An Introduction to the Theory and Practice of Multigrid Methods: a summer school in Jyväskylä

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Contents

Disclaimer

I have written down these notes (mostly) "on the fly" during the 26$^{\text{th}}$ Summer School in Jyväskylä. Their main purpose is a personal use for the future, thus they may not be really clear, they can be a bit messy and they are NOT (in any possible way) the official notes of the course, held by Professor Johannes Kraus. Feel free to contact me for any mistake and/or suggestions at negriporzio@student.unipi.it. You can find these notes at `http://poisson.phc.unipi.it/~negriporzio/amat.html` (the main site is in italian).

1 MGM for variational problems

1.1 Linear stationary iterative methods

We want to consider a system of linear algebraic equations

$$Ax = b \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix (SPD) (or Hermitian in the complex case). Let B be an approximation of A^{-1} and consider the iteration

$$x_{k+1} = Gx_k + d \qquad k = 0, 1 \ldots, \tag{1.2}$$

where $G = I - BA$ is the iteration matrix/iterator or the error propagation matrix. If x^* is a solution of (1.1), then x^* is a fixed point of (1.2). The error is given by

$$e_k = x_k - x^* = Ge_{k-1},$$

thus

$$e_k = G^k e_0 \tag{1.3}$$

Definition 1.1. The iteration (1.2) is called *convergent* if for any initial point $x_0 \in \mathbb{R}^n$,

$$\lim_{k\to\infty} x_k = x^*$$

Definition 1.2 (Spectral radius)**.** Let $A \in \mathbb{R}^{n\times n}$ and let λ_i for $i = 1, \ldots, n$. Then the *spectral radius* is defined by

$$\rho(A) = \max_i \lambda_i.$$

Proposition 1.3. *Let* $G \in \mathbb{R}^{n\times n}$*. Then iteration* (1.2) *is convergent if and only if* $\lim_{k\to\infty} G^k = 0$ *and if and only if* $\rho(G) < 1$.

Proof. Easy. Exercise □

Remark 1.4. It can be shown that $\lim_{k\to\infty} \left\| G^k \right\|^{1/k} = \rho(G)$.

Classical iterative methods (Gauss-Seidel, Jacobi) are based on the splitting $A = M - N$. In this case $G = M^{-1}N = I - M^{-1}A$.

Example 1.5. Let $A = D - L - U$, where D, $-L$, $-U$ are the diagonal, the lower and the upper triangular part of A respectively. The *Jacobi method* is characterized by $M = D$ and $N = L + U$. The *Gauss-Seidel* is characterized by $M = D - L$. For example, the iteration of GS becomes

$$(D - L)x_{k+1} = Ux_k + b \tag{1.4}$$

Definition 1.6. A matrix $A = (a_{ij})$ is called *weakly diagonally dominant* if

$$|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|$$

and there exists an index i_0 such that the inequality is strict

Definition 1.7. A matrix A is called *irreducible* if there exists no permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{k \times k}$, with $k \geq 1$.

Theorem 1.8 (Row sum criterion)**.** *Let $A \in \mathbb{R}^{n \times n}$ be an irreducible weakly diagonally dominant matrix. Then GSS and Jacobi methods converge.*

Proof. Easy. Check the spectral radius of the iteration matrix and use Gerschgorin theorem. It can be found in D.$\tilde{\text{B}}$reuss, Finite elements, Cambridge University Press. □

Example 1.9. The successive over-relaxation method (SOR) is defined by

$$Dx_{k+1} = \omega(Lx_k + Ux_k + b) + (1 - \omega)Dx_k, \qquad k = 0, 1, \ldots, \tag{1.5}$$

with $\omega \in]0, 2[$.

Theorem 1.10. *If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix with positive diagonal entries, then the SOR method converges if and only if A is symmetric positive definite.*

Proof. Same book as before. □

1.2 A model problem by Courant

We want to solve the following Poisson problem

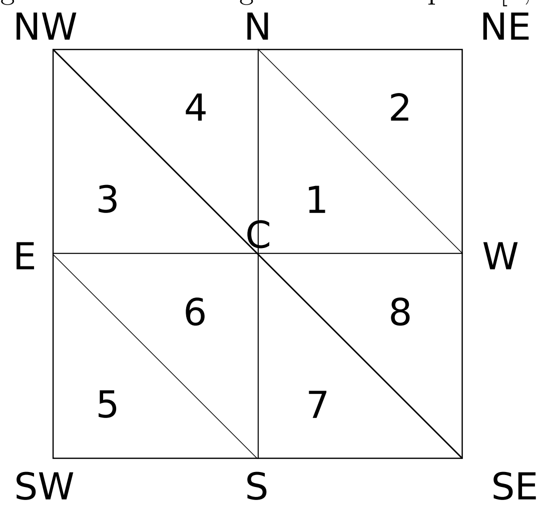
$$\begin{cases} -\Delta u = f & \text{in } \Omega =]0, 1[^2 \\ u = 0 & \text{in } \Gamma = \partial\Omega \end{cases}. \tag{1.6}$$

$\bar{\Omega}$ is partitioned by a uniform mesh of isosceles right-angled triangles T as depicted in 1. For (1.6) we want to use an appropriate Galerkin methods with piece-wise continuous trial and test functions

$$v_h \in V_h := \{u \in C(\bar{\Omega})\colon u_{|T} \text{ is linear } \forall T \in \mathcal{T}_h\}.$$

Every function $v_h \in V_h$ is determined on every triangle $T \in \mathcal{T}_h$ uniquely by its three function values in the vertices of T. Moreover, every $v_h \in V_h$ is

Figure 1: Partitioning of the unit square $[0,1]^2$.



determined uniquely globally by its values in all the $N := (n-1)^2$ interior vertices (nodes) of $\mathcal{T}_h$. We then choose a basis $\{\Psi_i\}_{i=1}^N$ of V_h such that $\Psi_i(x_j, y_i) = \delta_{ij}$ and thus we have $\dim V_h = N$.

The Galerkin method then reads: Find $v_h \in V_h$ such that

$$a(u_h, v_h) := \int_\Omega \nabla u_h \nabla v_h \, dx = \int_\Omega f u_h \, dx =: F(u_h) \quad \forall u_h \in V_h \tag{1.7}$$

Substituting u_h in the basis coordinates and testing for all the vector of the basis one finds that (1.7) is equivalent to the linear system

$$Au = f, \tag{1.8}$$

where A is the *stiffness matrix* $A_{ij} = a(\Psi_i, \Psi_j)$. In order to determine the entries A_{ij} we observe that for a basis function Ψ_c that takes the value 1 in the node (vertex) c we have

$$\begin{aligned} A_{ii} = a(\Psi_c, \Psi_c) &= \int_{1\ldots8} (\nabla\Psi_c)^2 \, dxdy \\ &= \int_{1+3+4} [\partial_1 \Psi_c^2 + \partial_2 \Psi_c^2] \, dxdy = \cdots = 4 \end{aligned}$$

For the off-diagonal entries we obtain

$$a(\Psi_c, \Psi_N) = a(\Psi_c, \Psi_N) = a(\Psi_c, \Psi_N) = a(\Psi_c, \Psi_N) = -1 \tag{1.9}$$

$$a(\Psi_c, \Psi_{NW}) = \cdots = a(\Psi_c, \Psi_{SE}) = 0 \tag{1.10}$$

In summary the linear system reads

$$4x_{ij} - x_{(i+1)j} - x_{(i-1)j} - x_{i(j+1)} - x_{i(j-1)} = b_{ij} \quad \text{for } 1 \leq i, j \leq n-1,$$

where the convention is to drop all the indices 0 and n (boundary condition).

Obviously the matrix A is irreducible and weakly diagonally dominant, therefore GSS and Jacobi methods are convergent, but extremely slow. We may also note that A is the same matrix we would obtain from the *finite differences method.*

1.3 Smoothing property of classical iterative methods

We study the Jacobi methods as a typical example. For the model problem we considered, we have that

$$G_j = D^{-1}(L - U) = I - D^{-1}A = I - \frac{1}{4}A,$$

so G_j has the same eigenvectors of A. We can denote them by $z^{l,m}$, with $1 \le l, m \le n - 1$. Using trigonometric identities, it can be easily shown that

$$\begin{aligned} Az^{l,m} &= (4 - 2\cos\frac{l\pi}{n} - 2\cos\frac{m\pi}{n})z^{l,m} \\ z_{i,j}^{l,m} &= \sin\frac{il\pi}{n}\sin\frac{jm\pi}{n} \end{aligned} \tag{1.11}$$

The spectral radius of G_j is obtained in $l = m = 1$ and $\rho(G_j) = \cos\frac{\pi}{n} = 1 - O(n^{-2})$. One can also show that GSS has the same asymptotic rate of convergence, although is a bit faster.

For the Jacobi method with relaxation parameter the eigenvalues read

$$\lambda^{l,m} = (\frac{1}{4}\cos\frac{l\pi}{n} + \frac{1}{4}\cos\frac{m\pi}{n} + \frac{1}{2}) \quad \text{for } \omega = \frac{1}{2} \tag{1.12}$$

Remark 1.11. After a few iteration with the Jacobi method (i.e. $\omega = 1$) the error contains only components for which l and m are both small (close to 1) or both large (close to n). The latter error components will also be reduced efficiently if one inserts a step with $\omega = \frac{1}{2}$. Then only smooth components (components with large wave length) or low frequency components of the error will remain after a few relaxation steps. For these smooth error components the reduction factor will only be $1 - O(h^2)$

For smoothing in practice one uses the *Richardson method* (which is characterized by $G_r = I - \frac{1}{\lambda_{\max}}A$), Jacobi, GSS, SOR and the linear stationary iterative methods based on incomplete factorization (or block variants of the above mentioned).

1.4 Two-grid methods

MG methods are based on the following idea: First one carries out a few iterations with the classical iterative scheme, which is called *relaxation*; after the high-frequency components have been "removed" (significantly reduced in amplitude) the residual is transferred to a coarser grid and the equation is solved there (eventually we solve it by the same method recursively).

Leading principles:

1. Smooth functions can be well approximated on coarse grids
2. Smooth error components "appears" more oscillatory on coarse grids and can be reduced more efficiently there by relaxation.

Assume we want to solve the problem

$$a(u_h, v_h) = F(u_h) \qquad \forall u_h \in V_h \subset V$$

that comes from a conforming FEM for an elliptic boundary value problem (BVP). Note that if $a(\cdot,\cdot)$ is a symmetric, coercive and bounded bilinear form and $F(\cdot) \in V'$, then by Lax-Milgram lemma it has a unique solution $v_h \in V_h$ which is also the unique solution of the minimization problem

$$\min_{u_h \in V_h} J(u_h), \tag{1.13}$$

where $J(u) = \frac{1}{2}a(u,u) - F(u)$.

Notation: We denote the smoothing operator by S. The k-th cycle of the two-grid method is then defined by the following algorithm (Two-grid methods). Let $u_h^{(k)} \in V_h$ be a given approximation of the solution u_h of (1.7):

1. Smoothing: Apply ν smoothing steps to $u_h^{(k)}$ and obtain $u_1^{(k,1)} := S^\nu u_h^{(k)}$
2. Coarse-grid correction: Compute the solution ω_H of the variational problem on a coarser grid with mesh size $H > h$ using the minimization form:
$$J(u_h^{(k,1)} + \omega_H) = \min_{u_H \in V_H} J(u_h^{(k,1)} + u_H),$$
(or solving the variational problem itself) where $V_H \subset V_h$ and set $u_h^{(k+1)} = u_h^{(k,1)} + w_H$.

Remark 1.12. The parameter ν determines the number of smoothing (relaxation) steps the algorithm performs. For elliptic problems and conforming FEM it is usually sufficient to use $\nu \leq 3$.

1.5 The multigrid algorithm

For simplicity we consider here Lagrangian Finite Elements and conforming discretization using nested FE spaces.

We start with a coarsest triangulation $\mathcal{T}_{h_0}$ with mesh size h_0 of the domain Ω. For simplicity we assume Ω is a polygonal domain, therefore the triangulation is exact. Next, every triangle $T \in \mathcal{T}_{h_0}$ is refined by subdividing it in 4 congruent triangles in the following triangulation $\mathcal{T}_{h_1}$ with mesh size $h_1 = \frac{h_0}{2}$. Repeating this procedure, we define a sequence of nested triangulation $\{\mathcal{T}_{h_i}\}$. For sake of notation, from now on we will set $\mathcal{T}_l := \mathcal{T}_{h_l}$. For each triangulation, we create a finite element space of piece-wise polynomial (piece-wise linear) functions V_i with the property of nestedness:

$$V_1 \subset V_2 \subset \cdots \subset V_L \subset V. \tag{1.14}$$

We call $\{V_i\}_{i=1}^L$ the nested spaces, while V is a conforming space of continuous functions, usually $H^1(\Omega)$. The following algorithm describes the k-th cycle of the approximate solution of (1.14) at level l, i.e., in V_l [MGM$_l$]. Let $u_l^{(k)}$ be an approximation of the solution of (1.14) u_l in V_l:

1. *Pre-smoothing*: Apply ν_1 smoothing steps to $u_l^{(k)}$, $u_l^{(k,1)} = S^{\nu_1} u_l^{(k)}$.
2. *Coarse-grid correction*: Compute the solution ω_{l-1} of the variational problem
$$J(u_l^{(k,1)} + \omega_{l-1}) = \min_{v_{l-1} \in V_{l-1}} J(u + v_{l-1}) \tag{1.15}$$
If $l = 1$ we compute the solution of (1.15) exactly and set $v_{l-1} = \omega_{l-1}$. Otherwise we compute an approximate solution by applying ν steps of MGM$_{l-1}$ at level $l-1$ using initial guess $u_{l-1}^{(0)} = 0$. Set $u^{(k,2)} = u_l^{k,1} + v_{l-1}$.
3. *Post smoothing*: Apply ν_2 steps of smoothing to $u^{(k,2)}$ and obtain
$$u_l^{(k,3)} = S^{\nu_2} u_l^{(k,2)}$$
We set $u_l^{(k+1)} := u_l^{(k,3)}$.

Remark 1.13. For $l = 1$ we solve the coarse-grid problem exactly. For $l > 1$ the coarse-grid problem is solved approximately and thus the MG iteration can be viewed as a perturbed two-grid iteration.

Remark 1.14. The parameter ν determines the amount of work is spent on the coarse-grid correction steps. If we set $\nu = 1$ we have the so-called *V-cycle*, while $\nu = 2$ is the so-called *W-cycle*. If $\nu = 1$ the coarsest grid is visited once, while if $\nu = 2$, it is visited 2^{L-1} times.

Remark 1.15. Sometimes post-smoothing is skipped, i.e., one chooses $\nu_2 = 0$. The V-cycle is often performed symmetrically, with $\nu_1 = \nu_2$.

The problem (1.15) to be solved in the coarse-grid corresponding step can be written as

$$a(u_l^{(k,1)} + w_{l-1}, v_{l-1}) = F(v_{-1}) \qquad \forall v_{l-1} \in V_{l-1} \tag{1.16}$$

and in a matrix form as

$$A_{l-1}y_{l-1} = b_{l-1}. \tag{1.17}$$

In order to find A_{l-1} and b_{l-1} we use that $V_{l-1} \subset V_l$, i.e., each basis function $\Psi_j \in V_{l-1}$ can be represented as a linear combination of the basis functions $\Phi_i \in V_l$:

$$\Psi_j = \sum_{i=1}^{N_l} r_{ij}\Phi_i \tag{1.18}$$

Due to (1.16) we have

$$a(\omega_{l-1}, u_{l-1}) = F(u_{l-1}) - a(u_{l-1}^{(k,1)}, u_{l-1}) \qquad \forall u_{l-1} \in V_{l-1} \tag{1.19}$$

Using (1.18) in (1.19) and rearranging the sums one finds that 1.18 takes the form of

$$RA_lR^Ty_{l-1} = Rd_l, \tag{1.20}$$

where $d_l = d$ is the vector defined by its components

$$d_i = F(\Phi_i) - \sum_{k=1}^{N_l} a(\Phi_k, \Phi_i)x_k$$

with $u_l^{(k,1)} = \sum_{k=1}^{N_l} x_k\Phi_k$. The matrix R is defined by the relation (1.18).

Denoting by P the matrix representation of the injection $\mathcal{I} : V_{l-1} \to V_l$, we have that $R = P^T$ is the matrix representation of the adjoint operator $\mathcal{I}^* : V_l^* \to V_{l-1}^*$. Then

$$A_{l-1} = RA_lP = P^TA_lP$$

and $b_{l-1} = P^Td_l$ with the defect $d_l = b_l - Ax_l^{(k,1)}$. So coarse grid correction in matrix form can be written as

$$x_l^{(k,2)} = x_l^{(k,1)} + Py_{l-1}, \tag{1.21}$$

where y_{l-1} is the solution of $Ay_{l-1} = P^Td_l$. We can now rewrite the algorithm in a matrix form: given an approximation x_l^k of the solution of $A_lx_l = b_l$

1. *Pre-smoothing*: $x_l^{(k,1)} = S^{\nu_1}x_l^{(k)}$

2. *Coarse-grid correction*: compute the defect $d_l = b_l - Ax_l^{(k,1)}$ and its restriction $b_{l-1} = P^Td_l$. Let y_{l-1}^* be the solution of $Ay_{l-1} = b_{l-1}$ where $A_{l-1} = P^TAP$. If $l = 1$ we set $y_{l-1} = y_{l-1}^*$, otherwise we compute an approximation y_{l-1} of y_{l-1}^* by performing μ steps of MGM_{l-1} at level $l - 1$ with the initial guess $x_{l-1}^{(0)} = 0$. Set $x_l^{(k,2)} = x_l^{(k,1)} + Py_{l-1}$.

3. *Post smoothing*:

$$x_l^{(k,3)} = S^{\nu_2}x_l^{(k,2)}$$
$$x_l^{(k+1)} = x_l^{(k,3)}$$

2 Convergence analysis of MG methods

Classical convergence analysis is based on a smoothing property of the form

$$\|S^{\nu} v_h\|_X \leq C_S h^{-\beta} \frac{\|v_h\|_Y}{\nu^{\gamma}} \tag{2.1}$$

and an approximation of the form

$$\|v_h - v_H\|_Y \leq C_A h^{\beta} \|v_h\|_X \qquad \forall v_h \in V_h, \tag{2.2}$$

where v_H is a coarse-grid correction of v_h. For ν large enough $C_S C_A \frac{1}{\nu} \leq 1$. From now on we will use the following

Assumption 2.1.

1. The BVP is H^1 (or H_0^1) elliptic.
2. The BVP is H^2 regular, i.e. the solution is in H^2.
3. The spaces V_l come from conforming discretization and are nested
4. We use a nodal basis for V_l, with $l = 0, 1, \ldots, L$.

2.1 Discrete norms and smoothing property

Definition 2.2. Let $A \in \mathbb{R}^{N \times N}$ be SPD and $s \in \mathbb{R}$. With the euclidean inner product $(\cdot, \cdot)$ in $\mathbb{R}^N$, we define the norm

$$|||x|||_s = (x, A^s x)^{1/2}. \tag{2.3}$$

Let $\{(z_i, \lambda_i)\}_{i=1}^N$ be the orthonormal eigenpairs of A. The eigenvectors form a basis of $\mathbb{R}^N$ due to our hypothesis, thus

$$A^s x = \sum_{i=1}^{N} c_i \lambda^s z_i \tag{2.4}$$

and further

$$(x, A^s x) = \sum_{i=1}^{N} c_i^2 \lambda^s \tag{2.5}$$

From (2.5) it follows that $|||x|||_s = \left\| A^{s/2} x \right\|$.

Remark 2.3. The norm $||| \cdot |||_s$ has the following properties:

1. $||| \cdot |||_0 = \|\cdot\|$.
2. Let $t, r \in \mathbb{R}$ and $s = \frac{t+r}{2}$. Then

   $$\left|(x, A^s y)\right| = \left|(A^{t/2} x, A^{r/2} y)\right| \leq |||x|||_t |||y|||_r$$

   Thus

   $$|||x|||_s \leq \sqrt{|||x|||_t |||x|||_r}$$

3. Let $\alpha > 0$ be the ellipticity (coercivity) constant. Then

$$\frac{|||x|||_t}{\alpha^{-t/2}} \geq \frac{|||x|||_s}{\alpha^{-s/2}} \qquad \text{for } t \geq s$$

4. If $Ax = b$ then $|||x|||_{s+2} = \|b\|$

Lemma 2.4. *Let $\omega \geq \rho(A)$ and $s \in \mathbb{R}$, $t > 0$. Then, for the Richardson iteration $x^{(\nu+1)} = (I - \frac{1}{\omega}A)x^{(\nu)}$ there holds*

$$|||x^{(\nu)}|||_{s+t} \leq c\nu^{-t/2}|||x^{(0)}|||_s \qquad \text{with } c = \left(\frac{t\omega}{2e}\right)^{t/2},$$

with e being the Euler number.

Proof. Exercise. □

Now we would like to answer how the discrete norms $|||\cdot|||_s$ are related to Sobolev norms. We consider first the case $s = 0$, i.e. the standard euclidean norm.

Lemma 2.5. *Let $\mathcal{T}_h$ be a uniform triangulation of $\Omega \subset \mathbb{R}^n$ and V_h denote the corresponding space associated with a family of affine finite elements. The nodal basis functions are assumed to be scaled such that*

$$\Psi(z_j) = h^{n/2}\delta_{ij} \tag{2.6}$$

For $v_h \in V_h$ let $|||v_h|||_0 = \|v_h\|$ be the euclidean norm of the coefficient vector and $\|v_h\|_{0,\Omega}$ be the $L^2(\Omega)$ norm of the corresponding function. Then we have

$$c^{-1}\|v_h\|_{0,\Omega} \leq \|v_h\| \leq c\|v_h\|_{0,\Omega}$$

i.e. the $L^2(\Omega)$ and the euclidean norm are equivalent. In addition the constant c does not depend on h.

Proof. In the following we identify finite element functions $v_h \in V_h$ and their vectors of expansion coefficient with respect to the basis $\{\Psi_i\}_{i=1}^N$. Details of the proof are left as exercise. On the reference element $T_{\text{ref}} \subset \mathbb{R}^n$ one has

$$\|v_h\|_{0,\Omega}^2 \simeq h^n \sum_{z_i \in T} (u_h(z_i))^2$$

i.e.,

$$c_1\|v_h\|_{0,T}^2 \leq h^n \sum_{z_i \in T} (v_h(z_i))^2 \leq c_2\|v_h\|_{0,T}$$

□

Now we consider the case of $s = 1$. This is even easier. Indeed, we have:

$$|||v_h|||_1{}^2 = (v_h, Av_h) = a(v_h, v_h) \quad .$$

From the ellipticity and boundedness of the bilinear form $a(\cdot,\cdot)$ we get immediately

$$c^{-1}\|v_h\|_{1,\Omega} \leq |||v_h|||_1 \leq c\|v_h\|_{1,\Omega}\,, \tag{2.7}$$

where

$$\|v_h\|_{1,\Omega}^2 = \int_\Omega v^h + \nabla v_h^2\, dx$$

is the usual Sobolev norm.

Lemma 2.6. *Let the hypothesis of Lemma 2.5 be satisfied. Then the extremal eigenvalues of the stiffness matrix of an H^1-elliptic BVP satisfy*

$$\lambda_{min} \geq c^{-1}, \quad \lambda_{max} \leq ch^{-2}, \quad \mathcal{K}(A_h) \leq c^2 h^{-2}. \tag{2.8}$$

Proof. Exercise. □

Using Lemma 2.4 with $s = 0$ and $t = 2$ and the estimate from Lemma 2.6, it follows the smoothing property given in Proposition 2.7.

Proposition 2.7. *The Richardson iteration $x^{(\nu+1)} = (I - \frac{1}{\omega}A_h)x^{(\nu)}$, with $\omega = \rho(A_h)$, satisfies*

$$|||x^{(\nu)}|||_2 \leq \frac{c}{\nu} h^{-2} |||x^{(0)}|||_0 \tag{2.9}$$

2.2 Approximation property

Assuming H^2-regularity of the BVP, the error of the coarse-grid correction can be estimated in the $|||\cdot|||_s$ norm. An important tool in the proof is an estimate of the form

$$\|u - u_h\|_{0,\Omega} \leq ch\|u - u_h\|_{1,\Omega}\,, \tag{2.10}$$

which follows from finite element analysis (Aubin-Nitsche duality argument)

Lemma 2.8 (Approximation property)**.** *For $u \in V_h$, let u_H be the solution of the variational problem*

$$a(u - u_H, \omega) = 0 \qquad \forall \omega \in V_H$$

(for example $H = 2h$). Moreover, let Ω be convex or let its boundary be smooth. Then we have

$$\|u - U_H\|_{1,\Omega} \leq cH|||v|||_2 \tag{2.11}$$

$$\|u - U_H\|_{0,\Omega} \leq cH|||v - U_H|||_2 \tag{2.12}$$

Proof. Left as exercise.

□

2.3 Convergence of the two-grid method

We have already shown in the last section that there holds the smoothing property (2.1) and the approximation property (2.2) in the particular case

$$\|\cdot\| = \|\cdot\|_2\,, \quad \|\cdot\|_Y = |||\cdot|||_0, \quad \beta = 2, \gamma = 1. \tag{2.13}$$

Theorem 2.9 (Convergence of the two-grid method)**.** *Under the usual assumptions 2.1, with Richardson smoother and $\rho(A_h) \le \omega \le \bar{c}\rho(A_h)$ satisfies*

$$\left\| u_1^{(k,1)} - u_1 \right\|_{0,\Omega} \le \frac{c}{\nu_1} \left\| u_1^k - u_1 \right\|_{0,\Omega}, \tag{2.14}$$

where the constant c is independent by h and ν_1.

Proof. After ν_1 pre-smoothing steps with the Richardson method we have

$$u_1^{(k,1)} - u_1 = (I - \frac{1}{\omega} A_h)^{\nu_1} (u_1^{(k)} - u_1)$$

and Lemma 2.8 (smoothing property) yields

$$\left\| u_1^{(k,1) - u_1} \right\|_2 \le \frac{c}{\nu_1} |||u_1^{(k)} - u_1|||_0. \tag{2.15}$$

The approximation $u_1^{(k,2)} = u_1^{(k,1)} + u_H$ after coarse-grid correction solves

$$a(u_1^{(k,1)} + u_H, v_h) = F(v_h) \qquad \forall v_h \in V_h.$$

Moreover, the exact solution $u_1 = u_H$ satisfies

$$a(u_1, v_H) = F(v_H) \qquad \forall v_H \in V_H.$$

Since $V_H \subset V_h$ subtraction on V_H yields:

$$a(u_1^{(k,1)} + u_H - u_1, v_H) = 0 \qquad \forall v_H \in V_H$$

Now Lemma 2.8 (approximation property) results in

$$\begin{aligned} \left\| u_1^{(k,2)} - u_1 \right\|_{0,\Omega} &= \left\| u_1 - u_1^{(k,1)} - u_H \right\|_{0,\Omega} \\ &\le \tilde{c} H \|v - U_H\|_{1,\Omega} \\ &\le c H^2 |||v|||_2 \\ &= c H^2 \left\| u_1^{(k,1)} - u_1 \right\|_2, \end{aligned} \tag{2.16}$$

where $v := u_1 - u_1^{(k,1)}$. Here we neglect the effect of post-smoothing, i.e., we just use

$$|||(I - \frac{1}{\omega} A_h)^{\nu_2} x|||_s \le |||x|||_s,$$

from which follows

$$|||u_1^{(k,3)} - u_1|||_0 \le |||u_1^{(k,2)} - u_1|||_0.$$

In view of the equivalence of the discrete and the continuous norm we therefore have

$$\left\| u_1^{(k,3)} - u_1 \right\|_{0,\Omega} \le c \left\| u_1^{(k,2)} - u_1 \right\|_{0.\Omega} \tag{2.17}$$

and finally, combining (2.15), (2.16), (2.17), we have

$$\begin{aligned}\left\| u_1^{(k+1)} - u_1 \right\|_{0,\Omega} &= \left\| u_1^{(k,3)} - u_1 \right\|_{0,\Omega} \le c_0 \left\| u_1^{(k,2)} - u_1 \right\|_{0,\Omega} \\ &\le c_1 H^2 |||u_1^{(k,1)} - u_1|||_2 \le \frac{c_2}{\nu_1} |||u_1^{(k)} - u_1|||_0 \\ &\le \frac{c_2}{\nu_1} \left\| u_1^{(k)} - u_1 \right\|_{0,\Omega}\end{aligned}$$

□

In matrix form the two-grid method can be also written in the form

$$u^{(k+1)} - u_1 = G(u_1^{(k)} - u_1), \tag{2.18}$$

with

$$\begin{aligned} G &= S^{\nu_2}(I - P A_H^{-1} P^T A_h) S^{\nu_1} \\ &= S^{\nu_2}(A_h^{-1} - P A_H^{-1} P^T) A_h S^{\nu_1}. \end{aligned} \tag{2.19}$$

So coarse-grid correction has the propagation matrix

$$I - P A_h^{-1} P^T A_h.$$

The smoothing property can be written as

$$\| A_h S^{\nu_1} \| \le \frac{c_s}{\nu_1} h^{-2} \tag{2.20}$$

and the approximation as

$$\left\| A_h^{-1} - P a_H^{-1} P^T \right\| \le h^2 c_A. \tag{2.21}$$

Together with the assumption that the smoother is convergent in the norm $\|\cdot\|$

$$\|S\| < 1$$

we get

$$\|G\| < c_s c_A \frac{1}{\nu_1} < 1$$

if ν_1 is large enough. In view of $|||Ax|||_0 = \|x\|_2$ one deduces from (2.20) and (2.21) the smoothing and approximation property from before.

2.4 Convergence of multigrid method

2.4.1 Convergence of the W-cycle MG method

The goal is to estimate the convergence rate ρ_l in

$$\left\| u_l^{(k+1)} - u_l \right\| \leq \rho_l \left\| u_l^{(k)} - u_l \right\|, \tag{2.22}$$

where $u_l \in V_l$ is the solution of (1.14) in V_l. Obviously

$$\left\| u_l^{(k,1)} - u_l \right\| \leq \left\| u_l^{(k)} - u_l \right\|, \tag{2.23}$$

where $u_l^{(k,1)}$ is the approximation after the Richardson approximation and $\|\cdot\| = |||\cdot|||_s$. Denote by $u_l^{(k,2)}$ the approximation after real (approximate) coarse-grid correction and by $\hat{u}_l^{(k,2)}$ the approximation after the exact coarse-grid correction. We have

$$\left\| \hat{u}_l^{(k,2)} - u_l \right\| \leq \rho_1 \left\| u_l^{(k)} - u_l \right\| \tag{2.24}$$

with the two-grid rate ρ_1. Using triangular inequality, one has

$$\left\| u_l^{(k,2)} - u_l \right\| \leq \left\| u_l^{(k,2)} - \hat{u}^{(k,2)} \right\| + \left\| \hat{u}^{(k,2)} - u_l \right\| \tag{2.25}$$

Now we assume that we know the convergence rate ρ_{l-1},i.e, the following inequality

$$\left\| u_{l+1}^{(k+1)} - u_{l+1} \right\| \leq \rho_{l-1} \left\| u_{l-1}^{(k)} - u_{l-1} \right\| \qquad u_{l-1} \in V_{l-1}$$

and conclude

$$\left\| u^{(k,2)} - \hat{u}_l^{(k,2)} \right\| \leq \rho_{l-1}^{\mu} \left\| u^{(k,1)} - \hat{u}_l^{(k,2)} \right\|. \tag{2.26}$$

Inserting (2.25) in (2.26) yields

$$\left\| u^{(k,2)} - \hat{u}_l^{(k,2)} \right\| \leq \rho_{l-1}^{\mu}(1+\rho_1) \left\| u^{(k)} - u_l \right\| \tag{2.27}$$

and together with (2.24)

$$\begin{aligned} \left\| u_l^{(k,2)} - u_l \right\| &\leq \left\| u_l^{(k,2)} - \hat{u}^{(k,2)} \right\| + \left\| \hat{u}^{(k,2)} - u_l \right\| \\ &\leq [\rho_{l-1}^{\mu}(1+\rho_1) + \rho_1] \left\| u_l^{(k)} - u_l \right\|, \end{aligned}$$

so with post smoothing it follows

$$\rho_l \leq \rho_1 + \rho_{l-1}^{\mu}(1+\rho_1). \tag{2.28}$$

With (2.28) we can prove the following theorem. Its hypothesis are a bit too strong and often they don't hold.

Theorem 2.10. *Assuming that the two-grid rate* ρ_1 *satisfies* $\rho_1 \leq \frac{1}{5}$*, the W-cycle method converges at a rate*

$$\rho_l \leq \frac{5}{3}\rho_1 \leq \frac{1}{3} \qquad \textit{for } l = 2, 3, \ldots \quad .$$

Proof. You need to use (2.28). It's really short. ☐

We want to improve the previous result by drawing an estimate in the energy norm. Since

$$a(u_l^{(k,1)} + \hat{u}^{(k,2)} - u_l^{(k,1)}, v_{l-1}) = F(v_{l-1}) \qquad \forall v_{l-1} \in V_{l-1}$$

and

$$a(u_l^{(k,1)} + u^{(k,2)} - u_l^{(k,1)}, v_l) = F(v_l) \qquad \forall v_l \in V_l,$$

it follows

$$a(\hat{u}^{(k,2)} - u_l, v_{l-1}) = 0 \qquad \forall v_{l-1} \in V_{l-1}$$

and hence

$$a(\hat{u}^{(k,2)} - u_l, u_l^{(k,1)} - \hat{u}_l^{(k,2)}) = 0.$$

We have thus

$$\left\| u_l^{(k,1)} - \hat{u}_l^{(k+2)} \right\|_a^2 = \left\| u_l^{(k,1)} - u_l \right\|_a^2 - \left\| \hat{u}_l^{(k+2)} - u_l \right\|_a^2. \tag{2.29}$$

This means that the error after coarse-grid correction is a-orthogonal to the coarse-space. Now from (2.26) and this orthogonality relation just mentioned, it follows

$$\begin{aligned} \left\| u_l^{(k+2)} - u_l \right\|_a^2 &= \left\| \hat{u}_l^{(k,2)} - u_l \right\|_a^2 + \left\| u_l^{(k,2)} - \hat{u}_l^{(k,2)} \right\|_a^2 \\ &\leq \left\| \hat{u}_l^{(k,2)} - u_l \right\|_a^2 + \rho_{l-1}^{2\mu} \left\| u_l^{(k,1)} - \hat{u}_l^{(k,2)} \right\|_a^2 \\ &\leq \left\| \hat{u}_l^{(k,2)} - u_l \right\|_a^2 + \rho_{l-1}^{2\mu} (\left\| u_l^{(k,1)} - u_l \right\|_a^2 + \left\| \hat{u}_l^{(k,2)} - u_l \right\|_a^2) \\ &= (1 - \rho_{l-1}^{2\mu}) \left\| \hat{u}_l^{(k,2)} - u_l \right\|_a^2 + \rho_{l-1}^{2\mu} (\left\| u_l^{(k,1)} - u_l \right\|_a^2) \end{aligned} \tag{2.30}$$

Using additionally (2.24) and (2.23) we finally obtain from (2.30)

$$\left\| u_l^{(k,2)} - u_l \right\|_a^2 \leq [(1 - \rho_{l-1}^{2\mu})\rho_1 + \rho_{l-1}^{2\mu}] \left\| u_l^{(k)} - u_l \right\|_a^2,$$

and thus

$$\rho_l^2 \leq \rho_1^2 + \rho_{l-1}^{2\mu}(1 - \rho_1^2) \tag{2.31}$$

Theorem 2.11. *Assuming that the two -grid rate satisfies* $\rho_1 \le \frac{1}{2}$*, the W-cycle method converges at a rate*

$$\rho_l \le \frac{6}{5}\rho_1 \le 0.6 \qquad \textit{for } l = 2, 3, \dots \tag{2.32}$$

Proof. For $l = 1$ there is nothing to prove. Assume now it is true for $l = k-1$. Then we have

$$\begin{aligned}
\rho_k^2 &\le \rho_1^2 + \rho_{k-1}^{2\cdot 2}(1-\rho_1^2) \\
&\le \rho_1^2 + \left(\frac{6}{5}\right)^4 (1-\rho_1^2) \\
&= \rho_1^2 \left[1 + \left(\frac{6}{5}\right)^4 (\rho_1^2(1-\rho_1^2))\right] \\
&\le \frac{36}{25}\rho_1^2 \le \frac{9}{25}
\end{aligned}$$

□

2.4.2 Convergence analysis of the V-cycle MG method

In order to prove a uniform bound $\rho_l \le \rho_\infty < 1$ for the convergence of the V-cycle, we need to refine our analysis. As before, let $\|\cdot\|$ denote the energy norm. Our goal is to prove the following theorem.

Theorem 2.12. *Under the assumptions 2.1 and if the Richardson smoother is used, the V-cycle MG method satisfies the estimate*

$$\left\|u_l^{(k+1)} - u_l\right\| \le \left(\frac{c}{c+2\nu}\right)^{1/2} \left\|u_l^{(k)} - u_l\right\|, \tag{2.33}$$

thus $\rho_l^2 \le \rho_\infty^2 \le (\frac{c}{c+2\nu})$*. Here* c *is a constant independent by* h *and* ν*.*

First we need to establish three preliminary results. We start with introducing a measure for the smoothness of a finite element function. For any $v_h \in V_h$ let

$$\beta = \beta(v_h) := \begin{cases} 1 - \rho(A_h)^{-1} \frac{{|||v_h|||_2}^2}{{|||v_h|||_1}^2} & v_h \neq 0 \\ 0 & v_h = 0 \end{cases} \tag{2.34}$$

Obviously $\beta \in [0, 1[$. For smooth functions ($|||v_h|||_2 \approx |||v_h|||_1$) we have that β is close to 1. For high-oscillatory functions ($|||v_h|||_2 \approx \rho(A_h)|||v_h|||_1$) we have that β is close to 0.

Lemma 2.13. *Let* S *denote the iteration matrix of the Richardson smoother. Then*

$$|||S^\nu v|||_1 \le [\beta(S^\nu v)]^\nu |||v|||_1 \qquad \forall v \in V_h$$

Proof. Let $v = \sum_{i=1}^{N} c_i \Phi_i$, where $\{\Phi_i\}$ denote the set of the orthonormal eigenvectors of $A := A_h$. Moreover let $\mu_i = 1 - \frac{\lambda_i}{\rho(A)}$. Now we set

$$p := \frac{2\nu + 1}{2\nu}, \quad q := 2\nu + 1,$$

so $p^{-1} + q^{-1} = 1$, and

$$a_i := \lambda^{1/p} u_i^{2\nu} |c_i|^{2/p}, \quad b_i := \lambda_i^{1/q} c_i^{2/q}$$

such that

$$\begin{aligned} |a_i|^p &= \lambda^i \mu_i^{2\nu+1} |c_i|^2, \quad |b_i|^q = \lambda_i |c_i|^2 \\ |a_i b_i| &= \lambda_i \mu_i^{2\mu} |c_i|^2 \quad . \end{aligned}$$

Now we have

$$\sum_{i=1}^{N} \lambda_i \mu_i^{2\nu} |c_i|^2 \leq \left(\sum_{i=0} \lambda_i \mu_i^{2\nu+1} |c_i|^2 \right)^{\frac{2\nu}{2\nu+1}} \left(\sum_{i=0} \lambda_i |c_i|^2 \right)^{\frac{1}{2\nu+1}} \tag{2.35}$$

Then, by definition of S^ν, we find that (2.35) is equivalent to

$$\|S^\nu v\|^{2\nu+1} \leq \left\| S^{\nu+1/2} v \right\| \|v\| \tag{2.36}$$

Substituting $w := S^\nu v$ we obtain from (2.36)

$$\|S^\nu v\| \leq \frac{\left\| S^{1/2} w \right\|^2}{w^2} \|v\| \quad . \tag{2.37}$$

The particular choice of $S = I - \frac{A_h}{\rho(A_h)}$ implies that S is self-adjoint and S and A commute. Hence

$$\begin{aligned} \left\| S^{1/2} v \right\|^2 &= \|\!|S^{1/2} w|\!\|_1{}^2 = (w, ASw) \\ &= (w, Aw) - \frac{1}{\rho(A)} (w, A^2 w) = \beta(w) \|w\|^2 \end{aligned} \tag{2.38}$$

We obtain the thesis inserting (2.37) in (2.38). □

The measure β can also be used to get a refined estimate of the error after coarse-grid correction.

Lemma 2.14. *The error after exact coarse-grid correction satisfies*

$$\begin{aligned} \left\| \hat{u}_l^{(k,2)} - l \right\| &\leq \min\{c\rho^{-1/2}(A_h) \|\!|\hat{u}_l^{(k,2)} - u_l|\!\|_2, \quad \left\| u_l^{(k,1)} - u_l \right\| \} \\ &= \min\{c\sqrt{1 - \beta(u_l^{(k,1)} - u_l)}, \quad 1\} \left\| u_l^{(k,1)} - u_l \right\|, \end{aligned} \tag{2.39}$$

where $\|\cdot\|$ *denotes the energy norm.*

Proof. The approximation property (2.11) reads in the present situation

$$\left\| \hat{u}_l^{(k,2)} \right\| \leq c_1 h |||u_l^{(k,1)} - u_l|||_2$$

because

$$\|v - u_h\| \leq c_1 h |||v|||_2$$

gives for $v = -u_l^{(k,1)} + u_l$ and $u_H = -u_l^{(k,1)} + \hat{u}_l^{(k,2)}$ the exact equation above. Using $\rho(A_h) \leq c_2 h^{-2} \Leftrightarrow c_1 h \leq c\rho(A_h)^{-1/2}$ we obtain

$$\left\| \hat{u}^{(k,2)} - u_l \right\| \leq \min\{c\rho^{-1/2}(A_h)|||\hat{u}_l^{(k,2)} - u_l|||_2, \quad \left\| u_l^{(k,1)} - u_l \right\| \}$$

and we get the second equality eliminating $|||\cdot|||_2$ using the definition (2.34) of β. □

The previous lemmas 2.13 and 2.12 allow to establish an improved recursion formula for ρ_l.

Lemma 2.15 (Improved recursion formula)**.** *Let the assumptions of Theorem 2.12 be satisfied, then we have the following relation*

$$\rho_l^2 \leq \max_{0 \leq \beta \leq 1} \beta^{2\mu}[\rho_{l-1}^{2\mu} + (1 - \rho_{l-1}^{2\mu}) \min\{1, c^2(1-\beta)\}] \tag{2.40}$$

where $\mu = 1$ corresponds to the V-cycle, while $\mu = 2$ corresponds to the W-cycle, and c is the same constant of lemma 2.14.

Proof. From Lemma 2.13 we have

$$\left\| u^{(k,1)} - u_l \right\| \leq \beta^\nu \left\| u_l^{(k)} - u_l \right\| \tag{2.41}$$

with $\beta = \beta(u_l^{(k,1)} - u_l)$ defined as in (2.34). Lemma 2.14 for the same β yields

$$\left\| \hat{u}_l^{(k,2)} - l \right\| \leq \beta^\nu \min\{c\sqrt{1 - \beta(u_l^{(k,1)} - u_l)}, \quad 1\} \left\| u_l^{(k,1)} - u_l \right\|. \tag{2.42}$$

Inserting(2.42) and (2.41) in the estimate (2.30) we finally get

$$\begin{aligned} \left\| u_l^{(k,2)} - u_l \right\|^2 &\leq (1 - \rho_{l-1}^{2nu}) \left\| \hat{u}_l^{(k,2)} - u_l \right\|^2 + \rho_{l-1}^2 \left\| u^{(k,1)} - u_l \right\|^2 \\ &\leq \beta^{2\nu}[(1 - \rho_{l-1}^{2\mu}) \min\{c\sqrt{1-\beta}, \quad 1\} + \rho_{l-1}^{2\mu}] \left\| u_l^{(k)} - u_l \right\|, \end{aligned} \tag{2.43}$$

which proves our thesis since $0 \leq \beta \leq 1$. □

One can computes the convergence factors (i.e., bounds for ρ_l) according to formula (2.40). We have ($l \to \infty$ is the upper bound):

c	V-cycle $l = 1$	V-cycle $l = \infty$	W-cycle
0.5	0.1418	0.243	0.1437
1.0	0.217	0.448	0.2904

Now we can finally prove Theorem 2.10.

Theorem 2.10. We note that $\rho_0 =$ which proves the Theorem for $l = 0$. Now we assume that (2.33) holds for $l = k - 1$. So we insert $\rho_{k-1}^2 \leq \frac{c^2}{c^2+2\nu}$ in the formula (2.40) to get

$$\begin{aligned}
\rho_k^2 &\leq \max_{0\leq\beta\leq 1}\{\beta^{2\nu}[\frac{c^2}{c^2-2\nu} + (1 - \frac{c^2}{c^2+2\nu})c^2(1-\beta)]\} \\
&\leq \frac{c^2}{c^2+2\nu}\max_{0\leq\beta\leq 1}\{\beta^{2\nu}[1 + (\frac{c^2+2\nu}{c^2} - 1)c^2(1-\beta)]\} \\
&= \frac{c^2}{c^2+2\nu}\max_{0\leq\beta\leq 1}\{\beta^{2\nu}[1 + 2\nu(1-\beta)]\} \\
&= \frac{c^2}{c^2+2\nu}
\end{aligned}$$

where we can choose $c^2 = \max\{c_1, c_2^2\}$ and c_1 is constant in (2.33) and c_2^2 is the constant in (2.40). The last equality stands because the $\max\{\dots\}$ is achieved for $\beta = 1$ and it equals 1. □

2.4.3 Complexity of multigrid methods

The estimation of the computational work for one MG cycle is based on estimating the number of arithmetic operations for:

1. *Smoothing* in V_k.
2. *Prolongation* (i.e., interpolation) from $V_{k-1} \to V_k$.
3. *Restriction* from $V_k \to V_{k-1}$.

These components of the MG methods require a number of arithmetic operations that can be bounded by $C \cdot N_k$, where $N_k = \dim(V_k)$. The number of arithmetic operations for one application of the smoother is proportional to the number of nonzero entries in Ak, which in case of affine family of finite element is proportional to N_k, i.e., $O(N_k)$. *Prolongation* and *Restriction* typically have sparse matrix representations and hence the amount of work for each visit at level k can be bounded by $(\nu + 1)C \cdot N_k$, where $\nu = \nu_1 + \nu_2$ denotes the number of smoothing steps. The total work for one cycle can therefore be estimated by

$$(\nu + 1)C \sum_{i=0}^{L} N_i \leq \frac{1}{1-q}(\nu + 1)CN_L$$

for the V-cycle, and

$$(\nu + 1)C \sum_{i=0}^{L} \mu^{L-i} N_i \leq \frac{1}{1-q\mu}(\nu + 1)CN_L \tag{2.44}$$

for the ν-fold V-cycle, e.g., by

$$\frac{1}{1-2q}(\nu + 1)CN_L$$

for the W-cycle. Here $q < 1$ denotes a bound for the reduction factor for the number of the unknowns when proceeding to coarser and coarser levels, i.e.,

$$N_{l-1} \leq qN_l \qquad \text{for } l = 1, 2, \ldots, L \quad .$$

As we see from (2.44) the condition $q < 1$ will be in general sufficient to guarantee that each cycle has optimal computational complexity, i.e, the number of the operations is of order $O(N_L)$.

3 A more abstract view on multigrid theory

Consider a finite-dimensional complete vector space, which is endowed with two inner products $(\cdot, \cdot)$ and $a(\cdot, \cdot)$, and corresponding norms $\|\cdot\|_0$ and $\|\cdot\|$ respectively. Now let $V_L = V$ and assume that we are given a sequence of nested spaces

$$V_0 \subset V_1 \subset \cdots \subset V_L = V \quad .$$

Next consider the operators $A_k : V_k \to V_k$ defined by

$$(A_k\Psi, \Phi) = a(\Psi, \Phi) \qquad \forall \Psi, \Phi \in V_k.$$

Moreover, let the projections $P_k : V_L \to V_k$ and $Q_k : V_L \to V_k$ be defined by:

$$a(P_k u, v) = a(u, v) \qquad \forall v \in V_k, u \in V$$

and

$$(Q_k u, v) = (u, v) \qquad \forall v \in V_k, u \in V.$$

Remark 3.1. If $a(\cdot, \cdot)$ is the bilinear form in (1.14) then P_k is often referred to as elliptic projector or Ritz projector. If $(\cdot, \cdot)$ denotes the L^2-inner product, then Q_k is called L^2-projector.

Furthermore, let $R_k; V_k \to V_k$ denote smoothing operators, which in general do not have to be symmetric and let R_k^T be denote their adjoint operators with respect to $(\cdot,\cdot)$. As in Section 1 consider a linear stationary iteration method of the form

$$x^{(k+1)} = x^{(k)} + B(f - Ax^{(k)}). \tag{3.1}$$

We want to study the V-cycle MG method, which corresponds to the choice $B = B_L$, where B_L is recursively defined via the following algorithm, called *V-cycle MG preconditioner: recursive definition of* B_k

1. If $k = 0$ set $B_0 = A_0^{-1}$; otherwise we define the action of B_k on a vector g recursively by the following three steps, assuming B_{k-1} is known.
2. *Pre-smoothing*: $x^{(1)} R_k^T g$.
3. *Coarse-grid correction*: $x^{(2)} = x^{(1)} + y$, with
$$y = B_{k-1} Q_{k-1} \underbrace{(g - A_k x^{(1)})}_{\in V_{k-1}}.$$
4. *Post-smoothing* : $B_k g := x^{(2)} + R_k(g - A_k x^{(2)})$.

3.1 Product formula for the error propagation operator

We want to derive a formula for $E = I - B_L A_L$. For that reason denote by $K_k : V_k \to V_k$ the error propagation operator (EPO) of R_k, i.e., $K_k = I - R_k A_k$ and by K_k^* the adjoint operator with respect to the $a(\cdot,\cdot)$ inner product, i.e., (after some computations), $K_k^* = I - R_k^T A_k$. In view of the identity (exercise for the reader)

$$Q_{k-1} A_l = A_{k-1} P_{k-1} \qquad \text{on } V_l \text{ for } k \leq l \tag{3.2}$$

and following the Algorithm defined above, we find

$$\begin{aligned} x - x^{(2)} &= x - x^{(1)} - B_{k-1} Q_{k-1}(\underbrace{g}_{=A_k x} - A_k x^{(1)}) \\ &= (I - B_{k-1} Q_{k-1} A_k)(x - x^{(1)}) \\ &= (I - B_{k-1} A_{k-1} P_{k-1}) K_k^* x \end{aligned} \tag{3.3}$$

From step 4) of the algorithm, using $g = A_k x$, we obtain

$$\begin{aligned} (I - B_k A_k)x &= x - x^{(2)} - R_k A_k (x - x_x^{(2)}) \\ &= K_k (x - x^{(2)}) \\ &= K_k (I - B_{k-1} A_{k-1} P_{k-1}) K_k^* x \\ &= K_k [(I - P_k) + (I - B_{k-1} A_{k-1}) P_{k-1}] K_k^* x \end{aligned}$$

and because x was arbitrary we get

$$(I - B_k A_k) = K_k[(I - P_k) + (I - B_{k-1} A_{k-1}) P_{k-1}] K_k^*. \tag{3.4}$$

We extend the operator K_k to be defined on the entire space V_L (and denote the extended operator again as K_k):

$$K_k = I - R_k A_k P_k = I - T_k,$$

with

$$T_k := R_k A_k P_k.$$

In a similar way we can write

$$K_k^* = I - R_k^T A_k P_k = I - T_k^*,$$

with

$$T_k^* := R^T A_k P_k.$$

Using (3.4) it follows

$$\begin{aligned} I - B_k A_k P_k &= I - P_k + (I - B_k A_k) P_k \\ &= I - P_k + K_k[(I - P_{k-1}) + (I - B_{k-1} A_{k-1}) P_{k-1}] K_k^* P_k \end{aligned} \tag{3.5}$$

P_k is a projection, thus $P_k^2 = P_k$ and $(I - P_k)^2 = I - P_k$. Obviously we have

$$T_k P_k = T_k,$$

and thus

$$(I - T_k)(I - P_k) = I - P_k.$$

Moreover

$$\begin{aligned} a(T_k u, v) = a(u, \underbrace{T_k^* v}_{\in V_k}) &= a(P_k u, T_k^* v) \\ &= a(P_k u, T_k^* v) = a(u, P_k T_k^* v) \end{aligned}$$

, i.e., $T_k^* = P_k T_k^*$. Hence, $(I - P_k)(I - T_k^*) = I - P_k$. We therefore have

$$\begin{aligned} (I - P_k) &= (I - P_k)(I - T_k^*) \\ &= (I - T_k)(I - P_k)(I - T_k^*) \\ &= K_k^*(I - p_k) K_k^*. \end{aligned}$$

Due to (3.5) it follows that

$$
\begin{aligned}
I - B_k A_k P_k &= (I - T_k)(I - P_k)(I - T_k^*) \\
&\quad + (I - T_k^*)[I - B_{k-1} A_{k-1} P_{k-1}](I - T_k^*) P_k \\
&= (I - T_k)[I - P_k + P_k - B_{k-1} A_{k-1} P_{k-1}](I - T_k^*),
\end{aligned}
$$

where we use, among other equalities, $(I - T_k^*)P_k = P_k(I - T_k^*)$. Finally we get, using $B_o = A_0^{-1}$ and $P_L = I$.

$$
I - B_L A_L P_L = \left[\prod_{i=L}^{1}(I - T_i)\right](I - P_0)\left[\prod_{i=1}^{L}(I - T_i^*)\right] \tag{3.6}
$$

Proof of Identity (3.2). We remember that

$$
\begin{aligned}
(Q_{k-1}u, V_{k-1}) = (u, V_{k-1}) \qquad \forall u \in V_{k-1} \\
(A_l u_l, v_l) = a(u_l, v_l) \qquad \forall v_l \in V_l
\end{aligned} .
$$

Thus it follows

$$
\begin{aligned}
(Q_{k-1} A_l u, v_{k-1}) &= (A_l u, v_{k-1}) \\
&= a(u_l, v_{k-1}) \\
&= a(P_{k-1} u_l, v_{k-1}) \\
&= (A_k P_{k-1} u_l, v_{k-1})
\end{aligned}
$$

□

3.2 Assumptions for convergence analysis of the V-cycle MG method

Assumption 3.2. There exists a constant $C_R \geq 1$, independent of k (and thus independent of h) such that

$$
\frac{|||v|||_0{}^2}{\lambda_k} \leq C_R(\bar{R}_k u, u) \qquad \forall u \in V_k, \tag{3.7}
$$

where

$$
\begin{aligned}
\bar{R}_k &= (I - K_k^* K_k) A_k^{-1} \\
&= R_k^T + R_k - R_k^T A_k R_k
\end{aligned}
$$

denotes the *symmetrized smoother*.

Remark 3.3. Let $R_{k,\alpha} = \alpha \lambda^{-1} I$ and $K_{k,\alpha} = I - R_{k,\alpha} A_k$, then Assumption 3.2 is equivalent to the following statement: There exist a constant $\alpha \in]0, 1]$ such that

$$
a(K_k u, K_k u) \leq a(K_{k,\alpha})u, K_{k,\alpha} u) \qquad \forall u \in V_k.
$$

. The proof is left as exercise .

A generalization of Assumption 3.2 requires the property (3.7) to be satisfied only on a surface $\hat{V}_k \subset V_k$ (we are still assuming that $\{V_k\}$ are nested, but not necessarily $\{\hat{V}_k\}$). In this case one can define

$$\begin{aligned} &\hat{A}_k : \hat{V}_k \to \hat{V}_k \\ &(\hat{A}_k \Phi, \Psi) = a(\Phi, \Psi) \qquad \forall \Phi, \Psi \in \hat{V}_k \end{aligned}$$

and $\hat{P}_k$, $\hat{Q}_k$ similarly. The weakened 3.2 then reads

Assumption 3.4. Assume that $R_k = R_k \hat{Q}_k$ and

$$\frac{|||u|||_0{}^2}{\lambda_k} \leq C_R(\bar{R}_k u, u) \qquad \forall u \in \hat{V}_k, \tag{3.8}$$

where

$$\bar{R}_k = R_k^T + R_k - R_k^T \hat{A}_k R_k$$

Obviously $R_k = R_k \hat{Q}_k$ if R_k is symmetric since

$$\begin{aligned} (R_k \Phi, \Psi) &= (\Phi, \underbrace{R_k \Psi}_{\in \hat{V}_k}) \\ &= (\hat{Q}_k, R_k \Psi) = (R_k \hat{Q}_k \Phi, \Psi) \end{aligned}$$

Assumption 3.5. There exists a constant $\theta \leq 2$ such that

$$a(T_k u, T_k u) \leq \theta a(T_k u, u) \qquad \forall v \in V_k, \tag{3.9}$$

where $T_k = I - R_k A_k P_k$. Note that we will assume the same inequality in the case $\hat{V}_k$

Remark 3.6. The following consideration shows that Assumption 3.5 is quite natural: For the choice $T_k = \frac{\alpha}{\lambda_k} A_k P_k$, i.e., $R_k = \frac{\alpha}{\lambda}$ 3.5 requires $\alpha \in]0, 2[$ because

$$\begin{aligned} a(T_k u, T_k u) &\leq \frac{\alpha}{\lambda} a(T_k, u, A_k u) \\ &\leq \alpha a(T_k u, u) \leq \theta a(T_k u, u) \end{aligned}$$

This is reasonable because for the eigenfunction v_k corresponding to the largest eigenvalue λ_k of A_k, we have $(I - T_k)v_k = (I - \alpha)v_k$ and thus $I - T_k$ does not reduce the components for $\alpha \geq 2$.

Assumption 3.7. There exist linear operators $\bar{Q}_k : V \to V_k$, with $\bar{Q}_L = I$ and

$$\left\|(\bar{Q}_k - \bar{Q}_{k-1})u\right\|^2 \leq c\lambda^{k-1} a(u, u) \qquad \text{for } k = 1, 2, \ldots, L$$

and

$$a(\bar{Q}_k u, \bar{Q}_k u) \leq ca(u, u)$$

Remark 3.8. In some application the operators $\bar{Q}k$ may be chosen as the L^2-projection operators. Sometimes, however, a more "careful" choice is required in the convergence analysis.

3.3 A convergence result for the V-cycle

Theorem 3.9 (Quasi-optimal convergence)**.** *Let B_L be the V-cycle preconditioner defined in the corresponding algorithm and let the Assumptions 3.4, 3.5, 3.7 be satisfied, and* Range$(\bar{Q}_k - \bar{Q}_{k-1}) \subset \hat{V}_k$. *Then*

$$a(I - B_L A_L)v, v) \leq \left(1 - \frac{1}{c(L+1)}\right) a(v, v) \qquad \forall v \in V_L \tag{3.10}$$

Proof. First, using $\bar{Q}_L = I$, we rewrite

$$\begin{aligned} a(v, v) &= \sum_{i=1}^{L} a(v, (\bar{Q}_k - \bar{Q}_{k-1})v) + a(v, \bar{Q}_0 v) \\ &= \underbrace{\sum_{i=1}^{L} a(E_{k-1}v, (\bar{Q}_k - \bar{Q}_{k-1})v)}_{:=S_1} \\ &\quad \underbrace{+\, a(v, \bar{Q}_0 v) + \sum_{i=1}^{L} a((I - E_{k-1})v, (\bar{Q}_k - \bar{Q}_{k-1})v),}_{:=S_2} \end{aligned} \tag{3.11}$$

where $E_{-1} = I$ and $E_i = (I - T_i)E_{i-1}$. Finally

$$E_L = (I - T_L)(I - T_{L-1}) \ldots (I - T_0),$$

where, in view of (3.6)

$$I - B_L A_L = E_L E_L^*$$

since $(I - T_0)(I - T_0^*) = (I - P_0)(I - T_0^*) = I - P_0$. We estimate S_1 first:

$$\begin{aligned} S_1 &= \sum_{k=1}^{L} a(E_{k-1}v, (\bar{Q}_k - \bar{Q}_{k-1})v) \\ &= \sum_{k=1}^{L} a(\hat{A}_k \hat{P}_k E_{k-1}v, (\bar{Q}_k - \bar{Q}_{k-1})v) \\ &\leq \sum_{k=1}^{L} \lambda_k^{1/2} |||\hat{A}_k \hat{P}_k E_{k-1}v|||_0 \frac{1}{\lambda_k^{1/2}} |||(\bar{Q}_k - \bar{Q}_{k-1})v|||_0 \\ &\leq (\sum_{k=1}^{L} \lambda_k |||\hat{A}_k \hat{P}_k E_{k-1}v|||_0{}^2)^{1/2} (\sum \frac{1}{\lambda_k} |||(\bar{Q}_k - \bar{Q}_{k-1})v|||_0{}^2)^{1/2} \end{aligned} \tag{3.12}$$

Due to the assumption 3.7 we have

$$(\sum_{k=1}^{L} \lambda_k |||\hat{A}_k \hat{P}_k E_{k-1}v|||_0{}^2)^{1/2} \leq c(L+1)^{1/2} a(v, v)^{1/2}$$

and in view of $\bar{T}_k = \bar{R}_k A_k P_k = \bar{R}_k \hat{A}_k \hat{P}_k$ and using Assumption 3.4

$$(\sum \frac{1}{\lambda_k} |||(\bar{Q}_k - \bar{Q}_{k-1})v|||_0{}^2)^{1/2} \leq c_k^{1/2} (\sum_{k=1}^{L} (\underbrace{\bar{T}_k E_{k-1} v}_{\bar{R}_k u}, \underbrace{\hat{A}_k \hat{P}_k E_{k-1} v}_{u}))^{1/2}$$

Hence we obtain from (3.12)

$$S_1 \leq c(L+1)^{1/2} a(v,v) \left[\sum_{k=1}^{L} (\bar{T}_k E_{k-1} v, \hat{A}_k \hat{P}_k E_{k-1} v) \right]^{1/2} \tag{3.13}$$

It remains to estimate the second term S_2. For this purpose the following identities will be useful:

$$E_k - E_{k-1} = T_k E_{k-1} \Leftrightarrow E_k = (I - T_k) E_{k-1} \tag{3.14}$$

and

$$I - E_k = \sum_{l=0}^{k} T_l E_{l-1}, \tag{3.15}$$

which is obtained by summation of (3.14) for $l = 0$ to $l = k$. Rearranging S_2 yields

$$\begin{aligned}
S_2 &= \sum_{i=1}^{L} a((I - E_{k-1})v, (\bar{Q}_k - \bar{Q}_{k-1})v)_{:=S_2} + a(v, \bar{Q}_0 v) \\
&= \sum_{i=1}^{L} a((I - E_{k-1})v, \bar{Q}_k v)_{:=S_2} - \sum_{k=0}^{L-1} a((I - E_k)v, \bar{Q}_k v) + a(v, \bar{Q}_0 v) \\
&= \sum_{k=1}^{L-1} a((E_k - E_{k-1})v, \bar{Q}_k v) + a((I - E_{L-1})v, \underbrace{\bar{Q}_L}_{=I} v) \\
&- a((I - \underbrace{E_0}_{=0})v, \bar{Q}_0 v) + a(v, \bar{Q}_0 v) \\
&\underbrace{=}_{(3.14)+(3.15)} - \sum_{k=1}^{L-1} a(T_k E_{k-1} v, \bar{Q}_k v) + \sum_{k=0}^{L-1} a(T_k E_{k-1} v, P_k v) \\
&= \sum_{k=1}^{L+1} a(T_k E_{k-1} v, (P_k - \bar{Q}_k) v) + \underbrace{a(T_0 E_{-1} v, P_0 v)}_{=a(T_0 v, v) = a(P_0 v, v)} .
\end{aligned}$$

Applying Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
S_2 &= \sum_{k=1}^{L+1} a(T_k E_{k-1} v, (P_k - \bar{Q}_k) v) \\
&\leq \left(\sum_{k=1}^{L-1} a(T_k E_{k-1} v, T_k E_{k-1} v) \right)^{1/2} \left(\sum_{k=1}^{L-1} a(P_k - \bar{Q}_k) v, (P_k - \bar{Q}_k) v \right)^{1/2} \\
&+ a(P_0 v, P_0 v)^{1/2} a(v,v)^{1/2} \\
&\leq \sqrt{\sum_{k=0}^{L-1} a(T_k E_{k-1} v, T_k E_{k-1} v)} \sqrt{\sum_{k=1}^{L-1} a((P_k - \bar{Q}_k) v, (P_k - \bar{Q}_k) v) + a(v,v)}
\end{aligned}
$$

,

$$
\tag{3.16}
$$

where in the last inequality we have used $a_1 b_1 + a_2 b_2 \leq (a_1^2 + a_2^2)^{1/2} (b_1^2 + b_2^2)^{1/2}$ (note that in the last line the sum starts at $k = 0$). From the boundedness of P_k and assumption 3.7 we conclude

$$
\begin{aligned}
a((P_k, \bar{Q}_k) v, (P_k, \bar{Q}_k) v) &\leq a(P_k v, P_k v) + a(\bar{Q}_k v, \bar{Q}_k v) - 2a(P_k v, \bar{Q}_k v) \\
&\leq 2a(P_k v, P_k v) + a(\bar{Q}_k v, \bar{Q}_k v) \\
&\leq c a(v,v),
\end{aligned}
\tag{3.17}
$$

and the last inequality holds because P_k is a projection with respect to $a(\cdot, \cdot)$. On the other hand, Assumption 3.5 allows us to estimate

$$
a(\bar{T}_k E_{k-1} v, T_k E_{k-1} v) \leq c a(\bar{T}_k E_{k-1} v, E_{k-1} v). \tag{3.18}
$$

One can prove $c = \frac{\theta}{2-\theta}$ (Exercise). Using (3.18) and (3.17) in (3.16) we obtain

$$
S_2 \leq c(L+1)^{1/2} a(v,v)^{1/2} \left(\sum_{k=0}^{L} a(\bar{T}_k E_{k-1} v, E_{k-1} v) \right)^{1/2} \tag{3.19}
$$

Finally, from (3.13) and (3.19) in (3.11), we get

$$
\begin{aligned}
a(v,v) &\leq c(L+1) \sum_{k=0}^{L} a(\bar{T}_k E_{k-1} v, E_{k-1} v) \\
&\underbrace{=}_{\text{Exercise}} c(L+1) \left[a(v,v) - a(E_L v, E_L v) \right]
\end{aligned}
\tag{3.20}
$$

From (3.20) if follows immediately

$$a(E_L v, E_L v) \leq \left(1 - \frac{1}{c(L+1)}\right) a(v, v),$$

which, in view of $I - B_L A_L = E_L E_L^*$ completes the proof. $\square$

Remark 3.10. This theorem tells us that

$$\rho(I - B_L A_L) \leq \left(1 - \frac{1}{c(L+1)}\right)$$