 3.4.6 above, is not very practical. Therefore, from now on we adopt the following standard convention.

Notation 3.4.8. The *cardinality* $|L|$ of a language L is the cardinality of the set of L-formulas. If T is an L-theory, we set $|T| \coloneqq |L|$.

Note that $|T|$ is the cardinality of the deductive closure of T, since one can always write something like $\varphi \vee \top$ where φ is a sentence involving your favourite symbol of L.

So, for example, even if L is the empty language we have $|L| = \aleph_0$, and if $T = \{\forall x\ x = x\}$ and $|L| = \aleph_{17}$ then $|T| = \aleph_{17}$. Sometimes we will still write that $+\aleph_0$ for emphasis.

Chapter 4

28/03

4.1 Completeness

Proposition 4.1.1. Every complete Henkin L-theory T has a model M, which can be chosen such that for every $a \in M$ there is a closed L-term T such that $a = t^M$ (hence in particular $|M| \leq |L|$).

Proof. Let T be a complete Henkin L-theory, and let X be the set of closed[1] L-terms. Since T is a Henkin theory, L has plenty of constant symbols, and in particular $X \neq \emptyset$. Define an equivalence relation $\sim_T$ on X by

$$t \sim_T t' \coloneqq T \vdash t = t'$$

Let $M \coloneqq X/{\sim_T}$. This M satisfies the required cardinality bound, but as of now is only a set; we want to make it[2] into an L-structure. Since we would like to end up with a model of T it seems like a good idea do this in such a way that each term t is interpreted as $t/{\sim_T}$. Moreover, if for example R is a binary relation symbol in L, it seems reasonable to interpret it as the set of those pairs $(t_0/{\sim_T}, t_1/{\sim_T}) \in M^2$ such that $T \vdash R(t_0, t_1)$. This is indeed what we will do, except that we need to check that this is well defined, i.e. does not depend on the choice of representative t_i for $t_i/{\sim_T}$.

Let f be an n-ary function symbol[3]. It is easy to see, by using $\mathrm{In}_=$ and $\mathrm{El}_=$, that

$$T \vdash f(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{n-1})$$

Consider the formula $\varphi(y_0, \ldots, y_{n-1}) \coloneqq f(y_0, \ldots, y_{n-1}) = f(t_0, \ldots, t_{n-1})$. Note that every closed term can be substituted in place of each y_i, and we just said that $T \vdash \varphi([t_0/y_0, \ldots, t_{n-1}/y_{n-1}])$. By repeated applications of instances of $\mathrm{El}_=$, we find that if, for $i < n$, we have $T \vdash t_i = t_i'$, then also

$$T \vdash f(t_0', \ldots, t_{n-1}') = f(t_0, \ldots, t_{n-1})$$

Therefore, the following function $f^M : M^n \to M$ is well-defined:

$$f^M(t_0/{\sim_T}, \ldots, t_{n-1}/{\sim_T}) \coloneqq f(t_0, \ldots, t_{n-1})/{\sim_T}$$

[1] Recall that a term is closed iff it has no free variables.

[2] Of course using the same symbol for a structure and its domain is an abuse of notation, but it is very convenient and ever more widespread.

[3] Recall that the case $n = 0$ corresponds to constants.

Claim 4.1.2. For every L-term t, we have $t^M = t/{\sim_T}$.

Proof of the Claim. This is an easy induction on terms (exercise). $\square_{\text{CLAIM}}$

Similarly, if R is an n-ary-relation symbol, using $\mathrm{El}_=$ tells us that it is well-defined to set

$$R^M \coloneqq \{(t_0/{\sim_T}, \dots, t_{n-1}/{\sim_T}) \mid T \vdash R(t_0, \dots, t_{n-1})\}$$

This completes our construction of M. We now check that indeed $M \vDash T$.

By Corollary 2.1.6 and Proposition 2.1.9, we only need to check formulas involving no logical symbols besides $\bot, \to, \exists, =$. In order to be able to use completeness of T, it will be convenient to also deal with $\neg$; if you want, treat $\neg\varphi$ as an abbreviation for $\varphi \to \bot$.

Let $\varphi \in T$ be a sentence as above. We prove by induction on the number of symbols in $\bot, \to, \exists, \neg$ (sic[4]) appearing in φ that

$$T \vdash \varphi \iff M \vDash \varphi \tag{$*\varphi$}$$

Note that $\Longrightarrow$ would be enough[5], but having also $\Longleftarrow$ available from the inductive assumption will be used in completing the induction step.

Formally, what we are doing is fixing a valuation v and checking that $M \vDash \{v\}_M\varphi$. Because φ is a sentence, this happens for some valuation if and only if it happens for every valuation. As v will play no role for most of the proof, we temporarily suppress it from the notation, and simply write $M \vDash \varphi$.

Claim 4.1.3. If φ is atomic, then $(*\varphi)$ holds.

Proof of the Claim. We have three cases.

1. Because T is consistent, $T \nvdash \bot$, hence $T \vdash \bot \Longrightarrow M \vDash \bot$ is vacuously true. So is its converse, because no structure satisfies $\bot$.

2. For $\varphi \coloneqq R(t_0, \dots, t_{n-1})$, the fact that $T \vdash \varphi \iff M \vDash \varphi$ holds by definition of R^M.

3. If $T \vdash t_0 = t_1$, then by definition $t_0 \sim_T t_1$, hence by construction $M \vDash t_0 = t_1$. If instead $T \nvdash t_0 = t_1$, then $t_0 \nsim_T t_1$ and it follows from Claim 4.1.2 that $M \vDash \neg t_0 = t_1$. $\square_{\text{CLAIM}}$

Claim 4.1.4. If $(*\varphi)$ holds, then $(*\neg\varphi)$ does as well.

Proof of the Claim. Because T is complete,[6] we have $T \vdash \neg\varphi \iff T \nvdash \varphi$. After observing that $M \vDash \neg\varphi \iff M \nvDash \varphi$ holds by definition of Tarski semantics, we conclude by the inductive hypothesis. $\square_{\text{CLAIM}}$

Claim 4.1.5. If both $(*\varphi)$ and $(*\psi)$ hold, then $(*(\varphi \to \psi))$ also holds.

---

[4]In other words, for the purpose of indexing our induction we will not count occurrences of the symbol $=$. If you are wondering why we do not simply do induction on the length of the formula, the reason is that one step will involve replacing a variable with a possibly long closed term.

[5]... and $\Longleftarrow$ would then follow downstream by completeness of T...

[6]... which, recall, implies "consistent" by definition, ...

Proof of the Claim. By definition of $\vDash$ and inductive assumption,

$$M \vDash \varphi \to \psi \iff (M \vDash \varphi \Longrightarrow M \vDash \psi) \iff (T \vdash \varphi \Longrightarrow T \vdash \psi)$$

Now, if $T \vdash \varphi \to \psi$, then $T \vdash \varphi \Longrightarrow T \vdash \psi$ is true by $\mathrm{El}_\to$. We prove the converse implication by cases.

If $T \nvdash \varphi$, then $T \vdash \neg\varphi$ by completeness of T. This means $T \vdash \varphi \to \bot$, hence (exercise) $T \vdash \varphi \to \psi$.

In the case where $T \vdash \varphi$, by assumption we must also have $T \vdash \psi$. If $T \vdash \varphi \to \psi$ fails, by completeness $T \vdash (\varphi \to \psi) \to \bot$, so (exercise) $T, \varphi \to \psi \vdash \bot$. But then Exercise 1.4.3 together with $T \vdash \psi$ imply that $T, \varphi \vdash \bot$. As $T \vdash \varphi$, this violates consistency of T. $\square_{\text{CLAIM}}$

Recall now that at the start of the proof we fixed a valuation v but suppressed it from the notation. We now start mentioning it again.

To conclude the proof we need to show the conclusion for sentences of the form $\exists x\ \varphi(x)$. To no one's surprise, this is where we will really need to use the assumption that T is Henkin, so let c be such that $T \vdash (\exists x\ \varphi(x)) \to \varphi[c/x]$.

If $T \vdash \exists x\ \varphi(x)$, then by $\mathrm{El}_\to$ we have $T \vdash \varphi[c/x]$. By inductive assumption, $M \vDash \{v\}_M\varphi[c/x]$, so by Lemma 2.2.1 $M \vDash \{v[(\{v\}_M c)]/x\}_M\varphi(x)$, that is, $M \vDash \{v[c^M/x]\}\varphi(x)$, or if you prefer $M \vDash \{v[(c/{\sim_T})/x]\}\varphi(x)$, so by definition of $\vDash$ $M \vDash \exists x\ \varphi(x)$.

Conversely, assume that $M \vDash \{v\}_M \exists x\ \varphi(x)$. By construction of M there is a closed term t such that $M \vDash \{v[(t/{\sim_T})/x])\}_M\varphi(x)$. Since t is closed, we may legally substitute it for x, hence $\varphi[t/x]$ is well-defined. Moreover, $\{v\}_M t = t^M = t/{\sim_T}$ by Claim 4.1.2, hence $M \vDash \{v[(\{v\}_M t)/x]\}_M\varphi$, so by Lemma 2.2.1 $M \vDash \{v\}_M\varphi[t/x]$. Again by inductive assumption, $T \vdash \varphi[t/x]$, so by $\mathrm{In}_\exists$ we have $T \vdash \exists x\ \varphi(x)$. $\square$

Exercise 4.1.6. Complete the proof of Proposition 4.1.1 by filling in the steps marked as "(exercise)".

Theorem 4.1.7 (Completeness)**.** Every consistent L-theory T has a model. Equivalently, $T \vDash \varphi \Longrightarrow T \vdash \varphi$. Moreover, such a model may be chosen of size $|M| \le |T| + \aleph_0$.

Proof. Because no L-structure is a model of $\bot$, the first statement is the contrapositive of the case $\varphi = \bot$. We reduce to this case by observing the following.

By the rules RaA and $\mathrm{In}_\neg$, we have $T \vdash \varphi \iff T, \neg\varphi \vdash \bot$. By definition of Tarski semantics, $T \vDash \varphi$ if and only if every model of T is a model of φ or, in other words, iff there are no models of $T, \neg\varphi$. Again because no L-structure is a model of $\bot$, this means $T \vDash \varphi \iff T, \neg\varphi \vDash \bot$.

So let T be a consistent L-theory. By Proposition 3.4.6 there are $L' \supseteq L$ and a complete Henkin L'-theory $T' \supseteq T$, and by Proposition 4.1.1, there is a model $M' \vDash T'$. In particular, $M' \vDash T$ and, because T mentions no symbols in $L' \setminus L$, if $M \coloneqq M' \upharpoonright L$ then $M \vDash T$. The cardinality bound follows from those in Propositions 3.4.6 and 4.1.1, together with the fact that taking the reduct of a structure does not change its domain. $\square$

Remark 4.1.8. The cardinality bound in the Completeness Theorem can be strict. A simple example is obtained by taking any T, in your favourite L, such that $T \vdash \exists y\ \forall x\ x = y$.

Chapter 5

09/04

5.1 Corollaries of the Completeness Theorem

Corollary 5.1.1. The Completeness Theorem holds for propositional logic.

Proof. As in the proof of Corollary 2.2.4. □

The Soundness and Completeness Theorems, taken together, say that being consistent is the same as having a model. As anticipated, we can now prove that being complete is the same as being *the* theory of a model.

Corollary 5.1.2. T is a complete theory if and only if there is M such that $T = \mathrm{Th}(M)$.

Proof. One direction is Example 3.3.5. For the other direction, by the Completeness Theorem there is $M \vDash T$, which means $\mathrm{Th}(M) \supseteq T$. The inclusion cannot be strict by completeness and Corollary 3.3.4. □

Theorem 5.1.3 (Compactness Theorem)**.** A theory has a model if and only if each of its finite subsets does.

Proof. By the Completeness Theorem and Syntactic Compactness. □

Note that while the proof above uses Syntactic Compactness, which is a notion involving a proof system, the statement of the Compactness Theorem is purely semantic (and in fact, you have already seen a proof that does not use any proof system).

5.2 Digression: more completeness theorems

We have seen *a* Soundness Theorem and *a* Completeness theorem. There's more.

First of all, we have seen that one can work with a restricted system of logical symbols. The choice we made in this course is not the only sensible one. In particular, a popular choice when working semantically is to limit the logical symbols to $\neg, \wedge, \exists, \bot$.

Exercise 5.2.1 (Easy)**.** Prove that $\neg, \wedge$ is a complete set of connectives for propositional logic (that is, they suffice to interpret connectives with arbitrary truth tables).

Exercise 5.2.2 (Challenging)**.** Come up with a restriction of ND dealing only with $\neg, \wedge, \exists, \bot$. Prove Soundness and Completeness for it.

Hint. Work backwards: look at the proofs of Completeness and Soundness above and see what rules your system needs in order to be able to adapt them. □

One can do the opposite and introduce other logical symbols as abbreviations for existing ones, together with suitable rules for them. For example, adding $\nleftarrow$ to ND yields a pleasantly symmetric system (in ND, the symbol $\to$ is the only one without a dual, which would be $\nleftarrow$).

So far this is all syntactic sugar. Going a little bit deeper, one can also change the class of structures considered.

Example 5.2.3. As Figure 1.3 reminds us, our framework does not allow structures with empty domain. Nevertheless, it *is* possible (and, believe it or not, sometimes useful) to allow them. Tarski semantics stays essentially the same (but no language with constant symbols will have structures with empty domain). A modified form of Natural Deduction satisfies Soundness and Completeness relative to the class of possibly empty structures, see [Sta16].

But let us go deeper. What we have proven is the Completeness Theorem for single-sorted classical first-order logic with equality, with respect to Natural Deduction, the class of nonempty first-order structures, and Tarski semantics. More generally, one can define what is *a* logic, with its own formulas, notion(s) of structure, and semantics, and give a deduction system for it. For some of these choices, it is possible to prove Soundness and Completeness Theorems.

One of many generalisations is *second-order logic*, which expands first-order logic by allowing quantification over relations. Of course there's also third-order logic, where one can e.g. quantify over filters, etc. People also allow multiple sorts (basically what are called "types" in computer science), infinitary formulas (e.g. by allowing infinite disjunctions), generalised quantifiers (e.g. "there are uncountably many"), modal operators (e.g. "it is possible that"), and a number of other things. People even drop innocent-looking rules like Wk in order to build a logic that models the expenditure of resources, having proofs that "consume" formulas (*linear logic*), or restrict the logical symbols to $\exists, \wedge, \vee, \bot$ (no negation!) to have a *positive logic* that studies formulas preserved by homomorphisms, or have formulas where connectives are continuous real functions, quantifiers are inf and sup, equality is a distance and the scalar product of a Hilber space is a perfectly legit relation (sic[1]) symbol (*continuous logic*), or discard formulas altogether and look at syntax-free versions of logic based on category theory (*AECats*[2]).[3] And of course this is not a complete list.

Do not be fooled into thinking that every property of first-order logic will carry over to all of these settings, though. This example is classical.

[1]Yes, relation, not function.
[2]Abstract Elementary Categories.
[3]Although I have heard people arguing that category theory *is* syntax.

Example 5.2.4. Let T be second-order Peano arithmetic, where induction is formulated with a single formula quantifying over subsets. As you probably know, its only model (with the obvious semantics for second-order logic, where subsets are interpreted as subsets) is, up to isomorphism, $\mathbb{N}$. If we expand the language by a new constant symbol c, add the axioms $\{c \neq \underbrace{s(s(\dots}_{n \text{ timess}}(x))) \mid n \in \omega\}$, and call the resulting second-order theory T', then every finite subset of T' has a model ($\mathbb{N}$ with c interpreted as a large enough number). But T' has no model: it would have to be an expansion of $\mathbb{N}$ because $T' \supseteq T$ and T has no other models, but $\mathbb{N}$ has no place to accommodate c in such a way as to satisfy T'.

This shows that second-order logic is not compact, i.e. here the analogue of the Completeness Theorem fails. In particular, by looking at the proof of Theorem 5.1.3, we see that, whatever the definition of "a logic" may be, second-order logic does not admit any proof system that is simultaneously finitary (i.e. satisfies Syntactic Compactness) and complete with respect to the obvious semantics.

You may wonder whether the above is a reason to work with second-order logic. After all, being able to pin down an infinite structure up to isomorphism may seem like a desirable property. There are several good reasons *not* to do this. One is that, contrary to first-order logic[4], for second-order logic, whether a sentence has a model depends on the universe of set theory you live in.

Exercise 5.2.5. Write down a sentence in second-order logic that has a model if and only if the Continuum Hypothesis holds. Bonus points if you find one in the empty language.

This tells us that, in a sense, second-order logic is *too* expressive: it can talk about things happening outside of the structure we are studying! This is particularly bad if you are picking a logic to serve as a bedrock for a foundational theory, e.g. ZFC, as some questions then end up having "turtles all the way down" [Wik] kind of answers.

We will see shortly that first-order logic has way more limited expressive power. While this may seem like a shortcoming, it turns out that this lack of expressivity can be leveraged in creative —and useful!— ways.

Nevertheless, second-order logic has interesting uses in some contexts, for example in certain questions regarding finite structures that are intimately tied to computational complexity. And speaking of finite structures, I guess it is a good time to mention that those *can* be pinned down up to isomorphism by first-order logic.

Exercise 5.2.6. Let M be a finite structure in a finite language L.

1. Prove that there is a first-order L-sentence φ such that $N \vDash \varphi$ if and only if $N \cong M$.

2. What if L is infinite?

[4]Here I am sweeping under the rug an interesting subtlety you can read about in [HY13]. But I would recommend waiting until you have fully digested the Incompleteness Theorems before doing that.

5.3 Elementary equivalence

Of course, having the same complete theory, that is, satisfying the same sentences, deserves a name. And so does having the same models.

Definition 5.3.1. Two L-structures M and N are *elementarily equivalent*, written $M \equiv N$, iff $\mathrm{Th}(M) = \mathrm{Th}(N)$.

How does one prove that two structures are elementarily equivalent? If we happen to know that both are models of a certain theory, that we already know to be complete, then of course we are done. In general, this question can be quite difficult, although in some cases there is an easy way out. We will see a criterion in Proposition 6.4.5. On the other hand, in order to show that two structures are *not* elementarily equivalent, all we need to do is finding a suitable sentence distinguishing them.

Exercise 5.3.2. Find L_{ab}-sentences showing that the following abelian groups, viewed as L_{ab}-structures in the natural way, are pairwise not elementarily equivalent.

1. $\mathbb{Q}$
2. $\mathbb{Z}$
3. $\mathbb{Z}^2$
4. $\mathbb{R}/\mathbb{Z}$
5. $\mathbb{Z}_{(3)}$, the (reduct to L_{ab} of the) localisation of $\mathbb{Z}$ at the ideal (3).
6. $(\mathbb{Z}/3\mathbb{Z})^{\oplus\aleph_0}$, the direct sum of countably many copies of the cyclic group with 3 elements.
7. $\mathbb{Z}(3^\infty)$, the Prüfer group of all 3^n-th complex roots of unity, as n varies, with multiplication.

The list of groups in this exercise is not completely arbitrary. If you want to know more about what logic has to say about abelian groups, a good source is [Hod93, Appendix A.2].

5.4 Definable subsets in a structure

As anticipated, although it is sometimes convenient to only work with sentences, formulas with free variables have their uses. Notably, they can be used to define sets, by looking at what points are "solutions". Below, we follow the conventions stipulated in Notation 3.4.1.

Definition 5.4.1. The set *defined* by an L-formula[5] $\varphi(x_0, \dots, x_{n-1})$ in M is

$$\varphi(M) \coloneqq \{(a_0, \dots, a_{n-1}) \in M^n \mid M \vDash \varphi(a_0, \dots, a_{n-1})\}$$

A subset of M^n is *definable* (in M) iff it is defined by some L-formula.

[5] To be extremely precise, here we should be talking of pairs given by a formula φ and a sequence of length n of pairwise distinct free variables whose image includes all the free variables in φ. This fixes a correspondence with coordinates in M^n (for example, if M has a linear order $<$, we want to think of $x_0 < x_1$ as the above-diagonal of M^2), and to write things like $\varphi(x_0, x_1) \coloneqq x_0 = 0$ and think of them as defining a line M^2.

Notation 5.4.2. The *language of ordered abelian groups* is $L_{\mathrm{oag}} \coloneqq \{+, 0, -, \leq\}$, where $+$ is a binary function symbol, 0 a constant symbol, $-$ a unary function symbol, and $\leq$ a binary relation symbol. We will write e.g. $\mathbb{R}_{\mathrm{oag}}$ for the natural L_{oag} structure on $\mathbb{R}$, write $<$ with the obvious meaning, etc.

Example 5.4.3. In $\mathbb{R}_{\mathrm{oag}}$, the formula $\varphi(x_0, x_1, x_2) \coloneqq x_0 + x_1 = x_2$ defines the graph of addition.

Non-Example 5.4.4. The set $\mathbb{Z}$ is *not* definable in $\mathbb{R}_{\mathrm{oag}}$. We will prove this shortly.

Quite often it is necessary to look at formulas with *parameters* from some subset $A \subseteq M$. It means what you expect, but formally this is what one does.

Definition 5.4.5. Let M be an L-structure and $A \subseteq M$. Define a language $L(A) \supseteq L$ by adding to L a new constant symbol c_a for every $a \in A$. Expand M to an $L(A)$ structure M_A by interpreting each c_a with a. A subset of M^n is *A-definable*, or *definable over A* iff it is definable in M_A.

Example 5.4.6. In R, the set $\{(x_0, x_1) \in \mathbb{R}^2 \mid x_0 < x_1 + 5\}$ is definable over $\mathbb{Z}$ (or even just over $\{5\}$).

See the literature for more lists of examples, e.g. [Mar02, Section 1.3] has some nice, more convoluted (and more interesting!) ones.

Sometimes we say that a set is $\emptyset$-definable to emphasise that it is definable without using parameters. Depending on the context, people use the work *definable* to mean “definable over $\emptyset$” or “definable over M”. In these notes, we stick to the first meaning.

Remark 5.4.7. For fixed M, n, and $A \subseteq M$, the A-definable subsets of M^n form a boolean algebra, with the operations induced by the connectives $\wedge, \vee, \neg$, which of course correspond to intersection, disjunction, and complement of definable sets.

5.5 Definable sets in a theory

Notation 5.5.1. From now on, “theory” means “deductively closed consistent set of sentences”.[6] The “deductively closed” part may be dropped whenever it makes sense to do so (it should be clear from context).

In Section 5.4 we have talked about definable subsets of a structure M. In particular, we have implicitly fixed a complete theory, $\mathrm{Th}(M)$. More generally, to talk of sets definable with parameters A, we have fixed a certain $L(A)$-theory.[7] One can also talk about definable sets in an arbitrary consistent theory, although now, in general, it does not make sense to talk of parameters. Let us see how.

[6] You may object that a set of formulas cannot be simultaneously deductively closed and contain sentences only. The formula $x = x$ says that you are right. If you want, restrict the deductive closure to sentences only, say obtaining T_1: since T contains no free variables, the deductive closures (in the sense considering also formulas with free variables) of T and of T_1 coincide, cf. Section 2.3.

[7] Spoiler, it is the reduct of $\mathrm{ED}(M)$ to $L(A)$, see Definition 6.1.1.

Definition 5.5.2. Let T be an L-theory. Two L-sentences φ, ψ are *equivalent modulo* T iff, for every $M \vDash T$, we have $M \vDash \varphi \iff M \vDash \psi$. Two formulas[8] $\varphi(x_0, \ldots, x_n)$ and $\psi(x_0, \ldots, x_n)$ are *equivalent modulo* T iff, for all $M \vDash T$, we have $\varphi(M) = \psi(M)$.

Exercise 5.5.3. Two formulas $\varphi(x_0, \ldots, x_n)$ and $\psi(x_0, \ldots, x_n)$ are equivalent modulo T if and only if $T \vdash \forall x_0, \ldots, x_n\ (\varphi(x_0, \ldots, x_n) \leftrightarrow \psi(x_0, \ldots, x_n))$.

If we say that φ, ψ are *equivalent*, or *logically equivalent*, without specifying T (and without having a fixed T which is clear from context), we mean that they are equivalent modulo $T = \emptyset$. Note that, by Lemmas 2.3.2 and 2.3.4 and the Completeness Theorem, this is the same notion we encountered in Proposition 2.1.3.

So, two formulas with the same n free variables[9] are equivalent modulo T if and only if, in every $M \vDash T$, they define the same subset of M^n.

Now we can think of a definable set in T, in n free variables, as an element of the corresponding *Lindenbaum–Tarski algebra*: the boolean algebra whose elements are the classes of formulas $\varphi(x_0, \ldots, x_{n-1})$ modulo being equivalent modulo T, with the operations induced by $\wedge, \vee, \neg$. Observe that this also makes sense for $n = 0$, where we get the algebra of sentences modulo equivalence in T. Note the following.

Exercise 5.5.4. *T is complete (resp. consistent) if and only if the cardinality of its 0-th Lindenbaum–Tarski algebra is exactly (resp. at least) two.*

In general, the ultrafilters on the 0-th Lindenbaum–Tarski algebra[10] are naturally identified with the completions[11] of T. More generally, filters on this algebra correspond to consistent extensions of T. Those wondering where is the compactness in "Compactness Theorem " may like to know that the set of these ultrafilters carries a natural topology, and that the Compactness Theorem is precisely the statement that this topology is compact.

Ultrafilters on the n-th Lindenbaum–Tarski algebra of T are known as (complete) n-types and, if $n > 0$, form an interesting space also when T is complete. Types are usually studied at length in a model theory course, see for instance the notes [Men22] from one held a couple of years ago[12]. With types, one can prove statements like "for every $n \geq 1$, there is a complete countable theory with exactly n countable models up to isomorphism if and only if $n \neq 2$".

5.6 Substructures

Definition 5.6.1. We say that the L-structure M is a *substructure* of the L-structure N (and that N is an *extension*[13] of M), and write $M \subseteq N$, iff:

[8] Recall the pedantry from Footnote 5.

[9] Again, which may or may not actually appear in the formulas. If you really do not like this, take conjunctions with $(x_0 = x_0) \wedge \ldots \wedge (x_{n-1} = x_{n-1})$.

[10] The definition of "ultrafilter" that you have seen is the definition of "ultrafilter on $\mathscr{P}(X)$". In general, one may use essentially the same definition on a boolean algebra that is not necessarily the powerset of a set.

[11] Cf. Footnote 14.

[12] As of April 2024.

[13] Not to be confused with *expansion*, cf. Definition 3.2.1.

1. $M \subseteq N$ (more precisely, $\operatorname{dom} M \subseteq \operatorname{dom} N$);
2. for every n-ary function symbol $f \in L$, we have $f^M = f^N \upharpoonright M^n$;[14] and
3. for every n-ary relation symbol $R \in L$, we have $R^M = R^N \cap M^n$; in other words, for every $a_0, \ldots, a_{n-1} \in M$, we have $M \vDash R(a_0, \ldots, a_{n-1}) \iff N \vDash R(a_0, \ldots, a_{n-1})$.

Example 5.6.2. Seen as L_{oag}-structures with the usual interpretations, we have $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Q} \subseteq \mathbb{R}$.

Example 5.6.3. If G is a graph, viewed as an L_{graph} structure in the natural way, then a substructure of G is the same as an *induced* subgraph of G.

Non-Example 5.6.4. Let G be the complete graph on $\mathbb{N}$, and let H be a graph on $\mathbb{N}$ with no edge between 3 and 64. Then H is *not* a substructure of G.

Non-Example 5.6.5. Let P be a poset in the language $\{\sqsubseteq\}$, and suppose $a, b \in P$ are not comparable, that is, $P \vDash (\neg(a \sqsubseteq b)) \wedge (\neg(b \sqsubseteq a))$. Let P' be some linear order with domain P extending the order of P. For instance, take $(P, \sqsubseteq^P) \coloneqq (\mathbb{N} \setminus \{0\}, |)$, and $(P', \sqsubseteq^{P'}) \coloneqq (\mathbb{N} \setminus \{0\}, \leq)$, and set $a = 2$ and $b = 3$. Then P is *not* a substructure of P' (nor the other way around).

Example 5.6.6. If L is a relational language,[15] and M an L-structure, then every $A \subseteq M$ can be made into an L-substructure of M in a unique way.

Example 5.6.7. More generally, if $A \subseteq M$, and B is the closure of A under the functions and constants of L, then B can be (uniquely) made into a substructure of M. Of course, this substructure is called the *substructure of M generated by A*.

Definition 5.6.8. A map $\iota \colon M \to N$ is an *embedding*[16] iff it *preserves and reflects* atomic formulas, that is, for every atomic formula $\varphi(x_0, \ldots, x_{n-1})$ and $a_0, \ldots, a_{n-1} \in M$, we have

$$M \vDash \varphi(a_0, \ldots, a_{n-1}) \iff N \vDash \varphi(\iota(a_0), \ldots, \iota(a_{n-1})) \tag{5.1}$$

Exercise 5.6.9. Every embedding is an injective function.

Exercise 5.6.10. A subset A of M is the domain of a substructure of M if and only if there is an embedding with image A.

Remark 5.6.11. Colloquially, it is common to say that things like "an injective map is an embedding if and only if its image is a substructure". Strictly speaking, this is false: an embedding $M \to N$ contains more information than its image, and it is easy to see that a different injective map with the same image as an embedding may not be one. Details of this kind start becoming relevant in things like Exercise 6.2.3.

Exercise 5.6.12. Come up with an example of two structures and an injective map $f \colon M \to N$ such that $f(M)$ is a substructure of N but f is not an embedding.

[14] In particular, for every constant symbol $c \in L$, we have $c^M = c^N$.
[15] Recall that this means that L has only relation symbols.
[16] In italiano: embedding. C.f. Footnote 19.

We will not really use them, but it is worth mentioning that *morphisms* of L-structures are defined similarly, by dropping the requirement of injectivity and weakening (5.2) by dropping "reflecting", that is, replacing $\iff$ with $\implies$. For instance, in Non-Example 5.6.5, the identity map $\mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ is a (bijective) morphism (but not an embedding) $P \to P'$.

As usual, an *isomorphism* is a bijective morphism such that its inverse is again a morphism (equivalently, a bijective embedding[17]), and an *automorphism* of M is an isomorphism with domain and codomain M.

The following is a "you should do it exactly once in your life" exercise.

Exercise 5.6.13. Suppose that $f\colon M \to N$ is an isomorphism. For every formula (without parameters) $\varphi(x_0, \dots, x_{n-1}, y_0, \dots, y_{m-1})$ and tuple of parameters $(b_0, \dots, b_{m-1}) \in M^m$, we have

$$f(\varphi(M, b_0, \dots, b_{m-1})) = \varphi(N, f(b_0), \dots, f(b_{m-1}))$$

Hint. Work with $\bot, \land, \neg, \exists$. □

As unsurprising as Exercise 5.6.13 may sound, it allows us to justify Non-Example 5.4.4 very quickly.

Proof. Recall that in Non-Example 5.4.4 "definable" meant "definable over $\emptyset$". If $\mathbb{Z}$ was definable over $\emptyset$, by Exercise 5.6.13 it would be fixed setwise by every automorphism of $\mathbb{R}_{\mathrm{oag}}$. Multiplication by 2 is a counterexample. □

In fact, $\mathbb{Z}$ is not definable in $\mathbb{R}_{\mathrm{oag}}$, even if we allow parameters. This follows almost immediately from the fact that $\mathbb{R}_{\mathrm{oag}}$ has quantifier elimination. Alas, the latter is not something we will prove. Anyway, observe that the natural attempt at a formula (with parameters) defining it would use an *infinite* disjunction $\bigvee_{i\in\mathbb{Z}} x = i$. This is *not* a first-order formula.

While Exercise 5.6.13 tells us that isomorphisms preserve and reflect all formulas, for embeddings this is no longer true (example coming soon). Nevertheless, the set of formulas preserved (and reflected) by embeddings is a bit larger than just the atomic ones.

Definition 5.6.14. A formula $\varphi(x_0, \dots, x_n)$ is *quantifier-free* if no quantifier appears in φ.

Exercise 5.6.15. Let $n \in \omega$.[18] For every embedding $\iota\colon M \to N$, quantifier-free formula $\varphi(x_0, \dots, x_{n-1})$, and $a_0, \dots, a_{n-1} \in M$, we have

$$M \vDash \varphi(a_0, \dots, a_{n-1}) \iff N \vDash \varphi(\iota(a_0), \dots, \iota(a_{n-1}))$$

In particular, if $\varphi(x_0, \dots, x_{n-1})$ is quantifier-free and $a_0, \dots, a_{n-1} \in M \subseteq N$, then $M \vDash \varphi(a_0, \dots, a_{n-1}) \iff N \vDash \varphi(a_0, \dots, a_{n-1})$.

[17] Note that its inverse will automatically be an embedding.

[18] Including the case $n = 0$, that is, where φ is a sentence.

5.7 Elementary embeddings

In Exercise 5.6.15, the assumption that φ is quantifier-free is crucial:

Example 5.7.1. Let $L = \{<\}$, $M = (\mathbb{Z}, <)$ and $N = (\mathbb{Q}, <)$. Then $M \subseteq N$. Let $\varphi(x_0, x_1)$ be the formula $\exists y\ \big((x_0 < y) \land (y < x_1)\big)$. Then $\varphi(0, 1)$ holds in N, but not in M.

In particular, $\varphi(M) \neq \varphi(N) \cap M^2$. Even if $0, 1 \in M$, whether $\varphi(0, 1)$ holds or not depends on whether we check in M or in N.

Substructures where this never happens are called *elementary.*

Definition 5.7.2. A map $\iota\colon M \to N$ is an *elementary embedding*[19] iff it *preserves and reflects* all formulas, that is, for every formula $\varphi(x_0, \ldots, x_{n-1})$ and $a_0, \ldots, a_{n-1} \in M$, we have

$$M \vDash \varphi(a_0, \ldots, a_{n-1}) \iff N \vDash \varphi(\iota(a_0), \ldots, \iota(a_{n-1})) \tag{5.2}$$

We say that M is an *elementary substructure* of N (and N an *elementary extension* of M), written $M \preceq N$, iff M is a substructure of N and the inclusion map is an elementary embedding.

These easy observations are essentially exercises in spelling out definitions, but are quite important:

Remark 5.7.3. Let $M \subseteq N$.

- $M \preceq N$ if and only if, for every formula $\varphi(x_0, \ldots, x_{n-1})$ and $a_0, \ldots, a_{n-1} \in M$, we have

$$M \vDash \varphi(a_0, \ldots, a_{n-1}) \iff N \vDash \varphi(a_0, \ldots, a_{n-1})$$

- $M \preceq N$ if and only if they have the same $L(M)$-theory.
- In particular, if $M \preceq N$, then $M \equiv N$.
- $M \preceq N$ if and only if, for every formula $\varphi(x_0, \ldots, x_{n-1})$, we have $\varphi(M) = \varphi(N) \cap M^n$.[20] In other words whether a point of M belongs to an M-definable set or not is something that can be checked in an arbitrary elementary extension of M.

At the risk of offending someone, let me point out that isomorphisms are elementary embeddings. We will see a less trivial example in Example 6.1.6. Nevertheless, elementarity is really a condition on the embedding, and not just on the isomorphism type:

Example 5.7.4. Let $N \coloneqq (\mathbb{Z}, <)$ and $M = (2\mathbb{Z}, <)$. Then $M \subseteq N$, $M \cong N$, but $M \not\preceq N$, as can be checked by looking at the formula $\exists x\ 0 < x < 2$.[21]

[19] In italiano: *immersione elementare.* Perché nella Footnote 16 non ho messo direttamente "immersione" come traduzione di "embedding"? Perché c'è anche la nozione di "immersion", che è una via di mezzo fra embedding e immersione elementare.

[20] If you are particularly categorically-minded, you may like to think of definable sets as (a particular kind of) functors from the category of L-structures with elementary embeddings to the category of sets with injective maps.

[21] Which of course is an abbreviation for $\exists x\ ((0 < x) \land (x < 2))$, but I guess it's time to start being a bit less pedantic.

How does one check that a substructure is elementary?

Theorem 5.7.5 (Tarski–Vaught test)**.** Let N be an L-structure, and suppose that M is a subset of N. The following are equivalent.

1. M is the domain of an elementary substructure $M \preceq N$.
2. For all $\varphi(x, y_0, \ldots, y_{n-1})$ and all $b_0, \ldots, b_{n-1} \in M$, if there is $a \in N$ such that $N \vDash \varphi(a, b_0, \ldots, b_{n-1})$, then there is $a' \in M$ such that $N \vDash \varphi(a', b_0, \ldots, b_{n-1})$.

The statement of the Tarski–Vaught test (or criterion) looks very similar to the definition of $\preceq$. The difference is that, in order to check the condition in the criterion, we only need to look at which formulas are satisfied in N: there is no "$M \vDash$" in the statement; in fact, in the assumptions M is just a subset of N, and has not been given an L-structure (yet). This is subtle but important, as it allows arguments like the proof of Theorem 6.3.1 to go through.

Proof. (1) $\Rightarrow$ (2) follows easily from the definition of $\preceq$.

Towards proving (2) $\Rightarrow$ (1), observe that, if $f(y_0, \ldots, y_{m-1})$ is an m-ary function symbol of L and $b_0, \ldots, b_{m-1} \in M$, by using (2) with the formula $\varphi(x, y_0, \ldots, y_{m-1}) \coloneqq x = f(y_0, \ldots, y_{m-1})$, we find that M is closed under the function symbols of L. Therefore, M is the domain of a (unique) substructure M of N, cf. Example 5.6.7.

To show elementarity, we now need to show that, whenever $\varphi \in L(M)$ is a sentence, then

$$M \vDash \varphi \iff N \vDash \varphi \tag{5.3}$$

We argue by induction on formulas. If (5.3) holds for φ and ψ, then it is immediate to observe that it also holds for $\neg\varphi$ and for $\varphi \wedge \psi$. Let us consider the case $\exists x\ \varphi(x)$. If $M \vDash \exists x\ \varphi(x)$, then there is $a \in M$ such that $M \vDash \varphi(a)$. But $\varphi(a)$ has lower complexity, so by induction $N \vDash \varphi(a)$, and in particular $N \vDash \exists x\ \varphi(x)$, proving $\Longrightarrow$. For the converse, suppose $N \vDash \exists x\ \varphi(x)$; then there is $a \in N$ such that $N \vDash \varphi(a)$. By assumption, there is $a' \in M$ such that $N \vDash \varphi(a')$, and again by inductive hypothesis $M \vDash \varphi(a')$, hence $M \vDash \exists x\ \varphi(x)$. $\square$

Exercise 5.7.6. Prove in detail that if T has quantifier elimination, then: for every $M \vDash T$, $N \vDash T$, and $M \subseteq N$, we automatically have $M \preceq N$. Do not try to prove the converse.[22]

[22] Well, of course I will not physically try to stop you, but I should at the very least make you aware that the converse is false. The string to search for is "model complete".

Chapter 6

11/04

6.1 Diagrams

Recall the natural expansions by constants defined in Definition 5.4.5.

Definition 6.1.1. Let M be an L-structure.

1. Its *elementary diagram* $\mathrm{ED}(M)$ is the complete $L(M)$-theory of M_M.

2. Its *diagram* $\mathrm{diag}(M)$ is the subset of $\mathrm{ED}(M)$ given by atomic formulas and negations of atomic formulas.

Note that $\mathrm{ED}(M)$ is, by definition, always a complete $L(M)$-theory. On the other hand, $\mathrm{diag}(M)$ need not be (Exercise 6.1.5).

Exercise 6.1.2. If φ is a quantifier-free $L(M)$-sentence and $M_M \vDash \varphi$, then $\mathrm{diag}(M) \vdash \varphi$.

The point of these definitions is that models of the (elementary) diagram of M correspond to (elementary) extensions of M:

Proposition 6.1.3. Let M be an L-structure, and N be an $L(M)$-structure. Let $\iota\colon M \to N$ be the map $m \mapsto c_m^N$. Then:

1. ι is an embedding if and only if $N \vDash \mathrm{diag}(M)$.

2. ι is an elementary embedding if and only if $N \vDash \mathrm{ED}(M)$.

Proof. Exercise (easy). □

This is useful, because it allows us to build elementary extensions of M with certain properties by writing down suitable theories containing[1] $\mathrm{ED}(M)$.

As an application, here is a guided exercise.

Exercise 6.1.4. Let $L = \{s\}$, with s a unary function symbol. Show that in $(\mathbb{N}, s)$, where s is interpreted as the successor function $x \mapsto x + 1$, the graph of addition is not definable, not even with parameters.

[1]In languages larger than $L(M)$, since $\mathrm{ED}(M)$ is already complete.

Hint. Show that if the graph of addition is definable, then so is the set of even naturals, say by a formula $\varphi(x)$. Observe that $\mathbb{N} \vDash \forall x\, \varphi(x) \leftrightarrow \neg\varphi(s(x))$. Show there is a proper elementary extension of $(\mathbb{N}, s)$, say N. Show that any such N has an automorphism σ fixing $\mathbb{N}$ pointwise such that $\sigma(\varphi(N)) \neq \varphi(N)$ (setwise). Deduce that the set of even natural numbers is not definable over $\mathbb{N}$, hence that the graph of addition is not $\mathbb{N}$-definable either. □

Exercise 6.1.5. Find an M such that $\operatorname{diag}(M)$ is not complete.

Whether you realised it or not, you already know a way to produce (non-trivial) elementary extensions of a(n infinite) structure.

Example 6.1.6. If M is an L-structure and $\mathcal{U}$ an ultrafilter on some set I, then the *diagonal embedding* $a \mapsto (a)_{i\in I}/\mathcal{U}$ is an elementary embedding $M \to M^I/\mathcal{U}$.

Proof. Exercise. Hint: ultrapowers speak all languages; for all an ultrapower cares, L may as well be the language with every possible relation and function symbol on every cartesian power of M. □

6.2 Building models using magic

Let us look in detail at a simple proof by magic, the "magic" being the Compactness Theorem.

Example 6.2.1. There exists an elementary extension $M \succeq \mathbb{R}_{\mathrm{oag}}$ containing an element m with $m > \mathbb{R}$.

Proof. Let $L = L_{\mathrm{oag}}(\mathbb{R}) \cup \{c\}$, where c is a new constant symbol. Consider the set of L-sentences

$$\Phi \coloneqq \mathrm{ED}(\mathbb{R}_{\mathrm{oag}}) \cup \{c > r \mid r \in \mathbb{R}\}$$

If $\Phi_0 \subseteq \Phi$ is finite, then it contains only finitely formulas of the form $c > r$. Let $r_0 \in \mathbb{R}$ be larger than all these finitely many r. Expand $(\mathbb{R}_{\mathrm{oag}})_{\mathbb{R}}$ to an L-structure S by interpreting $c^S \coloneqq r_0$. Then, by construction, $S \vDash \Phi_0$, hence Φ_0 is consistent.

By compactness, Φ is consistent, hence there exists $M' \vDash \Phi$. Let $M \coloneqq M' \restriction L_{\mathrm{oag}}(\mathbb{R})$. By construction, $M \vDash \mathrm{ED}(\mathbb{R}_{\mathrm{oag}})$, so by Proposition 6.1.3 the map $r \mapsto c_r^M$ is an elementary embedding $R \to M$, which we may assume, for notational convenience, to be the inclusion. Now let $m \coloneqq c^{M'}$. By construction, for all $r \in \mathbb{R}$ we have $\Phi \vDash c > r$, hence $M' \vDash c > r$, that is, $M' \vDash m > r$, hence $M \vDash m > r$, and we are done. □

In this proof, I have freely confused $r \in \mathbb{R}$ with the corresponding constant symbol in $L_{\mathrm{oag}}(\mathbb{R})$ standing for it, but besides that I have been quite pedantic, and spelled out explicitly all the naming new constants and taking reducts. Usually, one leaves these details to the reader. Below, you will find some more statements that can be proven by compactness as exercises. I recommend doing at least a couple of them by spelling out all details in a similar fashion as above.

As is typical with compactness arguments, the proof above tells us very little about the structure we have proven to exist, but at least it allows us to conjure one basically out of thin air (hence the title of this subsection). Here is another standard compactness argument which allows us to conjure quite large things.

$$\begin{tikzcd} & B_0 \arrow[dr, dashed, "g_0"] & \\ A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & C \\ & B_1 \arrow[ur, dashed, "g_1"'] & \end{tikzcd}$$

Figure 6.1: This diagram is required by law to be drawn whenever the amalgamation property is mentioned.

Corollary 6.2.2 (Upward Löwenheim–Skolem Theorem)**.** Let T be a theory such that, for every $n \in \mathbb{N}$, there is $M \vDash T$ of cardinality at least n. Then, for every cardinal κ, there is $M \vDash T$ of cardinality at least κ.

Furthermore, if T has an infinite model M_0, we may also require that $M \succeq M_0$.

Proof. Expand the language L of T to L' by adding new constant symbols $\{c_\alpha \mid \alpha < \kappa\}$, and let Φ be the set of L'-sentences

$$\Phi \coloneqq T \cup \{c_\alpha \neq c_\beta \mid \alpha < \beta < \kappa\}$$

Every finite subset of Φ can only mention finitely many c_α, hence *by compactness* and our assumptions on T, the set Φ has a model. Its reduct to L is the required M.

For the "furthermore" part, argue as above, but replacing T with $\mathrm{ED}(M_0)$. $\square$

Here is another standard fact which can be proven by using compactness.

Exercise 6.2.3. Let T be an L-theory. Prove that the category of models of T, with arrows the elementary embeddings,[2] has the *amalgamation property*: whenever A, B_0, B_1 are models of T, and $f_i \colon A \to B_i$ are elementary embeddings, there are $C \vDash T$ and elementary embeddings $g_i \colon B_i \to C$ such that $g_0 \circ f_0 = g_1 \circ f_1$. See Figure 6.1.

Exercise 6.2.4. Suppose that L contains infinitely many unary relation symbols $(P_i)_{i \in I}$, and that T is an L-theory where the P_i are nonempty and pairwise disjoint, that is, such that,

1. for every $i \in I$, we have $T \vdash \exists x\ P_i(x)$, and
2. for every $i \neq j \in I$, we have $T \vdash \neg\exists x\ (P_i(x) \wedge P_j(x))$.

Prove that there are some $M \vDash T$ and $m \in M$ such that, for all $i \in I$, we have $M \vDash \neg P_i(m)$.

Here is a small list of some more (fairly standard) things you can prove with compactness and some thinking. It is a good idea to try doing these exercises, and maybe to look for more in the literature. It is also a good idea to do at least two of them in a level of detail comparable to the proof of Example 6.2.1 above.

[2]Exercise: the composition of two elementary embeddings is an elementary embedding.

Exercise 6.2.5. There is no L_{grp}-theory whose models are precisely the torsion groups (i.e. those where every element has finite order).

Exercise 6.2.6. There is no L_{graph}-theory whose models are precisely the graphs of finite diameter.

Exercise 6.2.7. There is an elementary extension of $(\mathbb{Z}, <)$ in which we may embed $(\mathbb{R}, <)$. Can such an embedding be elementary?

Exercise 6.2.8. Fix a theory T, a formula $\varphi(x)$, and suppose that, for every $M \vDash T$, the set $\varphi(M)$ is finite. Then it is *uniformly finite*, i.e. there is $n \in \omega$ such that, for every $M \vDash T$, the set $\varphi(M)$ has size at most n.

Exercise 6.2.9. Show that if the elements of a *definable family* are finite in all models, then they are uniformly finite. More precisely, show that if $\varphi(x, y_0, \dots, y_{k-1})$ is formula such that, for every $M \vDash T$ and every tuple of parameters $(b_0, \dots, b_{k-1}) \in M^k$, the set $\varphi(M, b_0, \dots, b_{k-1})$ is finite, then there is an upper bound n on its cardinality that does not depend on the choice M, nor on the choice of $b_0, \dots, b_{k-1}$.

Exercise 6.2.10. Show that the conclusion of Exercise 6.2.9 fails if we only check on a single model, in the following sense: find a structure M and a formula $\varphi(x, y_0, \dots, y_{k-1})$ such that for every $(b_0, \dots, b_{k-1}) \in M^k$ the $\{b_0, \dots, b_{k-1}\}$-definable set $\varphi(M, b_0, \dots, b_{k-1})$ is finite, but there are $N \succ M$ and a tuple of parameters $(b'_0, \dots, b'_{k-1}) \in N^k$ such that $\varphi(N, b'_0, \dots, b'_{k-1})$ is infinite.

Exercise 6.2.11. Let G be a graph and $n \in \omega$. Then G is colourable with n colours if and only if each of its finite induced subgraphs is.

Exercise 6.2.12. There is no L_{graph}-theory T such that the class of models of T is precisely the class of connected graphs.[3]

Exercise 6.2.13. Every partial order extends to a linear order.

Exercise 6.2.14. There is no first-order $\{\leq\}$-theory T such that the class of models of T is precisely the class of well-orders.

Exercise 6.2.15. If ZFC is consistent, then it has a model containing nonstandard natural numbers.

Exercise 6.2.16. If ZFC is consistent, then there are $M \vDash \mathsf{ZFC}$ and a sequence $(a_i)_{i<\omega}$ of elements of M such that $a_{i+1} \in a_i$.

Exercise 6.2.17. Understand why Exercise 6.2.16 does not contradict the fact every model of ZFC satisfies the Axiom of Foundation.

Possibly confusing hint. A model of ZFC is just a digraph (of a special kind). □

[3] Bonus exercise: show that there is a second-order L_{graph}-theory T such that the class of models of T is precisely the class of connected graphs.

6.3 Building models using bookkeeping

The "magic" from Section 6.2 (that is, compactness) is very useful when we want to build "large enough" objects. Sometimes, we want things not to be *too* large, and in that case different tools are needed. One of these, which we have already seen, is the Henkin construction. Another is the *Downward Löwenheim–Skolem Theorem.*

Theorem 6.3.1 (Downward Löwenheim–Skolem)**.** Let M be an L-structure and $A \subseteq M$. There is an elementary substructure $M_0 \preceq M$ with $A \subseteq M_0$ and $|M_0| \le |A| + |L|$.

Proof. We do an inductive construction, starting with $B_0 \coloneqq A$. For every L-formula $\varphi(x, y_0, \dots, y_{k-1})$ and tuple $b \in B_n^k$, if there is $m \in M$ such that $M \vDash \varphi(m, b_0, \dots, b_{k-1})$, put one such m in B_{n+1}. Note that $|B_{n+1}| \le |L(B_n)| = |B_n| + |L|$, hence inductively $|B_{n+1}| \le |A| + |L|$. Therefore, $M_0 \coloneqq \bigcup_{n\in\omega} B_n$ has the required cardinality.

By the Tarski–Vaught test (Theorem 5.7.5) we need to check that, whenever $b \in M_0^k$ and $M \vDash \exists x\ \varphi(x, b_0, \dots, b_{k-1})$, then there is $a \in M_0$ such that $M \vDash \varphi(a, b_0, \dots, b_{k-1})$. Since $(b_0, \dots, b_{k-1})$ is a finite tuple from $M_0 = \bigcup_{n\in\omega} B_n$, it must be contained in some B_n, and by construction we can find the required a inside B_{n+1}. $\square$

Theorem 6.3.1 has caused headaches to a number of people, for the following reason.

Example 6.3.2 (Skolem paradox)**.** If ZFC is consistent, then it has a countable model.

Proof. By downward Löwenheim–Skolem. $\square$

Exercise 6.3.3. Understand why this does not contradict the fact that ZFC proves the existence of uncountable sets.

Example 6.3.4. There is $M \preceq \mathbb{R}_{\mathrm{oag}}$ which is not complete, in the sense that it has a subset with an upper bound but no supremum.

Proof. By Löwenheim–Skolem, there is a countable $M \preceq \mathbb{R}_{\mathrm{oag}}$ with $\mathbb{Q} \subseteq M$. As every ordered abelian group embedded in the reals, M must be Archimedean, and it follows easily by considering the group generated by any $m \in M \setminus \{0\}$ that M is unbounded in $\mathbb{R}$. Therefore, if $r \in \mathbb{R} \setminus M$, the set $\{m \in M \mid m < r\}$ is bounded in M. But, since M includes $\mathbb{Q}$, it is dense in $\mathbb{R}$, hence $\{m \in M \mid m < r\}$ has no supremum in M. $\square$

6.4 Categoricity

The Löwenheim–Skolem theorems give us models with upper and lower bounds on size. What if we want a model that has *precisely* a given size? For most cardinals, this is possible.

Exercise 6.4.1. Let M be an infinite L-structure. If $A \subseteq M$ and κ is a cardinal with $|A| + |L| \le \kappa \le |M|$, then there is $M_0 \preceq M$ with $A \subseteq M_0$ and $|M_0| = \kappa$.

Exercise 6.4.2. Let M be an infinite L-structure. If κ is a cardinal with $\kappa \geq |M| + |L|$, then there is $N \succeq M$ with $|N| = \kappa$.

Definition 6.4.3. A theory is *categorical* iff it has a unique model up to isomorphism. It is *κ-categorical* iff it has a unique model of cardinality κ up to isomorphism.

Note that, by definition, "(κ-)categorical" implies "consistent". To avoid misunderstandings, please note that the second part of Definition 6.4.3 is *not* to be read as "it has a unique model, which furthermore has cardinality κ, and…". In other words, "κ-categorical" does not imply "categorical". Well, not for every κ. In fact, this is a characterisation of finite nonzero cardinals:

Exercise 6.4.4. A theory is categorical if and only if its unique model is finite.

In Exercise 6.4.4, as well as almost everywhere in these notes, T is assumed first-order. As everyone and their dog knows, second-order Peano arithmetic is categorical, but the set of natural numbers is infinite.[4]

What can we say about κ-categorical theories? If $\kappa = |L| = \aleph_0$, it turns out that the answer is "*really* a lot". For example, there are at least 12 characterisations of this, see e.g. [Men22, Theorem 5.4.2]. If $\kappa > |L|$, the answer turns out to be "a *tremendous* amount of things". Once again, feel free to read spoilers in [Men22]. Nevertheless, at this point of this course, we already have enough tools to say something interesting.

Proposition 6.4.5 (Łoś–Vaught test)**.** Let T be an L-theory with no finite models. If there is a cardinal $\kappa \geq |L|$ such that T is κ-categorical, then T is complete.

Proof. Let T_0, T_1 be two completions of T, and let $M_0 \vDash T_0$ and $M_1 \vDash T_1$. By Upward Löwenheim–Skolem there are $N_0 \succeq M_0$ and $N_1 \succeq M_1$ of size at least κ. By Exercise 6.4.1 there are $N_0' \prec N_0$ and $N_1' \prec N_1$ both of cardinality κ. By assumption $N_0' \cong N_1'$. But then $T_0 = T_1$ follows from the fact that

$$M_0 \equiv N_0 \equiv N_0' \equiv N_1' \equiv N_1 \equiv M_1 \qquad \square$$

Exercise 6.4.6. The *theory of infinite sets* is the theory in the empty language (so, the only atomic formulas are equalities between variables) axiomatised by $\{\varphi_n \mid n \in \mathbb{N} \setminus \{0\}\}$, where

$$\varphi_n \coloneqq \exists x_0, \dots, x_{n-1} \bigwedge_{i \neq j < n} x_i \neq x_j$$

Prove that the theory of infinite sets is complete.

Exercise 6.4.7. Let K be an infinite field. Let $L_{K-\mathsf{VS}}$ be the language of K-vector spaces (cf. Exercise 3.3.10), and let $K{-}\mathsf{VS}$ be the theory of nontrivial K-vector spaces. Prove that $K{-}\mathsf{VS}$ is complete.

Exercise 6.4.7 implies that, for every positive[5] m and n the vector spaces K^m and K^n are elementarily equivalent. So, for instance, there is no $L_{\mathbb{Q}-\mathsf{VS}}$-theory whose models are precisely the $\mathbb{Q}$-vector spaces of dimension 3.

[4] Ok, not *literally* everyone agrees on this. Search for "ultrafinitism".

[5] Natural numbers, or even cardinals.

Exercise 6.4.8. Let $k > 0$. Show that $\mathbb{Q}^k$, see as an $L_{\mathbb{Q}-\mathrm{VS}}$-structure in the natural way, is an *expansion by definitions* of its reduct to L_{ab}. In other words, for every $\lambda \in \mathbb{Q}$, find an L_{ab}-formula (without parameters) defining the graph of scalar multiplication by λ. You should be able to do this independently of k.

Exercise 6.4.9. Show that, in Exercise 6.4.8, $\mathbb{Q}^k$ and $\mathbb{Q}^k \upharpoonright L_{\mathrm{ab}}$ have the same elementary substructures, but different substructures.

Exercise 6.4.10. Deduce from Exercise 6.4.8 that $\mathbb{Q} \equiv \mathbb{Q}^k$ in L_{ab}.

Definition 6.4.11. The L_{ring}-theory ACF of *algebraically closed fields* is the theory axiomatised by the axioms of fields and, for every $n > 1$, an axiom saying that every monic polynomial of degree n has a root, namely:

$$\forall y_0, \dots, y_{n-1}\ \exists x\ x^n + \sum_{i=0}^{n-1} y_i x^i = 0$$

Exercise 6.4.12. Describe all completions of ACF.

Exercise 6.4.13. In the proof of Proposition 6.4.5, where did we use that T has no finite models? What happens if we remove this assumption?

Exercise 6.4.14. Find all cardinals κ such that $\mathrm{Th}(\mathbb{N}, s)$ (cf. Exercise 6.1.4) is κ-categorical.

Hints.

1. Depending on taste, you may find it easier to first do the same exercise for $\mathrm{Th}(\mathbb{Z}, s)$, where s is again the function $x \mapsto x + 1$.

2. Aim for the stars: axiomatise that theory and classify all of its models.

3. Knowing the statement of Exercise 6.1.4 will probably not help you in solving this. On the other hand, if you have solved Exercise 6.1.4 by following the hinted proof strategy, then you may have already accidentally classified all models.

□

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