Computational Aspects of Line and Toric Arrangements

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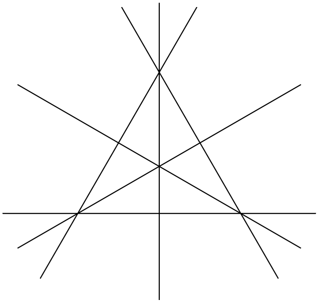
Hyperplane arrangements

Definition (Hyperplane arrangement)

Let V be a finite-dimensional vector space over a field $\mathbb{K}$. A hyperplane arrangement $\mathcal{A}$ is a (finite) collection of affine hyperplanes of V. The same definition can be given for a projective hyperplane arrangement in a projective space.



Basic definitions

- The complement of an arrangement $\mathcal{A}$ is the set
$$\mathcal{M}(\mathcal{A}) \coloneqq V \setminus \bigcup_{H \in \mathcal{A}} H.$$
- An arrangement $\mathcal{A}$ is central if
$$\bigcap_{H \in \mathcal{A}} H \neq \varnothing.$$
- The defining polynomial of an arrangement $\mathcal{A}$ is
$$Q_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H$$
where α_H is a linear form defining H.

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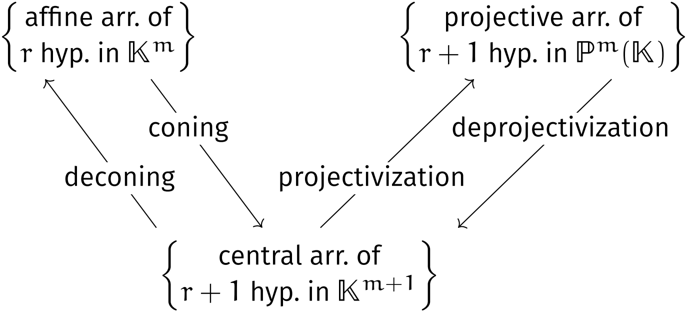
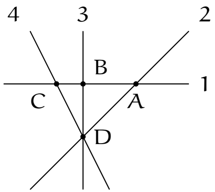
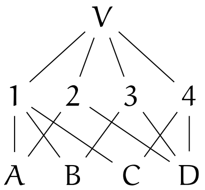


Intersection poset

Definition (Intersection poset)

The intersection poset $L(\mathcal{A})$ of an arrangement $\mathcal{A}$ is the set of all non-empty intersections of hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion. It includes V as the intersection of zero hyperplanes.





A, B, C, D are the *singular points* of $\mathcal{A}$. For a singular point P, its *multiplicity* $m(P)$ is the number of lines passing through it.

Combinatorial properties

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We say that a property of an arrangement $\mathcal{A}$ is combinatorial if it depends only on the intersection poset $\mathrm{L}(\mathcal{A})$.

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- The cohomology ring $\mathrm{H}^*(\mathcal{M}(\mathcal{A}); \mathbb{C})$ *is* combinatorial (Orlik-Solomon algebra).
- The fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ *is not* combinatorial (Rybnikov counterexample).

Local systems

Definition (Local system)

Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^m$, $M := \mathcal{M}(\mathcal{A})$, and let R be a commutative ring with unity. A rank-1 local system is a structure of $\pi_1(M)$-module on R.

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When $R = \mathbb{C}$, the action $\pi_1(M) \to \mathcal{A}ut(\mathbb{C}) \simeq \mathbb{C}^*$ factors through $H_1(M; \mathbb{Z})$, which is free abelian of rank n generated by $\beta_1, \ldots, \beta_n$, where β_i is a loop around a line of $\mathcal{A}$. In this case, the local system is defined by a choice of a non-zero complex number t_i for each β_i.

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We will denote by $\mathbb{C}_{\mathbf{t}}$ the local system defined by $\mathbf{t} := (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$, and with $H_*(M;\mathbb{C}_{\mathbf{t}})$ and $H^*(M;\mathbb{C}_{\mathbf{t}})$ respectively the homology and cohomology with coefficients in $\mathbb{C}_{\mathbf{t}}$.

Characteristic varieties

Definition (Characteristic variety)

Let $\mathcal{A}$ be an arrangement as before. The (first) characteristic variety is

$$\mathcal{V}(\mathcal{A}) \coloneqq \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim H^1(M; \mathbb{C}_{\mathbf{t}}) \geqslant 1\}.$$

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Theorem (Arapura '97)

$\mathcal{V}(\mathcal{A})$ is a union of (possibly translated) subtori of $(\mathbb{C}^*)^n$.

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Is $\mathcal{V}(\mathcal{A})$ combinatorial?

Resonance varieties

Let A be the Orlik-Solomon algebra associated with $\mathcal{A}$. Fix $a \in A^1$.
Left-multiplication by a gives $A^\bullet$ the structure of a cochain complex.

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$$\mathcal{R}(\mathcal{A}) \coloneqq \{a \in A^1 \mid \dim H^1((A^\bullet, a\,\cdot\,); \mathbb{C}) \geqslant 1\}.$$

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Tangent Cone Theorem (Cohen-Suciu ’99)

$\mathcal{R}(\mathcal{A})$ is the tangent cone of $\mathcal{V}(\mathcal{A})$ at $(1, \ldots, 1) \in (\mathbb{C}^*)^n$.

The “homogeneous part” of $\mathcal{V}(\mathcal{A})$ is combinatorial!

Our setting

From now we will suppose that $\mathcal{A}$ is an arrangement of $n+1$ projective lines in $\mathbb{P}^2(\mathbb{R})$. The defining polynomial $Q_{\mathcal{A}}$ belongs to $\mathbb{R}[X, Y, Z]$ and it is homogeneous of degree $n+1$.

Since the topology of $\mathcal{M}(\mathcal{A})$ in $\mathbb{P}^2(\mathbb{R})$ is easy to describe, we will consider the *complexified* arrangement $\mathcal{A}_{\mathbb{C}}$, which is the arrangement in $\mathbb{P}^2(\mathbb{C})$ defined by $Q_{\mathcal{A}}$, and study the complement $\mathcal{M}(\mathcal{A}_{\mathbb{C}}) \subseteq \mathbb{C}^2$.

Our setting

If $\mathcal{A}$ is an arrangement of $n+1$ projective lines in $\mathbb{P}^2(\mathbb{C})$, it is known that

$$H_1(\mathcal{M}(\mathcal{A}); \mathbb{C}) = \langle \beta_1, \ldots, \beta_{n+1} \mid \beta_1 \cdots \beta_{n+1} = 1 \rangle$$

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1. If $(t_1, \dots, t_{n+1})$ is a local system for $\mathcal{A}$, then $t_1 \cdots t_{n+1} = 1$.
2. Let $a\mathcal{A}$ be the arrangement of n lines in $\mathbb{C}^2$ obtained by sending ℓ_{n+1} to infinity. Then

$$\mathcal{V}(\mathcal{A}) = \{(\mathbf{t}, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid \mathbf{t} \in \mathcal{V}(a\mathcal{A}),\ t_1 \cdots t_{n+1} = 1\}.$$

Local and non-local components

Denote the lines of $\mathcal{A}$ with $[n+1] \coloneqq \{1,\dots,n+1\}$ and a singular point with the subset of $[n+1]$ indicating the lines passing through it. Let $S \subseteq \wp([n+1])$ be the set of the singular points.

For each $P \in S$ with $\#(P) \geqslant 3$, there is a *local* component of $\mathcal{R}(\mathcal{A})$ given by

$$C(P) \coloneqq \left\{ z \,\middle|\, \sum_{j=1}^{n+1} z_j = 0 \right\} \cap \bigcap_{j \notin P} \{ z \mid z_j = 0 \}$$

The *non-local* components admit a description in terms of *neighbourly partitions*.

Neighbourly partitions

Definition (Neighbourly partition)

A partition $\pi = (p_1 \mid \cdots \mid p_r)$ of $[n+1]$ is neighbourly if for all $i = 1, \ldots, r$ and for all $P \in S$

$$\#(p_i \cap P) \geqslant \#(P) - 1 \;\Rightarrow P \subseteq p_i.$$

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$$\#(p_i \cap P) \geqslant \#(P) - 1 \;\Rightarrow P \subseteq p_i.$$

If π is a neighbourly partition, define $C(\pi) \subseteq \mathbb{C}^{n+1}$ as

$$C(\pi) \coloneqq \left\{ z \;\middle|\; \sum_{j=1}^{n+1} z_j = 0 \right\} \cap \bigcap_{P \in \mathcal{P}} \left\{ z \;\middle|\; \sum_{j \in P} z_j = 0 \right\}$$

where $\mathcal{P} \coloneqq \{P \in S \mid \nexists\, p \in \pi \text{ s.t. } P \subseteq p\}$.

Neighbourly partitions

Proposition

If $\dim(C(\pi)) \geqslant 2$*, then* $C(\pi)$ *is a non-local component of* $\mathcal{R}(\mathcal{A})$*.*

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If π is a partition of a *subset* $B \subseteq [n+1]$, define *support* of π, $\mathrm{supp}(\pi)$, the set B.

Proposition

Let $\mathcal{B} \subseteq \mathcal{A}$ *be a subarrangement and let* π *be a neighbourly partition for* $\mathcal{B}$ *such that* $\dim(C(\pi)) \geqslant 2$*. Then*

$$C(\pi) \cap \bigcap_{j \notin \mathrm{supp}(\pi)} \{z_j = 0\}$$

is a non-local component of $\mathcal{R}(\mathcal{A})$*. All non-local components of* $\mathcal{R}(\mathcal{A})$ *arise from subarrangements of* $\mathcal{A}$ *this way.*

Combinatorics of the characteristic variety

For the homogeneous part of the characteristic variety $\mathcal{V}(\mathcal{A})$, we define *ideals* of $\mathbb{C}[T_1^{\pm 1}, \ldots, T_{n+1}^{\pm 1}]$ such that their varieties are the components of $\mathcal{V}(\mathcal{A})$.

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- If $P \in \mathcal{S}$ with $\#(P) \geqslant 3$, define

$$\mathcal{I}(P) \coloneqq \left(\prod_{j=1}^{n+1} T_j - 1 \right) + \left(T_j - 1 \mid j \notin P \right);$$

  this corresponds to a local component of $\mathcal{V}(\mathcal{A})$.

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- If π is a neighbourly partition, define

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  where $\mathcal{P} := \{P \in \mathcal{S} \mid \nexists\, p \in \pi \text{ s.t. } P \subseteq p\}$.

Combinatorics of the characteristic variety

Proposition

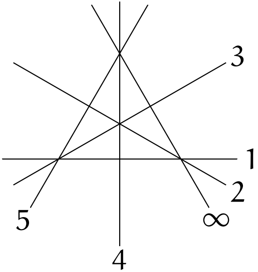
Let $\mathcal{B} \subseteq \mathcal{A}$ be a subarrangement and let π be a neighbourly partition for $\mathcal{B}$ such that $\dim(\mathcal{I}(\pi)) \geqslant 2$. Then the component passing through $(1, \ldots, 1)$ of the variety in $(\mathbb{C}^*)^{n+1}$ defined by the ideal

$$\mathcal{I}(\pi) + (T_j - 1 \mid j \notin \operatorname{supp}(\pi))$$

is a non-local component of $\mathcal{V}(\mathcal{A})$. All non-local components of $\mathcal{V}(\mathcal{A})$ passing through $(1, \ldots, 1)$ arise from subarrangements of $\mathcal{A}$ this way.

If $\mathcal{B} = \mathcal{A}$, we call the component *essential*.

Example: $A3$

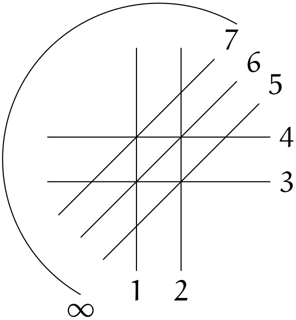


- 4 local components
- 1 component with equations

$$t_1 - t_4, \quad t_2 - t_5, \quad t_3 t_4 t_5 - 1$$

Example: B3x

This example was discovered by Suciu in 2002



- 7 local components
- 5 components of type $A3$
- 1 translated component with equations

$$t_6+1,\quad t_2-t_3,\quad t_1-t_4,$$
$$t_5t_7-1,\quad t_4t_7+t_3,$$
$$t_3t_5+t_4,\quad t_4^2-t_5,$$
$$t_3t_4+1,\quad t_3^2-t_7$$

Computing the characteristic variety

Let $\mathcal{A}$ be an arrangement of $n+1$ lines in $\mathbb{P}^2(\mathbb{R})$ and let $M = \mathcal{M}(\mathcal{A}_{\mathbb{C}})$.

- Alexander matrix from a presentation of $\pi_1(M)$;
- refined Salvetti complex (Salvetti-Settepanella '07; Gaiffi-Salvetti '09): algebraic complex that computes the homology of M with local coefficients.

Theorem

Let $\partial_2(t)$ be the 2-boundary map of the refined Salvetti complex that computes $H_*(M;\mathbb{C}_t)$. Then

$$\mathcal{V}(\mathcal{A}_{\mathbb{C}}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \mathrm{rk}([\partial_2](\mathbf{t})) < n-1\}.$$

Computing the characteristic variety

- Compute the primary decomposition of the ideal of all $(n-1) \times (n-1)$ minors.

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- Compute one $(n-1)\times(n-1)$ minor at a time, factoring it and computing “partial components”.

Remark

The number of minors is

$$n \cdot \binom{\nu}{n-1}$$

where $\nu = \sum_{P \in \mathcal{S}ing(\mathcal{A})} (m(P) - 1)$.

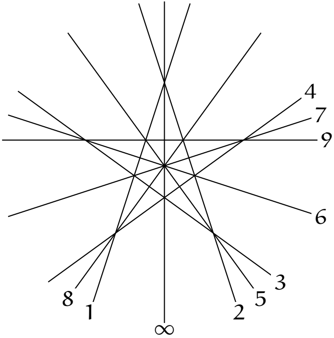
Computing the characteristic variety

CPU total time for computation of $\mathcal{V}(\mathrm{aB3x})$:

Computer	Processor	Benchmark	Time
`sedna`	Intel Atom N550	235	2 h 38 min 46 s
`lab6`	AMD A8-3850 APU	995	26 min 14 s
`lnx1`	Intel Xeon E5-2643 v4	2060	13 min 19 s

(benchmark: `www.cpubenchmark.net`, single thread, last checked on June 11, 2018)

$\mathcal{R}(10)$

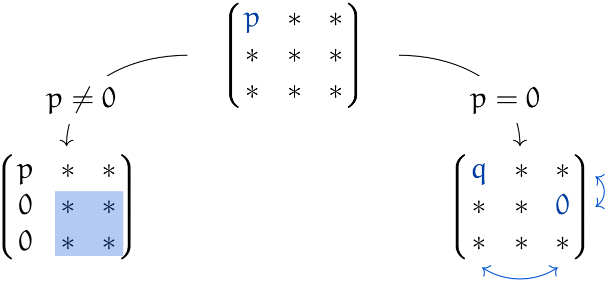


- 11 local components
- 10 components of type $A3$
- 4 translated components with equations

$$t_7 - t_8, \quad t_6 - t_8, \quad t_5 - t_8,$$
$$t_4 - t_9, \quad t_3 - t_9, \quad t_2 - t_9,$$
$$t_1 - t_9, \quad t_8^2 - t_9, \quad t_9^3 - t_8,$$
$$t_8 t_9 + t_9^2 + t_8 + t_9 + 1$$

A new algorithm

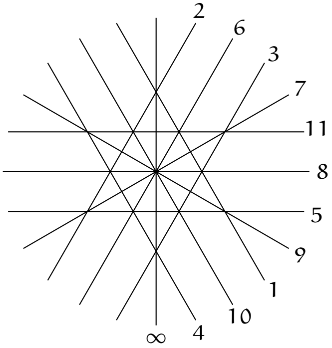
We developed a new algorithm that computes the characteristic variety through a series of *bifurcations*.



Comparison of the two algorithms

Computer	Old algorithm	New algorithm
`sedna`	2 h 38 min 46 s	2 min 21 s
`lab6`	26 min 14 s	27 s
`lnx1`	13 min 19 s	14 s

$\mathcal{R}(12)$



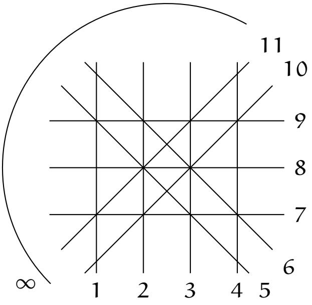
- 16 local components
- 23 comp. of type $A3$
- 1 comp. of type NonPappus
- 3 trans. comp. of type $B3x$
- 1 comp. with equations

$$t_8 - t_{11},\ t_7 - t_{10},\ t_6 - t_9,$$
$$t_5 - t_{11},\ t_4 - t_{10},\ t_3 - t_9,$$
$$t_2 - t_9,\ t_1 - t_{10},\ t_9 t_{10} t_{11} - 1$$

- 2 trans. comp. with equations

$$t_{10} + t_{11},\ t_9 + t_{11},\ t_8 + t_{11},$$
$$t_7 + t_{11},\ t_6 + t_{11},\ t_5 - t_{11},$$
$$t_4 - t_{11},\ t_3 - t_{11},\ t_2 - t_{11},$$
$$t_1 - t_{11},\ t_{11}^2 - t_{11} + 1$$

$\mathcal{A}(12, 2)$



- 14 local components
- 35 comp. of type $A3$
- 2 comp. of type $B3$
- 1 comp. of type NonPappus
- 10 trans. comp. of type $B3x$
- 8 trans. comp. of type $\mathcal{A}(10, 2)$
- 4 trans. comp. of type $\mathcal{A}(11, 1)$
- 2 trans. comp. with equations

$$t_{10} - t_{11},\ t_9 + 1,\ t_8 - t_{11} + 1,\\ t_7 + 1,\ t_6 - t_{11},\ t_5 - t_{11},\\ t_4 - t_{11} + 1,\ t_3 - t_{11} + 1,\\ t_2 - t_{11} + 1,\ t_1 - t_{11} + 1,\\ t_{11}^2 - t_{11} + 1$$

Double points partitions

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$. The *double points graph* $\Gamma(\mathcal{A})$ is the graph defined as follows:

- its vertex set is $\{H \mid H \in \mathcal{A}\}$;
- there is an edge $\{H_1, H_2\}$ iff $H_1 \cap H_2$ is a double point.

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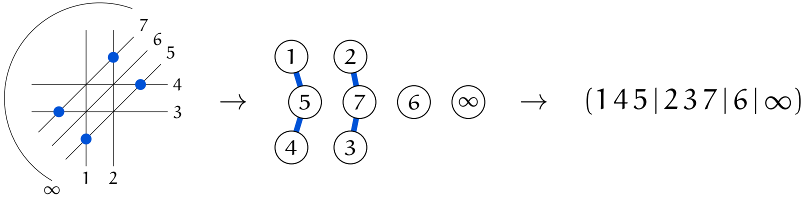
Definition

The double points partition of $\mathcal{A}$ is the partition $\Pi_{\mathcal{A}}$ induced by the connected components of $\Gamma(\mathcal{A})$.



Double points partitions

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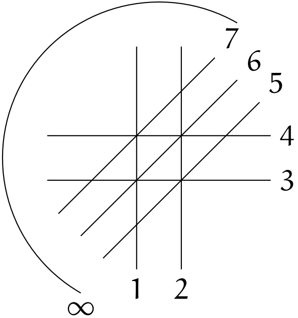
Proposition

For all arrangements $\mathcal{A}$ of which we computed $\mathcal{V}(\mathcal{A})$ except one, *if $\Pi_{\mathcal{A}}$ is the double points partition of $\mathcal{A}$, the essential translated components of $\mathcal{V}(\mathcal{A})$ (if exist) appear as the zero locus of one ideal of the primary decomposition of (the radical of) $\mathcal{I}(\Pi_{\mathcal{A}})$.*

Example: DPP of $B3x$

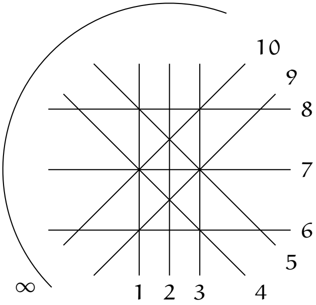


The primary decomposition of $\mathcal{I}(1\,4\,5\,|\,2\,3\,7\,|\,6\,|\,\infty)$ is $I_1 \cap I_2$, where

$$I_1 = \begin{pmatrix} t_6+1, & t_2-t_3, & t_1-t_4, & t_5t_7-1, & t_4t_7+t_3, \\ & t_3t_5+t_4, & t_4^2-t_5, & t_3t_4+1, & t_3^2-t_7 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} t_6-1, & t_2-t_3, & t_1-t_4, & t_5t_7-1, & t_4t_7-t_3, \\ & t_3t_5-t_4, & t_4^2-t_5, & t_3t_4-1, & t_3^2-t_7 \end{pmatrix}$$

Counterexample: $\mathcal{A}(11, 1)$



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$$t_9 - t_{10},\ t_8 + 1,\ t_7 - t_{10} + 1,\ t_6 + 1,\ t_5 - t_{10},\ t_4 - t_{10},\ t_3 - t_{10} + 1,\ t_2 + t_{10},\ t_1 - t_{10} + 1,\ t_{10}^2 - t_{10} + 1$$

Counterexample: $\mathcal{A}(11, 1)$

$$\Pi_{\mathcal{A}} = (2\,4\,5\,6\,7\,8\,9\,10 \,|\, 1 \,|\, 3 \,|\, \infty)$$

The two translated components are 0-dimensional, whereas all irreducible components of $\mathcal{Z}(\mathcal{I}(\Pi_{\mathcal{A}}))$ are 1-dimensional. However, the two points *do* belong to $\mathcal{Z}(\mathcal{I}(\Pi_{\mathcal{A}}))$.

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It turns out that the two translated components appear in the primary decomposition of

$$\mathcal{I}(\Pi_{\mathcal{A}}) + (t_2 t_4 t_9 - 1,\ t_2 t_5 t_{10} - 1)$$

which is the ideal generated by *all* the polynomials $\prod t_i - 1$ associated with singular points with multiplicity at least three.

The torus

Definition

A complex algebraic torus $\mathcal{T}$ is an affine variety isomorphic to $(\mathbb{C}^*)^n$.

Definition

A character of $\mathcal{T}$ is a group homomorphism $\chi\colon \mathcal{T} \to \mathbb{C}^*$ that is a morphism of algebraic varieties. The set of characters of $\mathcal{T}$ is a group $X^*(\mathcal{T})$ isomorphic to $\mathbb{Z}^n$.

Definition

A one-parameter subgroup of $\mathcal{T}$ is a group homomorphism $\lambda\colon \mathbb{C}^* \to \mathcal{T}$ that is a morphism of algebraic varieties. The set of one-parameter subgroups of $\mathcal{T}$ is a group $X_*(\mathcal{T})$ isomorphic to $\mathbb{Z}^n$.

Toric arrangements

Definition

A layer in $\mathcal{T}$ is a set of the form

$$\mathcal{K}(\Gamma, \varphi) := \{t \in \mathcal{T} \mid \chi(t) = \varphi(\chi) \text{ for all } \chi \in \Gamma\}$$

where $\Gamma < X^*(\mathcal{T})$ is a split direct summand and $\varphi \colon \Gamma \to \mathbb{C}^*$ is a homomorphism.

Definition

A toric arrangement $\mathcal{A}$ in $\mathcal{T}$ is a finite set of layers in $\mathcal{T}$.

Wonderful models

Wonderful models have been introduced by De Concini and Procesi in 1995 for subspace arrangements.

Definition

A projective wonderful model $Y_{\mathcal{A}}$ for $\mathcal{M}(\mathcal{A})$ is a smooth projective variety containing $\mathcal{M}(\mathcal{A})$ as a dense open set and such that the complement $Y_{\mathcal{A}} \setminus \mathcal{M}(\mathcal{A})$ is a divisor with normal crossings and smooth irreducible components.

Building the wonderful model

$$\begin{array}{ll} \mathcal{A} & \text{toric arrangement} \\ \downarrow & \\ X_{\mathcal{A}} & \text{“good” toric variety} \\ \downarrow & \\ Y_{\mathcal{A}} & \text{wonderful model} \end{array}$$

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A toric variety can be obtained from a *polyhedral rational fan* Δ in $V := X_*(\mathcal{T}) \otimes \mathbb{R}$.

Equal sign bases

There is a pairing $\langle \cdot, \cdot \rangle \colon X^*(\mathcal{T}) \times X_*(\mathcal{T}) \to \mathbb{Z}$.

Definition

Let Δ be a fan in V. A character $\chi \in X^*(\mathcal{T})$ has the equal sign property with respect to Δ if, for every cone $C \in \Delta$, either $\langle \chi, \mathbf{c} \rangle \geqslant 0$ for all $\mathbf{c} \in C$ or $\langle \chi, \mathbf{c} \rangle \leqslant 0$ for all $\mathbf{c} \in C$.

Definition

Let Δ be a fan in V and let $\mathcal{K}(\Gamma, \varphi)$ be a layer. A basis $(\chi_1, \dots, \chi_m)$ for Γ is an equal sign basis with respect to Δ if χ_i has the equal sign property for all $i = 1, \dots, m$.

$X_{\mathcal{A}}$ is "good" if it is projective, smooth and every layer of $\mathcal{A}$ has an equal sign basis w.r.t. the fan associated with $X_{\mathcal{A}}$.

Two algorithms

For each $\mathcal{K}_i = \mathcal{K}(\Gamma_i, \varphi_i) \in \mathcal{A}$, let $\chi_{i,1}, \ldots, \chi_{i,s_i}$ be a $\mathbb{Z}$-basis of Γ_i and let

$$\Xi = \bigcup_{\mathcal{K}_i \in \mathcal{A}} \{\chi_{i,1}, \ldots, \chi_{i,s_i}\}.$$

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1. Start with a smooth, projective fan and subdivide it so that the final fan is equal sign.
2. Start with an equal sign, projective fan and subdivide it so that the final fan is smooth.

DCG algorithm

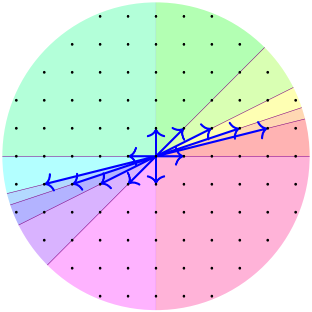
1. Start with the orthant fan (corresponding variety: $(\mathbb{P}^1)^n$).
2. Choose a vector $\chi \in \Xi$.
3. Repeat until there are no “bad” cones:
   3.1 Create the list of bad cones.
   3.2 Choose a bad cone and subdivide it.
4. Repeat for all vectors in Ξ, using the last computed fan as input.

Subdivision can be done in such a way that each computed fan is still smooth and projective.

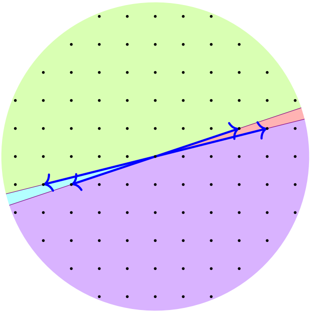
Smooth algorithm

1. Start with the fan generated by the vectors orthogonal to the ones in Ξ.
2. For each non-smooth cone of the fan, subdivide it in two cones. This can be done in such a way that at least one of them is smooth, and eventually both of them are.

Example



First algorithm



Second algorithm

$$\Xi = \{(-1, 3), (-1, 4)\}$$

Cohomology of the wonderful model

De Concini and Gaiffi (2018): presentation of $H^*(Y_{\mathcal{A}};\mathbb{Z})$.

- Cohomology ring $H^*(X_{\mathcal{A}};\mathbb{Z})$
- Well-connected building set $\mathcal{G}$
  - Poset of layers $\mathcal{C}(\mathcal{A})$
- Adapted bases for the elements of $\mathcal{C}(\mathcal{A})$

$$H^*(Y_{\mathcal{A}};\mathbb{Z}) \simeq H^*(X_{\mathcal{A}};\mathbb{Z})[T_G \mid G \in \mathcal{G}] \big/ \text{some relations}$$

Cohomology of the toric variety

Let X be a smooth complete toric variety with associated fan Δ and let $\mathcal{R}$ be the set of the primitive rays of Δ.

$$H^*(X; \mathbb{Z}) \simeq \mathbb{Z}[C_{\mathbf{r}} \mid \mathbf{r} \in \mathcal{R}] / (I_{SR} + I_L)$$

where

- I_{SR} is the *Stanley-Reisner ideal*

  $I_{SR} \coloneqq (C_{\mathbf{r}_1} \cdots C_{\mathbf{r}_k} \mid \mathbf{r}_1, \dots, \mathbf{r}_k$ do not belong to a cone of $\Delta)$;
- I_L is the *linear equivalence ideal*

$$I_L \coloneqq \left(\sum_{\mathbf{r} \in \mathcal{R}} \langle \beta, \mathbf{r} \rangle C_{\mathbf{r}} \;\middle|\; \beta \in X^*(\mathcal{T}) \right).$$

Poset of layers

Definition

Let $\mathcal{A}$ be a toric arrangement in the torus $\mathcal{T}$. The poset of layers $\mathcal{C}(\mathcal{A})$ is the set of all the connected components of the non-empty intersections of the layers of $\mathcal{A}$, partially ordered by reverse inclusion. It includes $\mathcal{T}$ as the intersection of zero layers.

To compute $\mathcal{C}(\mathcal{A})$ we use an algorithm by Lenz (2017). However he considers arrangements in the *real compact* torus $(S^1)^n$ instead of the complex algebraic torus $(\mathbb{C}^*)^n$. This is not a problem, as our definition is more general.

Building sets

Let $\mathcal{C}_0(\mathcal{A}) \coloneqq \mathcal{C}(\mathcal{A}) \smallsetminus \{\mathcal{T}\}$. For the sake of simplicity, assume that all the non-empty intersections of the layers of $\mathcal{A}$ are connected.

Definition

A subset $\mathcal{G} \subseteq \mathcal{C}_0(\mathcal{A})$ is a building set for $\mathcal{A}$ if for each layer $\mathcal{K} \in \mathcal{C}_0(\mathcal{A}) \smallsetminus \mathcal{G}$ the minimal elements of the set $\{G \in \mathcal{G} \mid G \supseteq \mathcal{K}\}$ intersect transversally and their intersection is $\mathcal{K}$.

Definition

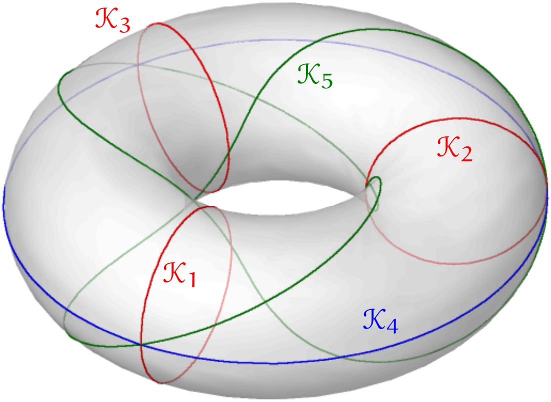
A building set $\mathcal{G}$ is well-connected if for any subset $\{G_1, \ldots, G_k\} \subseteq \mathcal{G}$, if the intersection $G_1 \cap \cdots \cap G_k$ has two or more connected components, then each of them belongs to $\mathcal{G}$.

Adapted bases

For every pair $(M, G) \in \mathcal{C}(\mathcal{A}) \times \mathcal{C}(\mathcal{A})$ with $G \subseteq M$, we choose a basis $(\beta_1, \ldots, \beta_s)$ for Γ_G such that $(\beta_1, \ldots, \beta_k)$, with $k \leqslant s$, is a basis for Γ_M. We omit the details here.

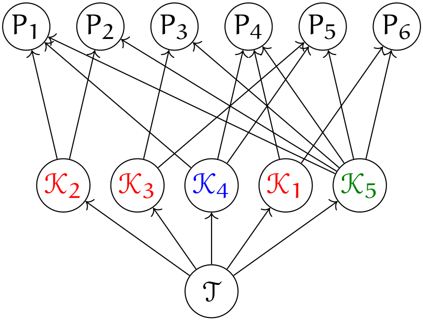
From these bases, we compute polynomials $P_G^M \in H^*(X_\mathcal{A}; \mathbb{Z})[Z]$, from which the relations for $H^*(Y_\mathcal{A}; \mathbb{Z})$ can be obtained.

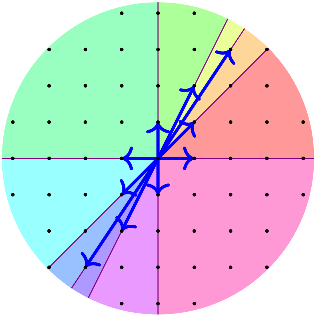
A two-dimensional example



$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

A two-dimensional example (cont'd)

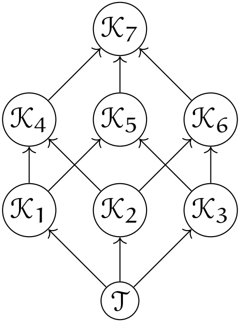




$$H^4(Y_{\mathcal{A}};\mathbb{Z}) \simeq \mathbb{Z}, \qquad H^2(Y_{\mathcal{A}};\mathbb{Z}) \simeq \mathbb{Z}^{14}, \qquad H^0(Y_{\mathcal{A}};\mathbb{Z}) \simeq \mathbb{Z}.$$

A five-dimensional example

$$\begin{aligned}
\mathcal{K}_1 &= \{(t_1,\dots,t_5) \in (\mathbb{C}^*)^5 \mid t_1 t_4^{-1} = 1\},\\
\mathcal{K}_2 &= \{(t_1,\dots,t_5) \in (\mathbb{C}^*)^5 \mid t_2 t_5^{-1} = 1\},\\
\mathcal{K}_3 &= \{(t_1,\dots,t_5) \in (\mathbb{C}^*)^5 \mid t_3 t_4 t_5 = 1\}.
\end{aligned}$$



$$\begin{aligned}
H^{10}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z},\\
H^{8}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z}^{29},\\
H^{6}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z}^{132},\\
H^{4}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z}^{132},\\
H^{2}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z}^{29},\\
H^{0}(Y_{\mathcal{A}};\mathbb{Z}) &\simeq \mathbb{Z}.
\end{aligned}$$

Future directions

- Line arrangements: investigate the link between the ideals generated by multiple points and the components of the characteristic variety.

Future directions

- Line arrangements: investigate the link between the ideals generated by multiple points and the components of the characteristic variety.
- Toric arrangements: generalize the construction, allowing arbitrary building sets as well as toric arrangements according to De Concini and Gaiffi's definition.

Thank you!

Thank you for your attention.