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New min-max frameworks for minimal submanifolds in dimension two or codimension two

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Abstract

In this dissertation we develop new variational methods with the aim to build *minimal* submanifolds Σ^k in a closed ambient Riemannian manifold $(\mathcal{M}^m, g)$, by means of *min-max* procedures. We will focus almost exclusively on the cases $k = 2$ and $k = m - 2$, namely minimal surfaces and minimal submanifolds of codimension 2.

After a general introduction, in the second chapter we present a recent min-max theory devised by the supervisor of this thesis, T. Rivière: starting from immersions which are critical for a suitable relaxation of the area, involving a power of the second fundamental form, this method builds, as the viscosity parameter tends to zero, a limit object satisfying a certain weak notion of minimality. The method applies to any min-max problem for the area functional in the space of immersions $\Sigma \to \mathcal{M}$, where Σ is a given closed surface. We revisit this theory and show how to adapt it to the *free boundary* case, relative to any submanifold $\mathcal{N}^n \subset \mathcal{M}$; the outcome is essentially an immersed manifold which is critical with respect to the constraint that the boundary is contained in $\mathcal{N}$.

In the following chapter we study axiomatically this new weak notion of minimal immersion, whose instances are called *parametrized stationary varifolds*. These special *varifolds* are induced by a parametrization and a Borel multiplicity; they enjoy a localization property for the stationarity, with respect to the domain. In spite of the lack of general regularity results for stationary varifolds, this last property can be exploited to show that the multiplicity is constant and the parametrization is smooth, without any assumption on the codimension.

The attention then moves to the study of the multiplicity when the parametrized varifold arises as a limit of critical or almost critical immersions for the relaxed functionals; in this case we show that the multiplicity is always equal to one. This result allows to obtain upper bounds on the Morse index of the limit minimal immersion, namely a bound on the instability of this map in the space of immersions. This fact would allow, by itself, to simplify the regularity theory, but its proof relies strongly on the theory contained in the previous chapters.

In the last chapter, which is completely independent of the rest of the thesis, we present another theory meant to produce minimal submanifolds in codimension 2. It is inspired by the recent use of the Allen–Cahn functional for maps $u : \mathcal{M} \to \mathbb{R}$, viewed as a relaxation of the $(m-1)$-area of the level sets of u: Rather than using the most immediate generalization

given by the Ginzburg–Landau energy for maps $u : \mathcal{M} \to \mathbb{C}$, which makes the asymptotic analysis difficult and not completely satisfactory, we study instead the Yang–Mills–Higgs functional for couples (u, ∇), where $u : \mathcal{M} \to L$ is a section of a given Hermitian line bundle $L \to \mathcal{M}$ and ∇ is a connection on it. This energy enjoys a natural gauge invariance for the symmetry group $U(1)$. The study of this functional brings to a simpler analysis of the energy concentration set and, differently from what happens with Ginzburg–Landau, it allows to obtain the integrality and concentration of the limit varifold.

The results contained in the third and fourth chapters have been obtained in collaboration with T. Rivière, while the ones from the last chapter come from a collaboration with D. Stern.

Sunto

Questa tesi di dottorato si occupa dello sviluppo di nuovi metodi variazionali volti a costruire sottovarietà *minime* Σ^k in un ambiente Riemanniano chiuso $(\mathcal{M}^m, g)$, tramite procedure di *min-max.* Ci dedicheremo quasi esclusivamente ai casi $k = 2$ e $k = m - 2$, ovvero superfici minime e sottovarietà minime di codimensione 2.

Dopo un'introduzione generale agli argomenti trattati, nel secondo capitolo presentiamo una recente teoria di min-max dovuta al relatore di questa tesi, T. Rivière: partendo da immersioni critiche per un opportuno rilassamento dell'area, contenente una potenza della seconda forma fondamentale, questo metodo costruisce, al tendere a zero del parametro di viscosità, un oggetto limite soddisfacente una certa nozione debole di minimalità. Il metodo si applica a un qualsiasi problema di min-max per l'area nello spazio delle immersioni $\Sigma \to \mathcal{M}$, dove Σ è una superficie chiusa fissata. Rivisitiamo questa teoria e mostriamo come adattarla al caso *free boundary*, relativamente a una qualsiasi sottovarietà $\mathcal{N}^n \subset \mathcal{M}$; l'oggetto che ne deriva è essenzialmente una superficie immersa critica per l'area rispetto al vincolo di avere il bordo contenuto in $\mathcal{N}$.

Nel capitolo successivo studiamo in modo assiomatico questa nuova nozione debole di immersione minima, le cui istanze vengono chiamate *varifold parametrici stazionari.* Questi speciali *varifold* sono indotti da una parametrizzazione e da una molteplicità Boreliana; godono di una proprietà di localizzazione per la stazionarietà, rispetto al dominio. Nonostante la scarsità di risultati di regolarità generali per varifold stazionari, quest'ultima proprietà può essere sfruttata per dimostrare che la molteplicità è costante e che la parametrizzazione è liscia, senza alcuna ipotesi sulla codimensione.

L'attenzione viene poi rivolta allo studio della molteplicità quando il varifold parametrico è un limite di immersioni critiche o quasi critiche per i funzionali rilassati; in questo caso mostriamo che la molteplicità è sempre uguale a uno. Questo risultato permette di ottenere stime dall'alto sull'indice di Morse dell'immersione minima limite, ovvero una misura della sua instabilità nello spazio delle immersioni. Ciò di per sé renderebbe più semplice la teoria di regolarità, ma la dimostrazione di questo fatto dipende fortemente dalla teoria contenuta nei capitoli precedenti.

Nell'ultimo capitolo, completamente indipendente dal resto della tesi, presentiamo un'altra teoria volta a produrre sottovarietà minime in codimensione 2. Questa è ispirata al recente utilizzo del funzionale di Allen–Cahn per mappe $u : \mathcal{M} \to \mathbb{R}$, pensato come

rilassamento dell'area $(m-1)$-dimensionale degli insiemi di livello di u. Anziché utilizzare la generalizzazione più immediata data dall'energia di Ginzburg–Landau per mappe $u : \mathcal{M} \to \mathbb{C}$, che rende l'analisi asintotica difficile e non interamente soddisfacente, studiamo invece il funzionale di Yang–Mills–Higgs per coppie (u, ∇), dove $u : \mathcal{M} \to L$ è una sezione di un fissato fibrato in rette complesse $L \to \mathcal{M}$, con una data struttura Hermitiana, e ∇ è una connessione sul fibrato. Questo energia gode di una naturale invarianza di gauge rispetto al gruppo di simmetria $U(1)$. Lo studio di questo funzionale porta a un'analisi dell'insieme di concentrazione dell'energia più semplice e, a differenza di quanto accade con Ginzburg–Landau, permette di ottenere l'integralità e la concentrazione del varifold limite.

I risultati contenuti nel terzo e quarto capitolo sono stati ottenuti in collaborazione con T. Rivière, mentre quelli dell'ultimo capitolo nascono da una collaborazione con D. Stern.

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Contents

1 | General introduction

1.1 The landscape

The main theme of this thesis is the variational construction of minimal submanifolds Σ^k in an assigned closed Riemannian manifold $(\mathcal{M}^m, g)$. While there is already a vast literature for hypersurfaces ($k = m - 1$) and geodesics ($k = 1$), with several satisfactory results, very little is known for the following two very general problems:

- develop a variational theory which allows to construct or exhibit *minimal submanifolds*, especially of saddle-point type, for general k and m; here "minimal submanifold" should not be necessarily intended in the strongest sense of smooth embedded submanifold, or not even immersed: we allow for appropriate weak notions of minimal objects for which one expects some kind of regularity;
- develop a regularity theory for the objects produced in this variational way.

We will provide some partial answers in the cases $k = 2$ and $k = n - 2$. Before stating them, we will briefly review some previous results in order to place this dissertation in perspective.

Some history of minimal submanifolds

In a Riemannian manifold $(\mathcal{M}^m, g)$, not necessarily closed, an embedded or immersed submanifold $\Sigma^k \subset \mathcal{M}^m$ is called *minimal* if the trace of its second fundamental form vanishes at every point. For hypersurfaces ($k = n - 1$) this means that, at every point, the principal curvatures sum up to zero.

Another characterization for $\mathcal{M} = \mathbb{R}^m$ is the following: expressing Σ locally as the graph of a smooth map $f : U \to \mathbb{R}^{m-k}$ with $U \subseteq \mathbb{R}^k$ open (up to ambient rotations), f is critical for the area functional, namely

$$\frac{d}{dt} \int_U \sqrt{1 + |\nabla(f + tg)|^2}\Big|_{t=0} = 0$$

whenever $\{g \neq 0\} \subset\subset U$. Note that the left-hand side is just the area of the graph of $f + tg$. A special feature of the case of hypersurfaces ($k = m - 1$) is that criticality can be upgraded to local minimality for graphs, namely

$$\int_U \sqrt{1 + |\nabla f|^2} \leq \int_U \sqrt{1 + |\nabla h|^2}$$

whenever $\{h \neq f\} \subset\subset U$. This feature comes from the fact that a minimal graph is calibrated by the $(m-1)$-form tangent to the foliation of $U \times \mathbb{R}$ made by vertical translations of the graph, which turns out to be closed (here we use the identification $\Lambda_{m-1}\mathbb{R}^m = \Lambda^{m-1}\mathbb{R}^m$).

The study of minimal surfaces began already in the eighteenth century with the work of Lagrange, who derived the Euler–Lagrange equations for minimal graphs. Lagrange also posed the famous Plateau problem, asking to find a surface in $\mathbb{R}^3$ with assigned boundary and least area; the problem is named after Plateau, who analyzed special cases empirically using soap films.

Along with this and other influential existence questions, such as the Björling problem, efforts were devoted to the classification of minimal surfaces in $\mathbb{R}^3$, at least with extra assumptions such as embeddedness (proper or not), completeness or simple connectivity (or finite topological type). Independently, around the same time in the 1860's, Enneper and Weierstrass were able to give an explicit representation for all minimal, simply connected immersed surfaces: up to reparametrization, we can assume that the immersion $\phi : \Omega \to \mathbb{R}^3$ is *conformal*, with $\Omega \subseteq \mathbb{C}$; then ϕ is a primitive of the real part of the $\mathbb{C}^3$-valued 1-form

$$f\Big(\frac{1}{2}(1-g^2), \frac{i}{2}(1+g^2), g\Big)\, dz,$$

for a suitable holomorphic function f and a suitable meromorphic function g, both defined on Ω.[1]

The study of minimal surfaces became more popular after the complete solution to the Plateau problem, found around 1930 by Douglas [33] and Radó [88]. Their methods exploit the fact that, for a conformal immersion, harmonicity and minimality are equivalent, a fact which is not useful for $k > 2$ due to the general lack of conformal reparametrizations. As we shall see below, the solution for general k (even when $m = k + 1$) requires a totally different technology which brought to the *theory of currents*, a part of the modern *geometric measure theory*.

Going back to the classification problem, Osserman [83] in 1963 showed that complete, orientable, immersed minimal surfaces in $\mathbb{R}^3$ of finite total curvature are conformally equivalent to a closed Riemann surface with finitely many punctures, with the Gauss map extending across them holomorphically. It was believed for a long time that complete, connected, properly embedded minimal surfaces of finite topological type must be either the plane, the catenoid or the helicoid. It was only in 1982 that Costa [25] found another example, with three ends. More examples were later found, including surfaces similar to the helicoid with arbitrary genus but only one end [52].

In the last decades, increasing efforts were offered to the understanding of minimal surfaces in closed manifolds. Two very influential questions for the model case of $\mathcal{M} = S^3$

[1]This representation misses only the case of a planar immersion. The fact that $\Omega \subseteq \mathbb{C}$ with ϕ conformal can be assumed by the uniformization theorem (which came actually later); note that the surface cannot be compact. Conversely, the formula always provides minimal, (weakly) conformal branched immersions if fg^2 is holomorphic.

were the Lawson conjecture and the Willmore conjecture, asking respectively whether the Clifford torus $\frac{1}{\sqrt{2}}(S^1 \times S^1) \subset S^3$ is the unique embedded minimal torus, and whether it minimizes the area among embedded minimal surfaces different from the equator, up to rotations. They were both solved recently, respectively by Brendle [19] in 2013 and Marques–Neves [74] in 2012, building also on previous work by other mathematicians.

Before going to the variational aspect of the story, which played an important role also in the resolution of the Willmore conjecture, let us mention that another important topic is the study of compactness of spaces of minimal submanifolds under certain "bounded complexity" assumptions, where the complexity can be understood in terms of area, Morse index and topology; the Morse index is a measure of the instability of the submanifold for the area, namely it is the dimension of a maximal subspace of infinitesimal variations where the second variation of the area becomes negative definite. The relation between these notions of complexity, depending on the ambient, is also an interesting and important subject. Compactness for *stable* (embedded or immersed) minimal hypersurfaces, namely those with nonnegative second variation, is related to the validity of a pointwise upper bound for the second fundamental form. Such inequality in its local version is essentially equivalent to the Bernstein conjecture that a complete, connected minimal hypersurface in $\mathbb{R}^m$ is a hyperplane. This conjecture is known to be true for immersed surfaces ($m = 3$), after the work by Fischer-Colbrie–Schoen [38].

The study of minimal submanifolds is not only interesting *per se*, but has also application in and outside mathematics: we mention its use in general relativity, namely in the proof of the positive mass theorem by Schoen–Yau [97] and in Bray's proof of the Penrose conjecture [18]. Another application, due to Colding–Minicozzi [24], enters the proof of the Poincaré conjecture, or more generally of Thurston's geometrization conjecture: at every time of the Ricci flow (with surgery), a minimal sphere is built whose area equals the infimum of all immersed spheres realizing a nontrival class in the second homotopy group; the existence of such sphere is used to show the "finite time extinction" of the second homotopy group. Both applications are of variational nature.

Variational construction of minimal submanifolds

The work of Douglas and Radó for the two-dimensional case ($k = 2$, $m = 3$) of the Plateau problem relies on considering immersed surfaces which are *parametrized* by the disk, in order to be able to assume that the parametrization is conformal, with the area then agreeing with the Dirichlet energy, thus shifting the problem to a more coercive functional. This technique cannot hope to be generalized to higher dimension and the community realized that a parametrization-free approach could be more convenient.

One is then led to seek a weak notion of submanifold compatible with the calculus of variations, namely a notion which is weak enough to ensure *compactness* of the set of competitors but rich enough to have meaningful definitions for *area* and *boundary*. A successful theory was proposed only in 1960 by Federer and Fleming, in [37]. Their *theory of currents* merged the abstract homological framework of general de Rham currents with the seminal analytic ideas of generalized hypersurface—thought as the boundary of a finite-perimeter set—proposed by Caccioppoli and De Giorgi, and of rectifiable set, introduced by Besicovitch and his school.

Currents of dimension k, in a similar spirit as distributions, are defined to be the topological dual of compactly supported smooth k-forms, once the latter space is endowed with a suitable structure of locally convex vector space. Oriented manifolds of dimension k, embedded or immersed, having locally finite area can be thought as currents, acting on k-forms just by integration. Given a current T, a good notion of boundary ∂T is given by duality: we let $\langle \partial T, \omega\rangle := \langle T, d\omega\rangle$, mimicking Stokes' theorem.

The area of a current, called *mass* and defined again by duality, is not necessarily finite but is certainly lower semicontinuous with respect to weak convergence, making currents a suitable framework for minimization problems like Plateau's.

Moreover, integral currents, namely currents T such that T and ∂T are rectifiable and have integer multiplicity a.e., satisfy a suitable weak compactness property, which is the celebrated compactness theorem by Federer–Fleming. This allows to "solve" the Plateau problem by applying the direct method of calculus of variations, shifting then the bulk of the work to the regularity theory, whose landmarks are mentioned later.

In a famous work, Sacks–Uhlenbeck [95] showed the existence of minimal (branched, immersed) *spheres* in any simply connected closed manifold, with arbitrary codimension ($k = 2$,$m \geq 2$). Their method exploits again the equivalence of minimality and harmonicity for conformal immersions. Rather than trying to find directly maps $u : S^2 \to M$ critical for the Dirichlet energy, for which the Palais–Smale property (described below) fails, they work with the perturbed functional

$$\int_{S^2} (1 + |du|^2)^\alpha, \quad \alpha > 1$$

and find critical points u_α, which they show to converge to a harmonic map as $\alpha \to 1$, up to points where the energy concentrates—a phenomen called *bubbling*. This delicate step relies on a "small-energy-regularity" theorem for the perturbed functionals, uniformly in $\alpha > 1$. With this method they also manage to find a minimizer for the area (or the Dirichlet energy) in certain free homotopy classes of maps $S^2 \to \mathcal{M}$ generating $\pi_2(\mathcal{M})$, when $\mathcal{M}$ is simply connected.

In order to find general critical points for the area, which are not necessarily minimizers but, rather, of "saddle" type, the general starting idea is to study a *min-max* variational problem. This principle, which goes back to the work [16] of Birkhoff on the existence of closed geodesics for any Riemannian metric on the sphere, is best seen in the so-called *mountain pass* situation: if we have a Banach space X and a nonlinear functional $f : X \to \mathbb{R}$

of class C^1 with the property that, say,

$$F(x_0), F(x_1) \leq 0, \quad F(x) \geq 1 \text{ for all } x \text{ with } \|x\| = 1,$$

with x_0 and x_1 lying inside and outside the unit ball, respectively, then we expect heuristically to have a saddle point x_s somewhere, with value $F(x_s) \geq 1$. Indeed, this situation is pictorially like having x_0 inside a volcano and x_1 outside of it, so that we expect a saddle point on the border. The value $\lambda = F(x_s)$ can be characterized as

$$\beta = \inf_{\gamma} \max_{t \in [0,1]} F(\gamma(t)), \quad \gamma \in C^0([0,1], X) \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) = x_1.$$

Moreover, we expect that, for a minimizing sequence of curves γ_j, any point $\gamma_j(t)$ achieving $\max_{t\in[0,1]} F(\gamma(t))$ becomes arbitrarily close to a critical point of F, as $j \to \infty$. These facts are true if F satisfies a technical condition, called *Palais–Smale condition*, namely

$$\{x_j\} \text{ is precompact whenever } F(x_j) \to \lambda,\ dF(x_j) \to 0.$$

This property is also needed for minimization problems. The standard way to show the above facts is then to use a negative (pseudo-)gradient flow of F.

We refer to [9] for a broad introduction to this topic and a large collection of examples implementing this idea.

The main issue is then how to implement a min-max construction in the setting of minimal submanifolds.

The work by Sacks–Uhlenbeck [95] fell short of providing general min-max critical spheres—this is due to the lack of a technical ingredient called the analysis of *neck regions*, which would provide a complete understanding of the energy concentration issue. Instead, in the work [24] motivated by the study of the Ricci flow, Colding–Minicozzi analyzed directly the Dirichlet energy in the mountain pass situation described above, in order to find a min-max minimal sphere realizing the so-called *width* of M.

Their method involves *harmonic replacements*, replacing directly pieces of each sphere $\gamma(t)$ with energy-minimizing ones (with the same boundary values) in order to have a sort of discretized gradient flow. This is very similar in spirit to the work of Birkhoff on closed geodesics on the sphere. In this framework they manage to analyze completely the bubbling issue. But, as the work of Sacks–Uhlenbeck, this work exploits the fact that harmonic maps from the sphere are automatically conformal, and hence minimal; the same is used in the two-dimensional solution of the Plateau problem, where the domain is the disk. In the latter situation one allows for certain order-preserving reparametrizations of the given boundary; for the minimizing map, this then allows to have inner variations which also shift the boundary of the disk, enabling one to deduce the conformality of the map.

The search for more minimal surfaces, possibly with a topology different from the sphere, was in part motivated by the following influential conjecture by Yau [111].

Conjecture (Yau [111, Problem 88], 1982). *Does any closed Riemannian 3-manifold contain infinitely many (immersed or embedded, closed) minimal surfaces?*

A successful theory reaching the existence of at least one embedded minimal hypersurface, in ambient dimension $2 < m < 8$, was proposed by Almgren and his student Pitts [7, 87]: within the theory of currents, using cycles mod 2 they produce an *almost minimizing varifold* in the limit. The notion of *varifold*, for which the reader may consult [98, Chapters 4 and 8], differs from the one of current in that, while also retaining good compactness properties, the mass becomes continuous under weak convergence: this property is essential to guarantee that the limit object attains the min-max value—on the other hand, lower semicontinuity of the mass for currents is just good enough for minimization problems.

General k-varifolds are Radon measures on the Grassmannian bundle $\mathrm{Gr}_k(\mathcal{M})$ of k-planes tangent to $\mathcal{M}$. Having a measure on this bundle, rather than just on the base manifold, is important in order to have a good notion of *varifold pushforward* under a diffeomorphism $F : \mathcal{M} \to \mathcal{M}$—involving the Jacobian of dF along k-planes—compatible with the assignment $\Sigma \mapsto F(\Sigma)$ when the varifold is represented by an embedded submanifold Σ^k. In turn, this notion of pushforward is essential in order to define *stationary* varifolds, namely varifolds whose mass is invariant at first order, under the action of diffeomorphisms.

An important class of varifolds is formed by the integer rectifiable ones, namely those varifolds which can be represented as a countable superposition, with positive integer coefficients, of k-rectifiable sets. This is the kind of varifolds which is most commonly used and studied, since it is more concrete than the general definition but still enjoys compactness properties. The fundamental reference for general varifolds is Allard's doctoral work [3], where the compactness of integer stationary varifolds—or more generally of integer rectifiable varifolds with locally bounded first variation—is proved, along with some regularity results, rectifiability criteria for general varifolds, and other important estimates.

In their work, Almgren and Pitts study the space of integral *cycles*, i.e. integral currents with no boundary, and a suitable modification of them, namely cycles with coefficients in $\mathbb{Z}_p$ for $p \geq 2$, both equipped with the so-called *flat topology*.

The idea of replacement is again present in their work: it is used both to obtain a meaningful object which should be the desired hypersurface and to investigate its regularity. The object that they produce is an *almost minimizing varifold*: as will be mentioned also in the next subsection, this technical notion is what allows to recover the full regularity.

The Almgren–Pitts theory is rather technical and uses discretized families in the min-max, together with a discretized notion of continuity called *fineness*. Variants of the Almgren–Pitts theory which circumvent the need of discretization were proposed by Simon–Smith, for $n = 3$, and by De Lellis–Tasnady [32] in general dimension. The Simon–Smith theory has the advantage of giving an effective control of the *genus* of the resulting minimal surface. For an introduction to this theory, the reader can consult [23].

The use of the Almgren–Pitts framework led to the solution of several long-standing problems, including the Willmore conjecture [74] and the Yau conjecture itself, which was first established in the positive Ricci case [76], then for Baire-generic metrics by Marques–Neves and collaborators [72, 57], and finally in full generality by Song [101].

This theory was used also to construct *free boundary* minimal hypersurfaces: given an ambient $\mathcal{M}^m$ and a submanifold $\mathcal{N}^{m-1}$—usually $\mathcal{M}$ has no boundary, or $\mathcal{N}$ is precisely $\partial\mathcal{M}$—they are hypersurfaces Σ^{m-1} with boundary, embedded or immersed in $\mathcal{M}$, which are critical for the $(m-1)$-area under the constraint $\partial\Sigma \subseteq \mathcal{N}$. This is equivalent to the fact that Σ is minimal and meets $\mathcal{N}$ orthogonally along $\partial\Sigma$.

The most studied case is $(\mathcal{M}, \mathcal{N}) = (\overline{B}^3, S^2)$ with the Euclidean metric. In [62], using an equivariant version of the Simon–Smith theory, Ketover constructed free boundary minimal surfaces in the ball with arbitrarily big genus and three boundary components. In the same spirit, a very recent work [20] constructs surfaces with connected boundary and arbitrary genus.

In the work by Li–Zhou and collaborators [68, 45], the Almgren–Pitts theory for hypersurfaces in arbitrary dimension is adapted to the free boundary case. We also mention the theory by De Lellis–Ramic [28] for a similar min-max theory in the free boundary case.

Several other techniques are used to construct free boundary minimal submanifolds, including notably desingularization methods—which are used also in the closed case—and the study of extremal eigenvalue problems; for a survey of recent results, we invite the reader to consult [67].

Recently, in the closed case, another approach using the Allen–Cahn functional was proposed by Guaraco [46]. This theory, which started with the works of Modica [79] for minimizers and Hutchinson–Tonegawa [55] for general critical points, interprets a minimal hypersurface as a limit interface of a phase transition, hence as a limit of level sets of functions which are critical for rescalings of the Allen–Cahn functional, which should then be seen as a relaxation of the area for the level sets. This approach seems to be at least as powerful as Almgren–Pitts; the additional structure given by having a sequence of smooth *critical* functions converging to the limit already allowed to obtain finer results: see, e.g., [22, 13]. We will return to this topic in the next section.

In codimension two, interesting attempts have been made by Cheng and Stern using the Ginzburg–Landau energy for complex valued maps [21, 102]. This functional, which appears formally identical to Allen–Cahn—the latter being just Ginzburg–Landau for scalar maps—exhibits a totally different behavior in terms of energy concentration, due to the dominance of the angular part of the map in the Dirichlet term. This component forces the asymptotic analysis to take place on infinitely many scales, making the study very challenging. A different attempt, based on rescalings of the Yang–Mills–Higgs energy for sections and connections of a Hermitian line bundle, was proposed by the author and Stern [86] and is part of the present thesis. In this last framework, the asymptotic analysis becomes much simpler and quite similar to the Allen–Cahn setting, although a regularity theory still lacks.

Yet another framework, which will be presented in the next chapters, was introduced by Rivière [91]. It concerns minimal surfaces, but works in arbitrary codimension. As in the classical works [33, 95], it uses parametrizations $\Phi : \Sigma^2 \to (\mathcal{M}^m, g)$. On the other hand, the area is not immediately relaxed with the Dirichlet energy; rather, one uses the functional

$$E'_\sigma(\Phi) := \operatorname{area}(\Phi) + \sigma^2 \int_\Sigma (1 + |\mathrm{II}^\Phi|^2)^p \operatorname{vol}_\Phi$$

for $\sigma > 0$ and a fixed exponent $p > 1$, where the norm of the second fundamental form II^Φ and the area element vol_Φ are with respect to the metric $\Phi^* g$ induced by Φ. By studying critical points for E'_σ, one hopes to get a limit minimal immersion regardless of the topology of the closed surface Σ, while in [95] one can just reach a harmonic map—whose minimality is not guaranteed unless Σ is a sphere. As for the free boundary case, minimality holds automatically only if Σ is a disk—a fact already exploited to solve Plateau's problem; in fact, we mention that the same approach developed in [95] was used to build free boundary minimal disks in [104]. Note that E'_σ is invariant under diffeomorphisms of the domain, whereas the Dirichlet energy is only conformally invariant.

The main outcomes of this theory is that certain critical maps for E'_σ converge, in the varifold sense, to a *parametrized stationary varifold*, as $\sigma \to 0$ along a suitable sequence. A precise statement will be given in the next section.

Regularity issues

The interior regularity for area minimizing currents in codimension one was a crowning achievement of geometric measure theory, due to the combined contributions of De Giorgi, Fleming, Almgren, Simons and Federer (see, e.g., [98, Chapter 7]); this theory completed the solution to Plateau's problem in codimension one, at least ignoring boundary regularity. The latter was studied by Hardt–Simon [49].

In arbitrary codimension, an optimal interior regularity result has been achieved in a big and deep work by Almgren [8], which was later revisited and simplified by De Lellis and Spadaro [8, 29, 30, 31]. We refer to [27] for some very recent developments concerning the regularity up to the boundary.

On the other hand, everywhere regularity for integer stationary varifolds does not hold without additional assumptions, not even in low dimension: one can consider for instance the union of two intersecting lines in the plane, or the union of three half-lines emanating from a point, with an angle $\frac{2\pi}{3}$ between any two of them.

So far, the regularity theory of integer stationary varifolds is still very incomplete and well understood only in special situations. An example is the structure theorem by Allard–Almgren in the one-dimensional case [5], which says that such varifolds are locally a finite graph of geodesic curves with multiplicity, obeying a natural balancing condition at each node.

A very important result is Allard's regularity theorem [3, Section 8], which roughly says that regularity holds near points where the density does not jump to a higher value compared to neighboring points; it can be seen as a nontrivial modification of De Giorgi's regularity theory for sets which locally minimize the perimeter. A consequence is the almost everywhere regularity for varifolds with multiplicity one, and the regularity on a dense open subset of the support in general. The latter is still the best known result without additional assumptions.

Pitts was able to obtain a satisfactory theory for varifolds arising via his min-max framework in codimension one, reaching in particular the full regularity for ambient dimension $m < 8$—together with subsequent work by Schoen and Simon—by introducing the stronger concept of *almost minimizing* varifold [87, Chapter 3]. Loosely speaking, the definition requires that one cannot locally deform the varifold in order to decrease the mass, unless the mass reaches a higher value at some time during the deformation—actually the definition is given in a discretized fashion and needs to replace the varifold with a current which is close to it in the weak topology.

For the stable, codimension one case, another important result, which completed Pitt's work, is the Schoen–Simon regularity theorem [96] under the assumption that the singular set has locally finite $\mathcal{H}^{n-2}$-measure. This was recently reduced to an optimal assumption in a monumental work by Wickramasekera [109]—used in the regularity theory for the Allen–Cahn approach—which essentially shows that the only onstruction to a very small (codimension 8 in the ambient) singular set is given by the presence of classical singularities. These are a generalization of the trivial examples given above for the plane, namely they consist of smooth hypersurfaces meeting along a common boundary.

As for the special class of parametrized stationary varifolds considered in [91, 84], we defer a discussion of their regularity to the next sections of this introduction.

1.2 Results from this thesis

In the following subsections, except for the first one, we will briefly describe the main results contained in this dissertation. Some hints about the techniques will be given in the next section.

A viscous relaxation of the area functional

A new relaxation of the area was studied by Rivière [91]. As already mentioned, the corresponding min-max framework can produce immersed minimal surfaces ($k = 2$) without a priori restrictions on the genus, on the codimension $m - 2$ or on the number of parameters in the min-max. Specifically, for a fixed $\sigma > 0$, choosing (e.g.) $p = 2$ one first finds an immersion $\Phi : \Sigma \to \mathcal{M}^m$ which is critical for the perturbation

$$E'_\sigma(\Phi) = \operatorname{area}(\Phi) + \sigma^2 \int_\Sigma (1 + |\mathbb{I}^\Phi|^2)^2 \operatorname{vol}_\Phi$$

of the area functional, where Σ is a fixed closed oriented surface. This functional enjoys a sort of Palais–Smale condition up to diffeomorphisms.

Considering any sequence $\sigma_j \downarrow 0$, one gets a sequence $\Phi_j : \Sigma_j \to M$ of conformal immersions, where Σ_j denotes Σ endowed with the conformal structure induced by Φ_j.

Assuming for simplicity that we are dealing with a constant conformal structure, the sequence Φ_j is then bounded in $W^{1,2}$ and we can consider its weak limit Φ_∞, up to subsequences.

At this stage of the theory, it is still not clear whether the strong $W^{1,2}$-convergence holds, even away from a finite bubbling set. However, in [91] it is shown that, if the sequences σ_j and Φ_j are carefully chosen so as to satisfy a certain *entropy condition*, then the immersions Φ_j converge to a *parametrized stationary varifold*. More precisely, the following holds.

Theorem. *Let (Φ_j) be a sequence of immersions, with $\Phi_j : \Sigma \to \mathcal{M}$ critical for E'_{σ_j} and $\sigma_j \to 0$. Assume that $\int_\Sigma \log(\sigma_j^{-1})\sigma_j^2(1 + |\mathbb{I}^{\Phi_j}|^2)^2 \operatorname{vol}_{\Phi_j} \to 0$ and that the area of Φ_j is bounded by a constant. Then, up to subsequences, the varifolds in $\mathcal{M}$ induced by Φ_j converge to a parametrized stationary varifold.*

This last notion is defined in a later subsection.

The main difficulty is the absence of a small-energy-regularity uniform in σ, as opposed to [95]. This is already true for a similar functional on curves: see [78], where explicit examples are shown.

As in that paper, the entropy condition $\log(\sigma_j^{-1})\sigma_j^2 \int_\Sigma (1 + |\mathbb{I}^{\Phi_j}|^2)^2 \operatorname{vol}_{\Phi_j} \to 0$ provides the extra information needed to obtain a satisfactory limit object. This condition can be ensured by means of a very general device which applies to certain relaxed functionals, due to Struwe.

The most important intermediate step in the proof of the theorem consists in establishing a lower bound for $\frac{\mu_j(B_r(p))}{r^2}$ for suitable ambient balls $B_r(p) \subset \mathcal{M}$, independently of σ, with μ_j denoting the area measure of Φ_j on $\mathcal{M}$. While the convergence $\sigma_j^2 \int_\Sigma (1 + |\mathbb{I}^{\Phi_j}|^2)^2 \operatorname{vol}_{\Phi_j} \to 0$ is enough to have a stationary limit, the stronger entropy condition is fully exploited in the proof of this lower bound.

A modification for the free boundary version

In the next chapter we study instead a similar energy for surfaces with boundary; namely, replacing σ with σ^2 for conveniency, we work with the energies

$$E_\sigma(\Phi) := \operatorname{area}(\Phi) + \sigma \operatorname{length}(\Phi|_{\partial\Sigma}) + \sigma^4 \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi,$$

where Σ is a fixed compact surface with (possibly nonempty) boundary and $\Phi : \Sigma \to \mathcal{M}^m$ is a smooth immersion with the constraint $\Phi(\partial\Sigma) \subseteq \mathcal{N}$, for a given closed submanifold $\mathcal{N}^n \subset \mathcal{M}$. The parameter σ should be thought dimensionally as a length. The length term is added in order to have the aforementioned lower bound for the area also in this case.

The treatment will be self-contained and, along the way, we will simplify many arguments from the original paper [91]. The main result that we get is similar to the closed case.

Theorem. *Given a sequence Φ_j of immersions which are σ_j^5-critical for E_{σ_j}, have bounded area and satisfy the condition*

$$\sigma_j^4 \log \sigma_j^{-1} \int_\Sigma |\mathbb{I}^{\Phi_j}|^4 \operatorname{vol}_{\Phi_j} + \sigma_j \log \sigma_j^{-1} \operatorname{length}(\Phi_j|_{\partial\Sigma}) \to 0,$$

there exists a subsequence such that the induced varifolds converge to a parametrized free boundary stationary varifold *for the couple $(\mathcal{M}, \mathcal{N})$. Moreover, the connected components Σ_i of its domain have $\chi(\Sigma_i) \geq \chi(\Sigma)$ and $g(\Sigma_i) \leq g(\Sigma)$.*

In this statement $\chi(\cdot)$ is the Euler characteristic and $g(\cdot)$ is the genus. For an immersion Φ, the assertion that Φ is τ-critical for E_σ means that $|dE_\sigma(\Phi)[w]| \leq \tau \|w\|_\Phi$ for all infinitesimal variations w, with respect to a suitable Finsler structure on the space of $W^{2,4}$ immersions $\Phi : \Sigma \to \mathcal{M}$ satisfying $\Phi(\partial\Sigma) \subseteq \mathcal{N}$.

The simplifications in the presentation given in the next chapter show more generally that, for the energy

$$k\text{-area}(\Phi) + \sigma^p \int_\Sigma |\mathbb{I}^\Phi|^p \operatorname{vol}_\Phi$$

on immersions $\Phi : \Sigma^k \to \mathcal{M}^m$, with Σ^k a closed k-manifold and $p > k$, the stationarity of the limit varifold holds regardless of the domain dimension k. Again, one has to assume an almost criticality for the maps Φ_j, as well as $\sigma^p \int_\Sigma |\mathbb{I}^\Phi|^p \operatorname{vol}_\Phi \to 0$.

Parametrized stationary varifolds and their regularity

Parametrized stationary varifolds, introduced in [91, 84], are two-dimensional varifolds admitting a *parametrization* in the following sense: given a Riemann surface Σ, they are induced by a weakly conformal map $\Phi \in W^{1,2}(\Sigma, \mathcal{M})$, together with a multiplicity function $N \in L^\infty(\Sigma, \mathbb{N} \setminus \{0\})$ *on the domain.*

They are required to satisfy a natural stationarity property: namely, we assume that, for *almost all* domains $\omega \subseteq \Sigma$, the varifold induced by the map $\Phi\big|_\omega$ with the multiplicity function $N\big|_\omega$ is stationary in the complement of the compact set $\Phi(\partial\omega)$.

In the free boundary case, we require that Φ maps $\partial\Sigma$ to $\mathcal{N}$ and that the above holds for a.e. domain $\omega \subset\subset \Sigma \setminus \partial\Sigma$. We also require that, for a.e. $\omega \subseteq \Sigma$, the induced varifold is free boundary stationary outside $\Phi(\partial\omega)$: this means that we can test the stationarity against vector fields tangent to $\mathcal{N}$ and supported outside $\Phi(\partial\omega)$. Note that $\partial\omega = \overline{\omega} \setminus \omega$ is the topological boundary of ω in Σ and does not include $\omega \cap \partial\Sigma$.

As already discussed, everywhere regularity for general integer stationary varifolds fails without additional assumptions, even in low dimension. In the present situation, regularity stems from a subtle interaction between stationarity and the topological information of being parametrized. The possibility of localizing the stationarity in the domain rules out automatically all classical singularities.

This localization property, for varifolds arising from the min-max framework, comes from the fact that we can choose $X(\Phi_j)\mathbf{1}_\omega$ as an infinitesimal variation for the (almost) critical map Φ_j, with X a vector field on $\mathcal{M}$ vanishing near $\Phi_j(\partial\omega)$ and tangent to $\mathcal{N}$ if ω intersects $\partial\Sigma$, in the free boundary case.

The following optimal regularity result from [84] will be presented in the third chapter.

Theorem. *The triple* (Σ, Φ, N) *is a parametrized stationary varifold in* $\mathcal{M}$ *if and only if* Φ *is a smooth, weakly conformal harmonic map and* N *is a.e. constant. In this case,* Φ *is a minimal branched immersion.*

In the statement we implicitly assume that Σ is connected and Φ is not (a.e.) constant. As discussed in the second chapter, there is a local version of this theorem which implies the regularity also in the free boundary case.

Theorem. *The triple* (Σ, Φ, N) *is a parametrized free boundary stationary varifold, for the couple* $(\mathcal{M}, \mathcal{N})$*, if and only if* Φ *is a smooth, weakly conformal harmonic map with* $\partial_\nu\Phi \perp T\mathcal{N}$ *along* $\partial\Sigma$ *and* N *is a.e. constant. In this case,* Φ *is a minimal branched immersion outside* $\partial\Sigma$.

A simple corollary is, for instance, the following.

Corollary. *Given any collection* $\mathcal{F}$ *of compact subsets of the space of smooth immersions* $(\Sigma, \partial\Sigma) \to (\mathcal{M}, \mathcal{N})$*, assuming* $\mathcal{F}$ *to be stable for isotopies of this space, the min-max value*

$$\beta := \inf_{A \in \mathcal{F}} \max_{\Phi \in A} \operatorname{area}(\Phi)$$

is the sum of the areas of finitely many free boundary minimal (branched) immersions $\Phi_{(i)} : \Sigma_{(i)} \to \mathcal{M}$*, whose domains are connected and have* $\chi(\Sigma_{(i)}) \geq \chi(\Sigma)$ *and* $g(\Sigma_{(i)}) \leq g(\Sigma)$.

Note that other min-max situations can be dealt with in the same way.

Multiplicity one

The result in [84], which is optimal for the class of parametrized stationary varifolds, left nonetheless open the question whether one can have $N > 1$ on some connected component of the domain. This question should be compared with the *multiplicity one conjecture* by Marques and Neves. Roughly speaking, it asks whether a minimal hypersurface Σ^{m-1} obtained from some min-max method should always have multiplicity one, at least for generic metrics.

Marques and Neves were able to prove this conjecture in the Almgren–Pitts theory for one-parameter sweepouts [75]. It was also recently established by Chodosh and Mantoulidis for bumpy metrics in 3-manifolds [22], in the setting of the Allen–Cahn level set approach, and by Zhou for hypersurfaces in any dimension $m < 8$, again for Baire-generic metrics, in the Almgren–Pitts setting [114].

The importance of this conjecture in relation to the Morse index of Σ is twofold. First of all, there is no satisfactory definition of Morse index for an embedded minimal hypersurface with multiplicity bigger than one: such an object could be thought as the limit of many qualitatively different sequences of multiplicity one hypersurfaces. Also, if one can establish a lower bound on the Morse index like

$$p \leq \sum_i n_i\big(\operatorname{index}(\Sigma_i) + \operatorname{nullity}(\Sigma_i)\big), \qquad \Sigma = \bigsqcup_i n_i \Sigma_i,$$

p being the number of (essential) parameters in the min-max, then the multiplicity one conjecture gives $n_i = 1$ and, hence, there are infinitely many *geometrically distinct* minimal hypersurfaces, provided there exists at least one for every value of p.

In the context of the viscosity approach, although Φ could still be a multiple cover of the image, a crucial advantage of having a parametrization at our disposal is that we have a good definition of Morse index and nullity, *provided* $N \equiv 1$.

In [85], which corresponds to the fourth chapter of this dissertation, the natural counterpart of the multiplicity one conjecture in the viscosity approach is established; namely we have the following result, in arbitrary codimension and without any genericity assumption.

Theorem. *We have* $N \equiv 1$.

Corollary. *If there is no bubbling or degeneration of the conformal structure induced by* Φ_j*, we have a strong* $W^{1,2}$*-convergence* $\Phi_j \to \Phi_\infty = \Phi$*. In general we have a* bubble-tree convergence $\Phi_j \to \Phi$*.*

The last corollary paves the way to obtain meaningful Morse index bounds. Using the results from [77] and [93], one can reach the following conclusion in the closed case.

Corollary. *Given a family* $\mathcal{F}$ *as above, the limit (possibly disconnected and branched) minimal immersed surface* $\Phi : S \to M$ *satisfies*

(i) $\beta = \operatorname{area}(\Phi)$,

(ii) $\operatorname{genus}(S) \leq \operatorname{genus}(\Sigma)$,

(iii) $\operatorname{index}(\Phi) \leq p$*, the number of min-max parameters.*

Codimension two minimal submanifolds from Yang–Mills–Higgs

Starting from the work of De Giorgi, Modica–Mortola and Sternberg for minimizers, a "level set" method to construct minimal hypersurfaces has been recently proposed, based on the rescalings of the Allen–Cahn functional

$$F_\varepsilon(v) := \int_{\mathcal{M}} \Big(\varepsilon|dv|^2 + \frac{1}{4\varepsilon}(1-v^2)^2\Big),$$

whose minimizers model a phase transition concentrating on a minimal interface of codimension one, as $\varepsilon \to 0$.

In their pioneering work, Hutchinson–Tonegawa [55] studied families of critical points v_ε of F_ε with bounded energy and showed, in particular, that their energy measures concentrate along a stationary, integer rectifiable $(m-1)$-varifold, whose support is the limit of the level sets $v_\varepsilon^{-1}(0)$.

These developments, together with the deep regularity work by Tonegawa–Wickramasekera [108] on stable solutions and subsequent work by Guaraco [46] and Gaspar–Guaraco [41], provided a PDE alternative to the Almgren–Pitts method. This new framework has already been used successfully to attack some profound questions concerning the structure of min-max minimal hypersurfaces.

It is natural to ask for similar theories in higher codimension, e.g. when $k = m-2$, based again on PDE methods. Attempts in this direction have been made by Cheng [21] and Stern [102], via the study of the Ginzburg–Landau energies

$$F_\varepsilon(v) := \frac{1}{|\log\varepsilon|}\int_{\mathcal{M}}\Big(|dv|^2 + \frac{1}{4\varepsilon^2}(1-|v|^2)^2\Big)$$

for complex-valued maps $v : \mathcal{M} \to \mathbb{C}$. While the Ginzburg–Landau approach can be employed successfully to produce nontrivial stationary *rectifiable* $(m-2)$-varifolds, based also on works by Lin–Rivière [71] and Bethuel–Brezis–Orlandi [15], it is not yet known whether the varifolds produced in this way are *integral*, nor is it known whether the full energies $F_\varepsilon(v_\varepsilon)$ of the min-max critical points converge to the mass of the limiting stationary varifold in the case $b_1(\mathcal{M}) \neq 0$, with $b_1(\cdot)$ denoting the first Betti number.

These difficulties point to the deeper fact that the Ginzburg–Landau functionals, though related to the $(m-2)$-area, do *not* provide a straightforward regularization of the latter. Indeed, they should be viewed mostly as a relaxation of the Dirichlet energy for singular maps to S^1 (away from singularities).

In [86], whose content is presented in the last chapter of the dissertation, we consider instead the Yang–Mills–Higgs energy

$$E(u,\nabla) := \int_{\mathcal{M}}\Big(|\nabla u|^2 + |F_\nabla|^2 + W(u)\Big)$$

and its rescalings

$$E_\varepsilon(u,\nabla) := \int_{\mathcal{M}}\Big(|\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \varepsilon^{-2}W(u)\Big),$$

for couples (u,∇) consisting of a section u of a given Hermitian line bundle $L \to \mathcal{M}$, and a metric connection ∇ on L. Here, the potential $W : L \to \mathbb{R}$ is given by

$$W(u) := \frac{1}{4}(1-|u|^2)^2,$$

while $F_\nabla \in \Omega^2(M,\mathfrak{u}(L)) \cong \Omega^2(M,\mathbb{R})$ denotes the curvature of ∇. These functionals have a natural $U(1)$ *gauge invariance*.

Taubes [105, 106] studied critical points (with $\varepsilon = 1$) for the trivial bundle $L = \mathbb{C}\times\mathbb{R}^2$ on the plane: he gave a complete classification, showing in particular that all finite-energy critical points (u,∇) solve the first order system

$$\begin{cases} \nabla_{\partial_1}u \pm i\nabla_{\partial_2}u = 0 \\ *F_\nabla = \pm\frac{1}{2}(1-|u|^2), \end{cases}$$

known as the *vortex equations.* Such solutions minimize the energy among pairs (u, ∇) with fixed vortex number $N := \frac{1}{2\pi}\int_{\mathbb{R}^2} F_\nabla \in \mathbb{Z}$, and carry energy exactly $2\pi|N|$.

In [86] we develop the asymptotic analysis as $\varepsilon \to 0$ for critical points of E_ε associated to an arbitrary line bundle $L \to \mathcal{M}$. The main result is the following, which describes the limiting behavior of the energy measures and curvatures F_{∇_ε}, for critical points $(u_\varepsilon, \nabla_\varepsilon)$ satisfying a uniform energy bound.

Theorem. *Let $L \to \mathcal{M}$ be a Hermitian line bundle over a closed, oriented Riemannian manifold $(\mathcal{M}^m, g)$ of dimension $m \geq 2$, and let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of critical pairs for E_ε with bounded energy. Then, as $\varepsilon \to 0$, the energy measures*

$$\mu_\varepsilon := \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\ \mathrm{vol}_g$$

converge (subsequentially) to the weight measure μ of a stationary, integer rectifiable $(m-2)$-varifold V. Also, for all $0 \leq \delta < 1$, $\mathrm{spt}(\mu) = \lim_{\varepsilon\to 0} \{|u_\varepsilon| \leq \delta\}$ in the Hausdorff topology.

Theorem. *The $(m-2)$-currents dual to the curvature forms $\frac{1}{2\pi}F_{\nabla_\varepsilon}$ converge (subsequentially) to an integral $(m-2)$-cycle Γ, with $|\Gamma| \leq \mu$.*

Roughly speaking, the first result says that the energy of the critical points concentrates near the zero sets $u_\varepsilon^{-1}(0)$ of u_ε as $\varepsilon \to 0$, which converge to a (possibly rather singular) minimal submanifold of codimension two.

Note that unit sections of a Hermitian line bundle are indistinguishable up to change of gauge: for a given unit section u of L, one can always choose a connection with respect to which u appears constant. Thus, while most of the energy of solutions v_ε to the complex Ginzburg–Landau equations falls on annular regions, relatively far from the zero set, where v_ε resembles a harmonic S^1-valued map, the energy $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)$ of a critical pair $(u_\varepsilon, \nabla_\varepsilon)$ for Yang–Mills–Higgs instead concentrates near the zero set $u_\varepsilon^{-1}(0)$. The integrand $|\nabla_\varepsilon u_\varepsilon|^2$ has exponential decay outside this region, allowing for a more effective blow-up analysis.

The advantages of this theorem over analogous results for the complex Ginzburg–Landau equations are the *integrality* of the limit varifold V—due ultimately to the aforementioned quantization of the energy of entire planar solutions—and the concentration of the *full energy measure* to V, independent of the topology of $\mathcal{M}$. Also, the analysis of this functional aligns much more closely with the work of Hutchinson–Tonegawa on the Allen–Cahn equations.

We also have the following general existence result, showing that nontrivial families satisfying the hypotheses of our main theorem arise naturally, from min-max constructions, on any line bundle (including, importantly, the trivial bundle) over any Riemannian manifold $\mathcal{M}$.

Theorem. *For any Hermitian line bundle $L \to \mathcal{M}$, there exists a family of critical pairs $(u_\varepsilon, \nabla_\varepsilon)$ with bounded energies $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon)$ and nonempty zero sets $u_\varepsilon^{-1}(0) \neq \emptyset$. In particular,* *the energy of these pairs concentrates (subsequentially) on a nontrivial stationary integral $(m-2)$-varifold V as $\varepsilon \to 0$.*

While in [86], for the case of a trivial bundle, we consider only one min-max construction, we mention that many more may be carried out in principle, due to the rich topology of the space

$$\mathfrak{M} := \{(u, \nabla) : 0 \not\equiv u \in \Gamma(\mathbb{C} \times \mathcal{M}),\ \nabla \text{ a Hermitian connection}\}/\mathcal{G},$$

where $\mathcal{G}$ is the gauge group of maps $\mathcal{M} \to S^1$.

As an application of our results, we obtain a PDE proof of this fact, first proved by Almgren (in any codimension) using his geometric measure theory framework.

Corollary. *Any Riemannian manifold of dimension $m \geq 2$ contains a stationary integral $(m-2)$-varifold.*

1.3 A glimpse of the techniques

Variational theory for the viscous relaxation of the area

As already mentioned earlier, the main difficulty is to prove a lower bound for the area of an almost critical immersed surface Φ in suitable balls $B_r(p)$ in the ambient, with p in the image of Φ. This is accomplished by studying how the ratio $\frac{\mu(B_s(p))}{s^2}$ behaves as s varies, with μ denoting the area measure of Φ on $\mathcal{M}$. While for $s < \sigma$ the boundedness of the quantity $\sigma^4 \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi$ is enough—in that, heuristically, magnifying by a factor s^{-1} we get an L^4-bound on the second fundamental form and we can apply directly the monotonicity formula—for $s > \sigma$ we have to use the almost criticality of Φ.

Namely, we use the same vector fields used to show the (approximate) monotonicity of $\frac{\mu(B_s(p))}{s^2}$ for free boundary minimal surfaces, in order to understand the growth rate of this ratio for our immersed surface. Oversimplifying, in the closed case the quantity $\frac{\sigma^4}{s} \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi$ appears among the error terms: since this has to be integrated between σ and r, this produces an error $\sigma^4 \log(\sigma^{-1}) \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi$, which is infinitesimal by hypothesis. In reality, the argument also requires a *maximal* bound

$$\sigma^4 \int_{\Phi^{-1}(B_s(p))} |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi \leq \delta \mu(B_s(p)) \quad \text{for all } s > 0.$$

We add the additional term $\sigma \operatorname{length}(\Phi|_{\partial\Sigma})$ in E_σ in order to deal with the additional challenge of having a nontrivial boundary $\Phi|_{\partial\Sigma}$. Due to this, we cannot use the monotonicity formula on a ball $B_s(p)$ (with $s < \sigma$) whose preimage intersects $\partial\Sigma$. In principle, one can impose a strong control of the boundary by adding a term involving the geodesic curvature of $\Phi|_{\partial\Sigma}$; however, this would still require to understand the topology of $\Phi^{-1}(B_s(p))$.

Rather, using a covering argument, we show that the set of points with distance less than σ from $\Phi(\partial\Sigma)$ has an area (i.e., the measure μ) controlled by $\sigma \operatorname{length}(\Phi|_{\partial\Sigma})$; this quantity is again infinitesimal as $\sigma \to 0$, so that this set can be ignored in the asymptotic analysis.

The next steps consist in the study of the area measures induced by Φ_j on the domain Σ. Assuming the maps to be conformal for a fixed conformal structure on Σ, one gets a weak limit Φ_∞ in $W^{1,2}$. By means of slicing arguments, we show that the limit measure can only have finitely many atoms, with a lower bound for their mass, and is absolutely continuous elsewhere. This is due to the fact that, for a domain ball B whose image of the boundary $\Phi(\partial B)$ has small diameter, either $\Phi|_B$ is close to a nontrivial stationary varifold, whose density is bounded below by virtue of the aforementioned lower bound, or its area is (eventually) bounded by the square of the diameter of $\Phi(\partial B)$. The limit map Φ_∞ is continuous away from the atoms.

The parametrized structure of the varifold is obtained with similar arguments; in order to show that the multiplicity function N is integer valued, we use a blow-up argument together with Allard's strong constancy lemma.

Finally, the lower bound on the mass of the concentration points allows to carry out a standard *bubble tree* analysis. The treatment of the situation where the induced conformal structure degenerates, as $j \to \infty$, is similar to the study of the concentration points.

Regularity of parametrized stationary varifolds

In order to study parametrized stationary varifolds, we first observe that the parametrization Φ is always continuous and the multiplicity function N admits an upper semicontinuous representative, although the latter could a priori fail to be everywhere an integer. Assuming for simplicity that N is integer valued, the strategy is then to prove the regularity locally, by induction on the maximum value of N. The regularity follows whenever N is a.e. constant, as was previously shown in [92].

It is crucial to study first the codimension zero case. If Φ takes values into $\mathbb{C}$ then a topological proof, together with induction, shows that N is a.e. constant. The topological ingredient is the fact that the domain cannot contain more than countably many disjoint *triods*, with a triod consisting of a connected compact set together with three regular curves emanating from it. Similarly we can show that, if Φ takes values into finitely many planes, then its image is contained in one of them and the map is holomorphic.

Following arguments similar to the ones used for the existence theory, we then show that one can form a *parametrized blow-up*, namely a parametrized varifold which is contained in the standard varifold blow-up, at certain points in the domain where the Dirichlet energy does not decay too fast. The blow-up is included in a polyhedral cone; hence, by the previous analysis, its parametrization is a holomorphic map. The image of the complement of these good points has Hausdorff dimension zero.

One would like to perform a blow-up at the boundary of the closed set where N attains the maximum and reach a contradiction. Apart from the fact that N is not really quantized, a serious problem is that a priori the set of good points is just a Borel set. However, the map parametrizing a blow-up has a controlled order of vanishing at the origin, in terms of the density of the varifold. This observation, by means of compactness arguments, gives some kind of openness for the set of good points, together with a control of the decay rate of the Dirichlet energy, allowing to conclude.

Multiplicity one

In order to show that $N = 1$ for parametrized varifolds arising from the min-max framework, the main idea is to define a sort of *macroscopic multiplicity*, on balls $B_\ell^Q(p)$ in a Euclidean space $\mathbb{R}^Q \supset \mathcal{M}^m$, *before* passing to the limit.

This macroscopic multiplicity is roughly the closest integer to the average of a *projected multiplicity*, issued by the map $\Pi \circ \Phi_j\big|_B$, where B is a small domain ball and Π is (the projection onto) a 2-plane close to the image of $\Phi_j\big|_{\partial B}$.

Then we use a continuity argument to show that this number stays constant as we pass from scale 1 to scale σ_j. At the latter scale we have a very clear understanding of the behavior of Φ_j, and in particular we are able to say that here the macroscopic multiplicity equals 1. Thus, the same holds at the original scale, and this is sufficient to get $N = 1$.

The comparison of two consecutive scales, as well as the fact that the projected multiplicity is well defined, are obtained through several compactness arguments, exploiting the fact that Φ_j resembles a parametrized stationary varifold for scales much smaller than 1 and much bigger than σ_j.

Asymptotic analysis for Yang–Mills–Higgs

A key ingredient is the improvement of the obvious $(m-4)$-monotonicity for the energy E_ε, which follows just from the inner variation formula and is a priori forced by the Yang–Mills term, to a sharp $(m-2)$-monotonicity. Namely, we want to show that the energy on a ball $B_r(p)$, normalized dividing by r^{m-2}, is (approximately) increasing in r.

This situation is in fact similar to the one for Allen–Cahn, where one wants to upgrade the trivial $(m-2)$-monotonicity, forced by the Dirichlet term and typical of Ginzburg–Landau energies for vector valued maps, to a sharp $(m-1)$-monotonicity.

We accomplish this by applying the Bochner identity for differential forms, deriving a partial differential inequality for the *discrepancy* $\varepsilon|F_\nabla| - \frac{1-|u|^2}{2\varepsilon}$. Under certain curvature assumptions on $\mathcal{M}$, we deduce immediately that $\varepsilon|F_\nabla| \le \frac{1-|u|^2}{2\varepsilon}$. With a sort of bootstrap, we can reach a pointwise upper bound for the discrepancy also in the general case. This estimate gives a natural balancing in the inner variation formula, from which the $(m-2)$-monotonicity follows.

The rectifiability of the limiting (generalized) varifold then follows from a rectifiability criterion by Ambrosio–Soner [10]. Integrality is proved by means of a blow-up, reducing to a sequence of critical pairs which are asymptotically invariant in all directions orthogonal to a plane; this can be done by exploiting the monotonicity formula, as in the work [69] for harmonic maps between manifolds. Here this step becomes slightly more complicated due to gauge invariance.

In order to show the existence of critical couples satisfying the assumptions of the main theorem, on any line bundle, we proceed as follows. For nontrivial bundles we show that the minimizers $(u_\varepsilon, \nabla_\varepsilon)$ of E_ε satisfy uniform energy bounds as $\varepsilon \to 0$. We also observe that, for a critical couple $(u_\varepsilon, \nabla_\varepsilon)$, if u_ε vanishes somewhere then the energy satisfies a lower bound independent of ε. For the trivial bundle, similarly to [102], we use instead a min-max over maps from the closed disk to the set of couples (u, ∇), with $u \equiv e^{i\theta}$ and $\nabla = d$ at the boundary point $e^{i\theta}$. We use the fact that each energy E_ε satisfies a Palais–Smale property up to change of gauge, for couples in a certain functional space; this is most conveniently proved using Coulomb gauges.

1.4 Open problems

We conclude the introduction with a list of interesting open questions immediately related to the previous results.

Concerning the viscous relaxation of the area, for a min-max with p parameters, the natural expected inequalities relating p with the *Morse index* and *nullity* of the resulting immersion would be

$$\mathrm{index}(\Phi) \leq p \leq \mathrm{index}(\Phi) + \mathrm{nullity}(\Phi),$$

where p is the "essential" number of parameters in the min-max, from the point of view of algebraic topology. A more tractable version of this question could be to show the same bounds if

$$\mathrm{index}(\Phi_j) \leq p \leq \mathrm{index}(\Phi_j) + \mathrm{nullity}(\Phi_j),$$

for a sequence of critical maps as in the existence part of the viscosity framework. The upper bound, however, seems to still require a more refined understanding of the convergence of Φ_j to the limit.

One can also ask if the number of *branch points* of the minimal immersion can be bounded in terms of the complexity of the min-max. It is not clear if there is a criterion to avoid branch points completely in codimension one, i.e. in 3-manifolds.

Finally, the natural question arises whether anything similar can be done for bigger domain dimension ($k > 2$). As we said, one still gets a stationary varifold in the limit. However, retaining a parametrization for the limit object does not look possible, due to the absence of a simple finite dimensional space of possible conformal structures (up to diffeomorphisms). It would be interesting to study adaptations of this method to manifolds with additional structure, such as special holonomy, restricting the attention to particular kinds of immersions. It is conceivable that a similar analysis may work in such situations.

As for the rescalings of the Yang–Mills–Higgs functional, a first problem left open in our analysis is to understand whether the limit cycle Γ is area-minimizing in its homology class and its weight agrees with the one of the limit varifold, for a sequence of minimizers $(u_\varepsilon, \nabla_\varepsilon)$. More generally, it would be interesting to develop a Γ-*convergence* theory for these energies, by mimicking e.g. [15].

Concerning the regularity theory, one may ask whether *stability* of $(u_\varepsilon, \nabla_\varepsilon)$ has any implication on the regularity for the limiting varifold V. One could also replace stability with an appropriate notion of *almost minimality*.

In three dimensions V is necessarily a geodesic network. It is thus natural to wonder what kind of conical singularities can arise in $\mathbb{R}^3$ from a min-max.

Positive results on the above open questions would give an improvement over the Almgren–Pitts theory, which is a purely geometric measure theory setting with no partial differential equation immediately available.

It is natural to ask for analogous theories in higher codimension. The fact that our approach, compared to Ginzburg–Landau, looks more appropriate is certainly connected to the fact that the configuration space $\mathfrak{M}$ introduced earlier resembles more closely the space of $(m-2)$-submanifolds: it is also interesting to remark that the homotopy groups $\pi_i(\mathfrak{M})$ are isomorphic to those of the space $\mathcal{Z}_{m-2}(\mathcal{M};\mathbb{Z})$ of integral $(m-2)$-cycles considered by Almgren. Finding other situations, possibly in ambients with special holonomy, where entire critical points solve a *first order* system like the vortex equations could be an initial hint that a similar analysis may carry over.

2 A viscous relaxation of the area for immersed surfaces, closed or with boundary

2.1 Introduction

Outline of the main results

In this chapter we study the energy

$$E_\sigma(\Phi) := \operatorname{area}(\Phi) + \sigma \operatorname{length}(\Phi|_{\partial\Sigma}) + \sigma^4 \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi,$$

where Σ is a fixed compact surface with (possibly nonempty) boundary, and $\Phi : \Sigma \to \mathcal{M}^m$ is a smooth immersion with the constraint $\Phi(\partial\Sigma) \subseteq \mathcal{N}$. The parameter σ should be thought dimensionally as a length. This energy is a modification of the one introduced in [91], as already mentioned in the first chapter.

We will fully exploit the invariance of E_σ under diffeomorphisms, namely the principle that every diffeomorphism invariant quantity should depend only on the shape of the immersed surface. In computing the first variation we will see that, using infinitesimal variations of the form $w = X(\Phi)$, all second-order terms involving w are expressible just in terms of the second fundamental form of Φ, as expected. A natural consequence of this is that the first variation of the relaxing terms $\sigma \operatorname{length}(\Phi|_{\partial\Sigma})$ and $\sigma^4 \int_\Sigma |\mathbb{I}^\Phi|^4 \operatorname{vol}_\Phi$, for such special ambient deformations, can be bounded in terms of these quantities themselves (and the ambient vector field X).

Also, working on a Finsler manifold $\mathfrak{M}$ of $W^{2,4}$ immersions, equipped with a norm on $T_\Phi\mathfrak{M}$ involving the induced metric $g_\Phi := \Phi^* g$, we observe that also $\|X(\Phi)\|_\Phi$ is bounded in terms of $E_\sigma(\Phi)$, X and σ. Since in the asymptotic analysis we will use only this particular kind of variations, we do not need to construct critical points of E_σ: it suffices to have $\|dE_\sigma(\Phi)\|_\Phi$ very small in terms of σ. Since such almost critical maps are easy to construct using pseudo-gradient flows and can be assumed, without loss of generality, to be smooth, this makes the chapter self-contained—except for the regularity theory in Section 2.7 which uses results from the following chapter.

These observations, detailed in Sections 2.2 and 2.3, represent a major simplification over the original work [91], which appeals to [14] for the Palais–Smale property of E'_σ and the regularity of critical points. The formulas obtained here are quite simple, independently of the ambient: differently from [91]—where $\mathcal{M}$ is assumed to be the round sphere S^3 in order to simplify the presentation—we can deal immediately with general closed manifolds $\mathcal{M}$ and $\mathcal{N}$.

Then, as in the closed case, the main difficulty is to prove a lower bound for the area of the immersed surface Φ in suitable balls $B_r(p)$ in the ambient; a brief discussion of the technique was given in Section 1.3.

The rest of the chapter adapts the remaining arguments from [91] and [92] to the free boundary case—again with some important simplifications. In Section 2.6 we study carefully what happens when the conformal structure induced by Φ degenerates as $\sigma \to 0$, which is more delicate and less well known for surfaces with boundary.

The following is the main result of this part of the dissertation.

Theorem 2.1.1. *Let $(\mathcal{M}^m, g)$ be a closed Riemannian manifold, $\mathcal{N}^n \subset \mathcal{M}$ a closed embedded submanifold (with $1 \le n < m$), and let Σ be a compact surface, possibly with boundary. Given a sequence Φ_k of immersions which are σ_k^5-critical for E_{σ_k}, have bounded area and satisfy the condition*

$$\sigma_k^4 \log \sigma_k^{-1} \int_\Sigma |\mathbb{I}^{\Phi_k}|^4 \operatorname{vol}_{\Phi_k} + \sigma_k \log \sigma_k^{-1} \operatorname{length}(\Phi_k|_{\partial\Sigma}) \to 0,$$

there exists a subsequence such that the induced varifolds converge to a parametrized free boundary stationary varifold *for the couple $(\mathcal{M}, \mathcal{N})$. Moreover, the connected components Σ_i of its domain have $\chi(\Sigma_i) \ge \chi(\Sigma)$ and $g(\Sigma_i) \le g(\Sigma)$. If Σ is closed, then the components Σ_i are closed, as well.*

In this statement $\chi(\cdot)$ is the Euler characteristic and $g(\cdot)$ is the genus. The last part of the statement follows from the analysis carried out in Section 2.6. We refer to Definition 2.5.9 for the precise description of this notion of parametrized varifold; the fact that one can localize the stationarity with respect to the domain stems from the fact that one can use variations $w = X(\Phi)$ also just on a domain $\omega \subset \Sigma$, extending w to vanish on the complement, provided X is supported far from $\Phi(\partial\omega)$.

Remark 2.1.2. This result applies also to a compact ambient manifold $\mathcal{M}$ with boundary $\mathcal{N}$, such as the flat unit ball $\overline{B}^3$; note that the (almost) criticality should be understood formally, for infinitesimal variations w which are sections of $\Phi^*T\mathcal{M}$, with $w(\partial\Sigma) \subseteq T\mathcal{N}$. Indeed, we can smoothly extend $\mathcal{M}$ to a closed Riemannian manifold.

Remark 2.1.3. It also applies to the case $\mathcal{M} = \mathbb{R}^m$, with $\mathcal{N} \subset \mathbb{R}^m$ a closed embedded submanifold: the lower bounds obtained in Section 2.4 (see also the proof of Proposition 2.5.1) show that the varifolds induced by Φ_k form a tight sequence, and the result then follows with the same proofs.

As for the regularity of the limit, we have the following.

Theorem 2.1.4. *For a parametrized free boundary stationary varifold $(\widetilde{\Sigma}, \Phi, N)$, the map Φ is smooth up to the boundary $\partial\Sigma$, where $\partial_\nu \Phi \perp T\mathcal{N}$. Also, on the components of $\widetilde{\Sigma}$ where Φ is not (a.e.) constant, the multiplicity N is constant and Φ is a branched minimal immersion outside $\partial\Sigma$.*

Remark 2.1.5. We stress that the limit (branched) immersion Φ is free boundary minimal in the sense that it meets the constraint $\mathcal{N}$ orthogonally along $\partial\widetilde{\Sigma}$. However, there could be points x in the interior $\operatorname{int}(\Sigma) = \Sigma \setminus \partial\Sigma$ with $\Phi(x) \in \mathcal{N}$—a possibility which cannot happen, e.g., for $(\overline{B}^3, S^2)$ (on the components where Φ is not constant); unlike the main result of [68], at such points the orthogonality is not guaranteed.

A simple corollary is, for instance, the following. Note that other min-max situations can be dealt with in the same way.

Corollary 2.1.6. *Given any collection $\mathcal{F}$ of compact subsets A of the space of smooth immersions $(\Sigma, \partial\Sigma) \to (\mathcal{M}, \mathcal{N})$, assuming $\mathcal{F}$ to be stable for isotopies of this space, the min-max value*

$$\beta := \inf_{A \in \mathcal{F}} \max_{\Phi \in A} \operatorname{area}(\Phi)$$

is the sum of the areas of finitely many free boundary minimal (branched) immersions $\Phi_{(i)} : \Sigma_{(i)} \to \mathcal{M}$, whose domains are connected and have $\chi(\Sigma_{(i)}) \geq \chi(\Sigma)$ and $g(\Sigma_{(i)}) \leq g(\Sigma)$. If Σ is closed, then the domains $\Sigma_{(i)}$ are also closed.

Organization of the chapter

We conclude the introduction with a very brief description of the structure of the present chapter.

- In Section 2.2 we show how to deduce Corollary 2.1.6 from Theorem 2.1.1, by introducing a Finsler manifold of maps and checking that it satisfies the conditions guaranteeing that Struwe's monotonicity trick applies;
- in Section 2.3 we compute the first variation of E_σ for special variations $X(\Phi)$, and use the resulting formula to show that the varifolds induced by the maps Φ_k converge, up to subsequences, to a free boundary stationary varifold;
- Section 2.4 is devoted to the proof of the lower bound for the area mentioned earlier, in various forms;
- in Section 2.5 we show several structure results for the (weak) limit of the area measures that Φ_k induces on Σ and we obtain Theorem 2.1.1, under the assumption that Φ_k induces a constant conformal structure on Σ and ignoring possible concentration points for the area;

- in Section 2.6 we remove the above assumption, studying carefully how to deal with all possible situations of degeneration of the conformal structure and describing how to recover the energy arising from concentration points, thus proving Theorem 2.1.1 in general;
- finally, Section 2.7 is devoted to the regularity part, namely the proof of Theorem 2.1.4.

2.2 Almost critical points for E_σ

Let $(\mathcal{M}^m, g)$ be a closed Riemannian manifold and $\mathcal{N}^n \subset \mathcal{M}$ a closed embedded submanifold, with $1 \le n < m$. For simplicity, we will assume without loss of generality that $\mathcal{M}$ is isometrically embedded in some Euclidean space $\mathbb{R}^Q$, although the proofs could be easily modified so as to avoid the Nash embedding theorem.

Also, let Σ be a compact surface, possibly with boundary $\partial\Sigma$. We will study the following relaxation of the area functional: given an immersion $\Phi : \Sigma \to \mathcal{M}$, we let

$$\begin{aligned} E_\sigma(\Phi) &:= \operatorname{area}(\Phi) + \sigma \operatorname{length}(\Phi|_{\partial\Sigma}) + \sigma^4 \int_\Sigma |\mathrm{II}^\Phi|^4 \operatorname{vol}_\Phi \\ &= \int_\Sigma \operatorname{vol}_\Phi + \sigma \int_{\partial\Sigma} \operatorname{vol}_{\Phi|_{\partial\Sigma}} + \sigma^4 \int_\Sigma |\mathrm{II}^\Phi|^4 \operatorname{vol}_\Phi . \end{aligned} \tag{2.2.1}$$

Here vol_Φ and $\operatorname{vol}_{\Phi|_{\partial\Sigma}}$ are the (two- and one-dimensional) volume forms of the induced metric $\Phi^* g$ on Σ and $\partial\Sigma$, which we will often identify with the corresponding measures. In the last term, II^Φ denotes the second fundamental form of Φ.

In order to construct almost critical maps for E_σ, with the constraint $\Phi(\partial\Sigma) \subseteq \mathcal{N}$, we introduce the topological space

$$\mathfrak{M} := \{\Phi \in W^{2,4}(\Sigma, \mathcal{M}) : \Phi \text{ is an immersion and } \Phi(\partial\Sigma) \subseteq \mathcal{N}\},$$

with the topology induced from $W^{2,4}(\Sigma, \mathcal{M})$, in turn induced from $W^{2,4}(\Sigma, \mathbb{R}^Q)$. Recall that $W^{2,4}(\Sigma, \mathbb{R}^Q)$ embeds into $C^1(\Sigma, \mathbb{R}^Q)$, so that the definition makes sense and $\mathfrak{M}$ is canonically a Banach manifold.

For each $\Phi \in \mathfrak{M}$, the tangent space $T_\Phi\mathfrak{M}$ identifies with the Banach space of $W^{2,4}$ sections $s : \Sigma \to T\mathcal{M}$ of the pullback bundle $\Phi^* T\mathcal{M}$, with $s \in T\mathcal{N}$ along $\partial\Sigma$.

Given $\Phi \in \mathfrak{M}$, we call $g_\Phi := \Phi^* g$ the metric that Φ induces on Σ. We endow $T_\Phi\mathfrak{M}$ with the following norm: we let

$$\|s\|_\Phi := \|s\|_{L^\infty} + \|\nabla s\|_{L^\infty} + \|\nabla^2 s\|_{L^4},$$

where ∇ is the pullback connection on $\Phi^* T\mathcal{M}$ and the norms are with respect to the metrics g on $T\mathcal{M}$ and g_Φ on $T^*\Sigma$. It is straightforward to check that this choice satisfies the requirements to be a Finsler structure on $\mathfrak{M}$ (see [42, p. 54] for the definition).

Proposition 2.2.1. *The Finsler manifold $\mathfrak{M}$ is complete.*

Recall that the distance between two elements $\Phi_1, \Phi_2 \in \mathfrak{M}$ (in the same connected component) is defined to be the infimum of $\int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\,dt$, as $\gamma : [0,1] \to \mathfrak{M}$ ranges among all piecewise C^1 curves from Φ_1 to Φ_2. It is a consequence of the Finsler structure axioms that it induces the original topology on $\mathfrak{M}$.

Proof. Let $(\Phi_k)_{k\geq 0}$ be a Cauchy sequence. Up to subsequences, we can assume that $\sum_k \operatorname{dist}(\Phi_k, \Phi_{k+1}) < \infty$. Hence, by definition we can find a piecewise C^1 curve $\Phi : [0,\infty) \to \mathfrak{M}$ of finite length, with $\Phi(k) = \Phi_k$ for every $k \in \mathbb{N}$. We will use the notation Φ_t in place of $\Phi(t)$. It suffices to show that Φ_t converges in $W^{2,4}$ as $t \to \infty$. With a perturbation argument, we can assume that $\Phi_t(x)$ is smooth in the couple (x,t).

Let $w_t := \frac{d\Phi_t}{dt}$. Since w_t is bounded pointwise by the summable (in t) quantity $\|w_t\|_{\Phi_t}$, we know that Φ_t converges in C^0 to a limit Φ_∞.

Let $g_t := g_{\Phi_t}$ be the metric induced by the immersion Φ_t on Σ. For a fixed $v \in T\Sigma$ we have

$$\frac{d}{dt} g_t(v,v) = \frac{d}{dt}|d\Phi_t[v]|^2 = 2\langle d\Phi_t[v], \nabla_v w_t\rangle$$

and, since $|\nabla_v w_t| \leq \|w_t\|_{\Phi_t}|v|_{g_t}$, we deduce that

$$\Big|\frac{d}{dt} g_t(v,v)\Big| \leq 2 g_t(v,v)\|w_t\|_{\Phi_t}.$$

Hence, for $v \neq 0$, the time derivative of $\log g_t(v,v)$ is bounded in L^1 on $[0,\infty)$. Thus there exists a constant $C > 0$ such that

$$C^{-2} g_0(v,v) \leq g_t(v,v) \leq C^2 g_0(v,v)$$

for all $t \geq 0$ and all $v \in T\Sigma$. As a consequence, for any $x \in \Sigma$ and any $v \in T_x\Sigma$

$$|\nabla_{\partial_t}(d\Phi_t[v])| \leq |\nabla w_t|_{g_t}|v|_{g_t} \leq C\|w_t\|_{\Phi_t}|v|_{g_0},$$

with ∇_{∂_t} being the covariant derivative along the curve $\Phi_t(x)$. Together with the C^0 convergence $\Phi_t \to \Phi_\infty$, this implies that actually $\Phi_t \to \Phi_\infty$ in C^1. Finally, given smooth vector fields X, Y on Σ,

$$\nabla_{\partial_t}\nabla_X(d\Phi_t[Y]) = \nabla_X\nabla_Y w_t + \operatorname{Rm}(d\Phi_t[X], w_t)(d\Phi_t[Y])$$

where $\operatorname{Rm}(V,W)Z = \nabla^2_{W,V}Z - \nabla^2_{V,W}Z$ is the Riemann tensor of $\mathcal{M}$. Again, thanks to the comparability between g_0 and g_t, the right-hand side is bounded in L^4 by $\|w_t\|_{\Phi_t}$, up to a multiplicative constant depending only on X, Y. This implies the convergence $\Phi_t \to \Phi_\infty$ in $W^{2,4}$. $\square$

The following variational result, essentially due to Struwe, is proved in [90]. Before stating it, we give a notion of *admissible family*.

Definition 2.2.2. Given a Banach manifold $\mathfrak{M}$, a nonempty family $\mathcal{F}$ of subsets of $\mathfrak{M}$ is said to be *admissible* if, for any continuous deformation $F:[0,1]\times\mathfrak{M}\to\mathfrak{M}$ with $F_0=\mathrm{id}_{\mathfrak{M}}$ and F_t a homeomorphism for all $0\le t\le 1$, we have $F_1(A)\in\mathcal{F}$ for all $A\in\mathcal{F}$ (where $F_t:=F(t,\cdot)$).

Proposition 2.2.3. *Assume $(E_\sigma)_{\sigma\ge 0}$ is a family of C^1 functionals on a complete Finsler manifold $\mathfrak{M}$, with $E_\sigma(x)$ differentiable in σ and $\sigma\mapsto E_\sigma(x)$, $\sigma\mapsto\frac{d}{d\sigma}E_\sigma(x)$ both increasing in σ, for every $x\in\mathfrak{M}$. Assume also that*

$$\|dE_{\sigma_j}(x_j)-dE_\sigma(x_j)\|\to 0 \tag{2.2.2}$$

whenever $1\ge\sigma_j\ge\sigma>0$, $\sigma_j\to\sigma$ and $\limsup_{j\to\infty}E_\sigma(x_j)<\infty$.

Then, for any admissible family $\mathcal{F}$, defining the min-max values

$$\beta(\sigma):=\inf_{A\in\mathcal{F}}\sup_{x\in A}E_\sigma(x),$$

there exist sequences $(\sigma_k)\subseteq(0,1)$ and $(x_k)\subseteq\mathfrak{M}$, with $\sigma_k\to 0$, such that

$$E_{\sigma_k}(x_k)-\beta(\sigma_k)\to 0,\quad \|dE_{\sigma_k}(x_k)\|<f(\sigma_k),\quad \sigma_k\log(1/\sigma_k)\frac{d}{d\sigma}E_\sigma(x_k)\Big|_{\sigma_k}\to 0,$$

where $f:(0,\infty)\to(0,\infty)$ is any function fixed in advance.

This statement is quite robust and can be adapted to other kinds of min-max problems, where one replaces admissible families with other notions.

Remark 2.2.4. Actually, in [90] the functional E_σ is assumed to be Palais–Smale, and the second conclusion becomes $dE_{\sigma_k}(x_k)=0$. Without this hypothesis, we can still find almost critical points x_k for E_{σ_k}, in the sense that we can require $\|dE_{\sigma_k}(x_k)\|$ to be as small as we want, with the same proof.

Proposition 2.2.5. *The functionals $(E_\sigma)_{\sigma\ge 0}$ previously defined satisfy the assumptions of Proposition 2.2.3.*

Before proving this fact, we make an important observation.

Proposition 2.2.6. *For X,Y vector fields on Σ we have $(\nabla d\Phi)(X,Y)=\mathrm{I\!I}^\Phi(\Phi_*X,\Phi_*Y)$.*

Proof. The left-hand side equals $\nabla_X(\Phi_*Y)-\Phi_*\nabla_XY$; since Φ is an isometry from (Σ,g_Φ) to the immersed surface Φ, the term $\Phi_*\nabla_XY$ equals (locally) the Levi-Civita connection $\nabla_{\Phi_*X}\Phi_*Y$ on this surface; the latter equals the orthogonal projection of $\nabla_X(\Phi_*Y)$ onto the tangent plane, since ∇ is the pullback of the Levi-Civita connection from $\mathcal{M}$. □

Proof of Proposition 2.2.5. We only need to check that (2.2.2) holds. We first show how to obtain an upper bound for $|dE_{\sigma'}(\Phi)[w]-dE_\sigma(\Phi)[w]|$, when $1\ge\sigma'\ge\sigma>0$.

If $\Phi \in \mathfrak{M}$ is a smooth map and (Φ_t) is a smooth variation (with $\Phi_0 = \Phi$), we compute

$$\frac{d}{dt}\int_\Sigma \mathrm{vol}_{\Phi_t}\Big|_{t=0} = \int_\Sigma \langle d\Phi, \nabla w\rangle \ \mathrm{vol}_\Phi,$$

where $w := \frac{d}{dt}\Phi_t\Big|_{t=0}$ belongs to $T_\Phi\mathfrak{M}$, and the scalar product (with respect to g_Φ) in the integral is bounded by $2\|\nabla w\|_{L^\infty} \le 2\|w\|_\Phi$.

With a similar computation for the length of $\Phi|_{\partial\Sigma}$, we get

$$\begin{aligned}\frac{d}{dt}E_\sigma(\Phi)\Big|_{t=0} &= \int_\Sigma (1+\sigma^4|\mathbb{I}^\Phi|^4)\langle d\Phi, \nabla w\rangle \ \mathrm{vol}_\Phi + \sigma\int_{\partial\Sigma}\langle d\Phi[\tau], \nabla_\tau w\rangle \ \mathrm{vol}_{\Phi|_{\partial\Sigma}} \\ &\quad + \sigma^4\int_\Sigma \frac{d}{dt}|\mathbb{I}^{\Phi_t}|^4\Big|_{t=0} \ \mathrm{vol}_\Phi,\end{aligned} \tag{2.2.3}$$

where τ is the unit vector (with respect to g_Φ) orienting $\partial\Sigma$.

Given a local orthonormal frame $\{e_1, e_2\}$, oriented as Σ, define $n_t := d\Phi_t[e_1] \wedge d\Phi_t[e_2]$. We have $|\mathbb{I}^{\Phi_t}| = |\nabla n_t|$ and

$$\nabla_{\partial_t}\nabla_X n_t = \nabla_X\nabla_{\partial_t} n_t + \mathrm{Rm}\left(d\Phi[X], \frac{d\Phi}{dt}\right)n_t,$$

where $\mathrm{Rm}(a,b)(c\wedge d) := (\mathrm{Rm}(a,b)c)\wedge d + c\wedge(\mathrm{Rm}(a,b)d)$ for vectors in $T\mathcal{M}$. At $t=0$ the above equals $\nabla_X\omega + R(d\Phi[X], w)n$, where $n := n_0$ and

$$\omega := \nabla_{e_1} w \wedge d\Phi[e_2] + d\Phi[e_1]\wedge\nabla_{e_2} w - \langle d\Phi[e_i], \nabla_{e_i} w\rangle n.$$

Using Proposition 2.2.6 we see that $|\nabla_X\omega| \le C|X|(|\nabla^2 w| + |\nabla w||\mathbb{I}^\Phi|)$.

Finally, the contribution of the metric g_{Φ_t} for the time derivative of $|\nabla n_t|^4$ is just $-4|\nabla n|^2\langle d\Phi\otimes\nabla w, \nabla n\otimes\nabla n\rangle$. Combining this fact with the preceding computations, we deduce that the time derivative of $|\mathbb{I}^{\Phi_t}|^4$ at $t=0$ is bounded by

$$|\mathbb{I}^\Phi|^3|\nabla^2 w| + |\mathbb{I}^\Phi|^4|\nabla w| + |\mathbb{I}^\Phi|^3|w| \tag{2.2.4}$$

up to a multiplicative constant depending on $\mathcal{M}$.

Thus, using (2.2.3), (2.2.4), Hölder's inequality and Young's inequality, we see that

$$|dE_{\sigma'}(\Phi)[w] - dE_\sigma(\Phi)[w]| \le C\frac{\sigma'-\sigma}{\sigma}E_\sigma(\Phi)\|w\|_\Phi + C(\sigma'-\sigma)E_\sigma(\Phi)^{3/4}\|w\|_\Phi$$

for $0 < \sigma \le \sigma' \le 2\sigma$. Since E_σ and $E_{\sigma'}$ are C^1 functionals, this bound holds for general $\Phi\in\mathfrak{M}$ and $w \in T_\Phi\mathfrak{M}$. Starting from this estimate, it is immediate to check that (2.2.2) is satisfied. □

Thanks to Proposition 2.2.5, letting $f(\sigma) := \sigma^5$ we can then find sequences of numbers $\sigma_k \to 0$ and maps $\Phi_k \in \mathfrak{M}$ satisfying the conclusions of Proposition 2.2.3. In particular,

$$\|dE_{\sigma_k}(\Phi_k)\|_{\Phi_k} < \sigma_k^5 \tag{2.2.5}$$

and

$$\sigma_k \log(1/\sigma_k)\,\mathrm{length}(\Phi_k|_{\partial\Sigma}) + \sigma_k^4 \log(1/\sigma_k) \int_\Sigma |\mathrm{I\!I}^{\Phi_k}|^4 \,\mathrm{vol}_{\Phi_k} \to 0. \tag{2.2.6}$$

Since smooth functions are dense in $\mathfrak{M}$, we can assume that the maps Φ_k are smooth.

In the following sections we will study the limit behavior of the measures $\nu_k := \mathrm{vol}_{\Phi_k}$ and the varifolds $\mathbf{v}_k$ induced by Φ_k. Note that the weight measure $|\mathbf{v}_k|$ equals $(\Phi_k)_*\nu_k$.

We conclude this section by discussing how Corollary 2.1.6 follows from Theorem 2.1.1.

Proof of Corollary 2.1.6. For any $A \in \mathcal{F}$, by compactness of A we have

$$\max_{\Phi\in A} E_\sigma(\Phi) \to \max_{\Phi\in A} \mathrm{area}(\Phi) \quad \text{as } \sigma \to 0.$$

Hence, the min-max value $\beta(\sigma)$ for E_σ converges to β. Although $\mathcal{F}$ is not stable under isotopies of $\mathfrak{M}$, Proposition 2.2.3 still applies since in its proof we can use a pseudo-gradient flow preserving the subset of smooth immersions. Taking then smooth maps Φ_k as above, the statement follows from Theorem 2.1.1, Theorem 2.1.4 and the fact that

$$\lim_{k\to\infty} \mathrm{area}(\Phi_k) = \lim_{k\to\infty} \beta(\sigma_k) = \beta. \qquad \square$$

2.3 First variation

In this section we will derive a particularly useful formula for the first variation of E_σ at $\Phi \in \mathfrak{M}$, for infinitesimal variations $w \in T_\Phi\mathfrak{M}$ of the form $X(\Phi)$, with X a smooth vector field on $\mathcal{M}$.

Let $\Phi \in \mathfrak{M}$ be a smooth map and $w \in T_\Phi\mathfrak{M}$ a smooth section of $\Phi^*T\mathcal{M}$, with $w \in T\mathcal{N}$ on $\partial\Sigma$. In the sequel, $\{e_1, e_2\}$ will be an oriented orthonormal basis at an arbitrary point of Σ, with respect to the induced metric g_Φ. The $(1,1)$-tensor $J : T\Sigma \to T\Sigma$, given by $Je_1 := e_2$ and $Je_2 := -e_1$, is parallel for this metric.

As in the proof of Proposition 2.2.5, we use the notation $n := \Phi_*e_1 \wedge \Phi_*e_2$ and we set $f := |\mathrm{I\!I}^\Phi|^2 = |\nabla n|^2$. We also define the sections $\widehat{I}$ and $\widehat{J}$ of $\Phi^*T\mathcal{M} \otimes T^*\Sigma$, as well as the section $\widehat{\mathrm{I\!I}}$ of $\Phi^*T\mathcal{M} \otimes T^*\Sigma \otimes T^*\Sigma$, by

$$\widehat{I}(v) := \Phi_*v,\ \ \widehat{J}(v) := \Phi_*(Jv),\ \ \widehat{\mathrm{I\!I}}(v,v') := \mathrm{I\!I}^\Phi(\Phi_*v, \Phi_*v'), \quad \text{for } v, v' \in T\Sigma.$$

Recall the following formula, which was computed in that proof:

$$\begin{aligned} dE_\sigma(\Phi)[w] &= \int_\Sigma (1+\sigma^4 f^2)\langle \widehat{I}, \nabla w\rangle + \sigma \int_{\partial\Sigma} \langle \Phi_*\tau, \nabla_\tau w\rangle \\ &\quad + 4\sigma^4 \int_\Sigma f\langle \nabla n, \nabla\omega + \mathrm{Rm}(d\Phi, w)n\rangle - 4\sigma^4 \int_\Sigma f\big\langle \widehat{I}\otimes\nabla w, \nabla n \otimes \nabla n\big\rangle, \end{aligned} \tag{2.3.1}$$

where we omit the volume forms and ω denotes the infinitesimal variation of n, namely

$$\omega = \nabla_{e_i} w \wedge \widehat{J}(e_i) - \big\langle \nabla w, \widehat{I}\big\rangle n.$$

When the variation w has the form $w = X(\Phi)$, using Proposition 2.2.6 we get

$$\begin{aligned} \nabla w &= \nabla X(\Phi)[\Phi_* \cdot] = \nabla X \circ \widehat{I}, \\ \nabla^2_{e_i,e_j} w &= \nabla^2 X(\Phi)[\Phi_* e_i, \Phi_* e_j] + \nabla X(\Phi)[\widehat{\mathrm{I\!I}}(e_i, e_j)]. \end{aligned} \tag{2.3.2}$$

For such special variations, (2.3.1) becomes

$$\begin{aligned} dE_\sigma(\Phi)[w] = &\int_\Sigma (1 + \sigma^4 f^2) \left\langle \widehat{I}, \nabla X \circ \widehat{I} \right\rangle + \sigma \int_{\partial\Sigma} \langle \Phi_* \tau, \nabla X[\Phi_* \tau] \rangle \\ &+ 4\sigma^4 \int_\Sigma f(\langle \nabla n, \nabla \omega \rangle + \langle \nabla n, \mathrm{Rm}(d\Phi, X(\Phi)) n \rangle) \\ &- 4\sigma^4 \int_\Sigma f \left\langle \widehat{I} \otimes (\nabla X \circ \widehat{I}), \nabla n \otimes \nabla n \right\rangle. \end{aligned} \tag{2.3.3}$$

We now write the term $\langle \nabla n, \nabla \omega \rangle$ in a way which will prove useful for our later work. Since $\langle \nabla_{e_i} n, n \rangle = 0$, we compute

$$\langle \nabla n, \nabla \omega \rangle = \left\langle \nabla n, \nabla (\nabla_{e_i} w \wedge \widehat{J}(e_i)) \right\rangle - \left\langle \nabla w, \widehat{I} \right\rangle |\nabla n|^2$$

and the first term equals $\left\langle \nabla_{e_j} n, \nabla^2_{e_j,e_k} w \wedge \widehat{J}(e_k) + \nabla_{e_k} w \wedge \widehat{\mathrm{I\!I}}(e_j, J e_k) \right\rangle$. Substituting the above formulas for ∇w and $\nabla^2 w$, we get

$$\begin{aligned} \langle \nabla n, \nabla \omega \rangle = &\langle \nabla_{e_j} n, \nabla^2 X[\Phi_* e_j, \Phi_* e_k] \wedge \widehat{J}(e_k) + \nabla X[\widehat{\mathrm{I\!I}}(e_j, e_k)] \wedge \widehat{J}(e_k) \\ &+ \nabla X[\Phi_* e_k] \wedge \widehat{\mathrm{I\!I}}(e_j, J e_k) \rangle - \left\langle \nabla X, \widehat{I} \right\rangle |\nabla n|^2. \end{aligned}$$

Thus,

$$f |\langle \nabla n, \nabla \omega \rangle| \leq C(\mathcal{M})(\|\nabla X\|_{L^\infty} f^2 + \|\nabla^2 X\|_{L^\infty} f^{3/2}). \tag{2.3.4}$$

We are now ready to state an initial consequence of this bound.

Definition 2.3.1. A k-varifold $\mathbf{v}$ on $\mathcal{M}$ is a *free boundary stationary varifold* for the couple $(\mathcal{M}, \mathcal{N})$ if it holds that

$$\frac{d}{dt} \|(F_t)_* \mathbf{v}\|(\mathcal{M}) \Big|_{t=0} = 0$$

whenever $(F_t)_{-\varepsilon < t < \varepsilon}$ is a family of diffeomorphisms of $\mathcal{M}$ with $F_t(\mathcal{N}) = \mathcal{N}$, $F_0 = \mathrm{id}$ and $F_t(x)$ smooth in the couple (t, x). We say that $\mathbf{v}$ is free boundary stationary *outside* a closed set $K \subseteq \mathcal{M}$ if the same holds for isotopies (F_t) such that $F_t|_U = \mathrm{id}$ for some neighborhood $U \supseteq K$.

Definition 2.3.2. We denote $\mathcal{X}_{fb}$ the linear space of smooth vector fields X on $\mathcal{M}$ which are tangent to $\mathcal{N}$, namely such that $X(p) \in T_p\mathcal{N}$ for all $p \in \mathcal{N}$.

Remark 2.3.3. With $X := \frac{d}{dt} F_t \Big|_{t=0}$, we have $X \in \mathcal{X}_{fb}$ and

$$\frac{d}{dt} \|(F_t)_* \mathbf{v}\|(\mathcal{M}) \Big|_{t=0} = \int_{(p,\Pi) \in \mathrm{Gr}_k(\mathcal{M})} \mathrm{div}_\Pi X \, d\mathbf{v}(p, \Pi),$$

where $\mathrm{Gr}_k(\mathcal{M})$ is the Grassmannian bundle made of couples (p, Π) with $p \in \mathcal{M}$ and $\Pi \subseteq T_p\mathcal{M}$ a k-plane. Conversely, given X tangent to $\mathcal{N}$, we can take F_t to be its flow. Hence, $\mathbf{v}$ is a free boundary stationary varifold if and only if

$$\int_{(p,\Pi)\in \mathrm{Gr}_k(\mathcal{M})} \mathrm{div}_\Pi X \, d\mathbf{v}(p, \Pi) = 0 \quad \text{for all } X \in \mathcal{X}_{fb}.$$

Similarly, $\mathbf{v}$ is free boundary stationary outside K if and only if the same holds for all $X \in \mathcal{X}_{fb} \cap C_c^\infty(\mathcal{M} \setminus K)$.

Given a sequence (Φ_k) as in Section 2.2, the following holds.

Theorem 2.3.4. *The varifolds $\mathbf{v}_k$ induced by Φ_k converge, up to subsequences, to a free boundary stationary varifold $\mathbf{v}_\infty$.*

A priori it is not clear whether $\mathbf{v}_\infty$ is integer rectifiable. This, together with a structure theorem for $\mathbf{v}_\infty$, will be proved later on.

Proof. Fix any $(F_t)_{-\varepsilon<t<\varepsilon}$ as above and consider the variation $(F_t \circ \Phi_k) \subseteq \mathfrak{M}$. The corresponding infinitesimal variation $w_k \in T_{\Phi_k}\mathfrak{M}$ is just $w_k = X \circ \Phi_k$. Hence, (2.3.3) and (2.2.5) give

$$\begin{aligned}
&\int_\Sigma (1 + \sigma_k^4 f_k^2)\Big\langle \widehat{I}_k, \nabla X \circ \widehat{I}_k \Big\rangle + \sigma_k \int_{\partial\Sigma} \langle (\Phi_k)_*\tau, \nabla X[(\Phi_k)_*\tau]\rangle \\
&+ 4\sigma_k^4 \int_\Sigma f_k(\langle \nabla n_k, \nabla \omega_k\rangle + \langle \nabla n_k, \mathrm{Rm}(d\Phi_k, X(\Phi_k))n_k\rangle) \\
&- 4\sigma_k^4 \int_\Sigma f_k \big\langle \widehat{I}_k \otimes (\nabla X \circ \widehat{I}_k), \nabla n_k \otimes \nabla n_k \big\rangle \\
&= o(\sigma_k^4 \|w_k\|_{\Phi_k}).
\end{aligned} \tag{2.3.5}$$

We now show that all terms where σ_k appears are infinitesimal as $k \to \infty$. Note that $\big|\big\langle \widehat{I}_k, \nabla X \circ \widehat{I}_k \big\rangle\big| \le 2\|\nabla X\|_{L^\infty}$, since the scalar product is with respect to the induced metric g_{Φ_k}. Hence, by (2.2.6),

$$\sigma_k^4 \int_\Sigma f_k^2 \big\langle \widehat{I}_k, \nabla X \circ \widehat{I}_k \big\rangle \to 0$$

and similarly the boundary term is also infinitesimal. Thanks to the boundedness of the area of Φ_k, the pointwise bound (2.3.4) and Hölder's inequality, we deduce that also the remaining terms in the left-hand side of (2.3.5) are infinitesimal, except for the first one.

We now estimate $\|w_k\|_{\Phi_k}$. Note first that $|w_k| \le \|X\|_{L^\infty}$ and $|\nabla w_k| \le \|\nabla X\|_{L^\infty}$. Also, from (2.3.2) we get

$$|\nabla^2 w_k| \le \|\nabla^2 X\|_{L^\infty} + \|\nabla X\|_{L^\infty} |\mathrm{I\!I}^{\Phi_k}|.$$

We deduce that $\sigma_k \|w_k\|_{\Phi_k} \to 0$.

Finally, $\big\langle \widehat{I}_k, \nabla X \circ \widehat{I}_k \big\rangle(x) = \mathrm{div}_{(\Phi_k)_*[T_x\Sigma]} X$, so that

$$\int_\Sigma \big\langle \widehat{I}_k, \nabla X \circ \widehat{I}_k \big\rangle = \int_{(p,\Pi)\in \mathrm{Gr}_2(\mathcal{M})} \mathrm{div}_\Pi X \, d\mathbf{v}_k(p, \Pi)$$

and, taking any subsequential limit $\mathbf{v}_\infty$, the claim follows. □

2.4 A lower bound for the area

In order to obtain more information for the asymptotic behavior of the measures ν_k and the varifolds $\mathbf{v}_k$ introduced at the end of Section 2.2, we first obtain (various versions of) a lower bound on the mass $\frac{|\mathbf{v}_k|(B_r(p))}{r^2}$. The main idea will be to mimick the proof of the monotonicity formula for stationary varifolds; since that proof uses vector fields in the ambient $\mathcal{M}$, we will be able to use formula (2.3.3), involving variations of the form $X(\Phi)$.

The statements contained in this section make it essential to require the decays $\sigma_k^4 \log \sigma_k^{-1} \int_\Sigma f_k^2 \operatorname{vol}_{\Phi_k} \to 0$, as well as $\sigma_k \log \sigma_k^{-1} \operatorname{length}(\Phi_k|_{\partial\Sigma}) \to 0$, guaranteed by Proposition 2.2.3.

Rather than dealing with the sequence (Φ_k), in this section all the statements concern a general smooth map $\Phi \in \mathfrak{M}$, with a fixed value of σ. Of course, in order for the results to be useful in the asymptotic analysis, the constants appearing in their statements will depend neither on Φ nor on σ.

Definition 2.4.1. In the following statements, we say that a smooth map $\Phi \in \mathfrak{M}$ is *ε-critical* for E_σ if $\|dE_\sigma(\Phi)\|_\Phi \le \varepsilon$, meaning that $|dE_\sigma(\Phi)[w]| \le \varepsilon \|w\|_\Phi$ for all $w \in T_\Phi\mathfrak{M}$.

Proposition 2.4.2. *Let Φ be σ^5-critical for E_σ, $x \in \Sigma$, and denote $p := \Phi(x)$. Assume $U \subseteq \Sigma$ is an open neighborhood of x. Defining the measures $\mu := (\Phi|_U)_*(\operatorname{vol}_\Phi)$ and $\lambda := (\Phi|_{\partial\Sigma\cap U})_*(\sigma \operatorname{vol}_{\Phi|_{\partial\Sigma}}) + (\Phi|_U)_*(\sigma^4 f^2 \operatorname{vol}_\Phi)$ on $\mathcal{M}$, assume also that*

$$\lambda(B_s(p)) \le \delta\mu(B_{5s}(p)) \quad \textit{for all radii } s > 0,$$

for some $0 < \delta < 1$. Given $r > s \ge \sigma$, if $B_r(p) \cap \Phi(\partial U) = \emptyset$ then we have

$$\frac{\mu(B_r(p))}{r^2} \ge (c - C\delta \log(r/s)) \frac{\mu(B_s(p))}{s^2} - C\sigma^2,$$

for some constants $c, C > 0$ depending on $\mathcal{M}$ and $\mathcal{N}$.

Note that ∂U is the topological boundary of U in Σ and therefore does not include $\partial\Sigma \cap U$. Recall that $f = |\mathbb{I}^\Phi|^2$.

Before delving into the proof, we state without proof an immediate but useful fact.

Proposition 2.4.3. *There exists a constant $c_F(\mathcal{M}, \mathcal{N})$ such that, for every $p \in \mathcal{M}$, there are coordinates*

$$\xi = (\xi_1, \dots, \xi_m) : B_{c_F}(p) \to \mathbb{R}^m$$

depending on the center p, satisfying

$$g_{ij}(0) = \delta_{ij}, \quad \|g_{ij}\|_{C^2} \le C(\mathcal{M}, \mathcal{N}), \quad \frac{1}{2} \operatorname{dist}(\cdot, p) \le |\xi| \le 2 \operatorname{dist}(\cdot, p) \tag{2.4.1}$$

for the Euclidean metric $|\cdot|$. When $p \in \mathcal{N}$ we also ask that the coordinates are adapted to $\mathcal{N}$, in the sense that $B_{c_F}(p) \cap \mathcal{N}$ corresponds to $\{\xi_{n+1} = \cdots = \xi_m = 0\}$.

Proof of Proposition 2.4.2. Without loss of generality, we can assume $r, \frac{\sigma}{s} \le c'$ for a constant $c' < c_F$ to be chosen later. Once we get the desired estimate with these constraints, the statement follows in general with possibly different values of c and C.

We will imitate the proof of the monotonicity formula, using now our equation (2.3.3). Assume first $B_r(p) \cap \mathcal{N} = \emptyset$. In this case we can find coordinates $\xi : B_r(p) \to \mathbb{R}^m$ as in Proposition 2.4.3. Given a decreasing cut-off function $\chi \in C_c^\infty([0,\infty))$, with $\chi = 1$ on $[0, 1/4]$ and $\chi = 0$ on $[1/2, \infty)$, for $0 < \tau < r$ we set $\chi_\tau := \chi(|\xi|/\tau)$ and $X_\tau := \chi_\tau \xi_i \frac{\partial}{\partial \xi_i}$.

Note that, by (2.4.1), we have $|\nabla X_\tau| \le C$, $|\nabla^2 X_\tau| \le C\tau^{-1}$ and

$$\operatorname{div}_\Pi(X_\tau) \ge (2 - C\tau)\chi_\tau + (1 + C\tau)\chi'(|\xi|/\tau)\frac{|\xi|}{\tau} \tag{2.4.2}$$

for any $p \in B_\tau(q)$ and any 2-plane $\Pi \subseteq T_p\mathcal{M}$ (recall that $\chi' \le 0$).

We now want to apply (2.3.3) with the infinitesimal variation $w := X_\tau(\Phi)\mathbf{1}_U$, which is admissible since X_τ vanishes near $\Phi(\partial U)$. By (2.3.4) we have

$$\sigma^4 f\langle \nabla n, \nabla \omega\rangle \le C\sigma^4(f^2 + \tau^{-1} f^{3/2})\mathbf{1}_{\operatorname{spt}(w)};$$

hence, the corresponding term in the first variation is bounded by

$$C\lambda(B_\tau(p)) + C\sigma\tau^{-1}\lambda(B_\tau(p))^{3/4}\mu(B_\tau(p))^{1/4} \le C(\delta + \sigma\tau^{-1})\mu(B_{5\tau}(p)).$$

Similarly, the curvature term in (2.3.3) is bounded by $C\sigma\mu(B_{5\tau}(p))$, while the last term is again bounded by $C\delta\mu(B_{5\tau}(p))$. Also, the boundary term vanishes since the support of w does not intersect $\partial\Sigma$.

Finally, as in the proof of Theorem 2.3.4, we have

$$\begin{aligned}\|w\|_\Phi &\le \|X_\tau\|_{L^\infty} + \|\nabla X_\tau\|_{L^\infty} + \Big(\int_{\operatorname{spt}(w)} (\|\nabla^2 X_\tau\|_{L^\infty} + \|\nabla X_\tau\|_{L^\infty} f^{1/2})^4 \operatorname{vol}_\Phi\Big)^{1/4} \\ &\le C + C\tau^{-1}\mu(B_\tau(p))^{1/4} + \sigma^{-1}\lambda(B_\tau(p))^{1/4} \\ &\le C + C(\tau^{-4} + \sigma^{-4})\mu(B_{5\tau}(p)),\end{aligned}$$

so that from (2.2.5) we get $|dE_\sigma(\Phi)[w]| \le C\sigma^5 + C\sigma\mu(B_{5\tau}(p))$ for $\tau \ge \sigma$.

Hence, defining $h(\tau) := \tau^{-2}\int_U \chi_\tau(\Phi)\operatorname{vol}_\Phi$, (2.3.3) and a straightforward computation give

$$h'(\tau) \ge -C(\delta + \sigma\tau^{-1})\tau^{-3}\mu(B_{5\tau}(p)) - C\tau^{-2}\mu(B_{5\tau}(p)) - C\sigma^2 \tag{2.4.3}$$

for $\tau \ge \sigma$. Call $\bar{r}$ the biggest radius in $[s, r]$ such that

$$\frac{\mu(B_{\bar{r}}(p))}{\bar{r}^2} \ge \frac{\mu(B_s(p))}{s^2}.$$

For $\frac{r}{5} \ge \tau \ge \bar{r} \ge \sigma$, (2.4.3) becomes

$$h'(\tau) \ge -C(\delta\tau^{-1} + \sigma\tau^{-2} + 1)\frac{\mu(B_s(p))}{s^2} - C\sigma^2.$$

Integrating this inequality between $8\overline{r}$ and $\frac{r}{5}$ we get

$$\frac{\mu(B_{r/5}(p))}{(r/5)^2} \geq h(r/5) \geq h(8\overline{r}) - C(\delta \log(r/s) + \sigma s^{-1} + r)\frac{\mu(B_s(p))}{s^2} - C\sigma^2 r,$$

unless $r < 40\overline{r}$, in which case the statement follows trivially. Since r and $\frac{\sigma}{s}$ are both bounded by c', observing that $h(8\overline{r}) \geq \frac{\mu(B_{\overline{r}}(p))}{64\overline{r}^2} \geq \frac{\mu(B_s(p))}{64s^2}$ we arrive at

$$\frac{\mu(B_{r/5}(p))}{(r/5)^2} \geq \Big(\frac{1}{64} - C\delta \log(r/s) - 2Cc'\Big)\frac{\mu(B_s(p))}{s^2} - C\sigma^2 r,$$

and the statement follows in this case, once we impose $2Cc' < \frac{1}{64}$.

If $r' := \mathrm{dist}(p, \mathcal{N}) < r$, we let q be a nearest point to p in $\mathcal{N}$ (hence, $q = p$ when $r' = 0$). If $r' \geq s$, we know that the claim holds with r' replacing r; so it follows also for r if either $s \geq r/8$ or $r' \geq r/8$, with possibly different constants. Assume in the sequel that $r', s < r/8$. For $\tau > 2r' + s$ we have

$$\lambda(B_\tau(q)) \leq \lambda(B_{2\tau}(p)) \leq \delta\mu(B_{10\tau}(p)) \leq \delta\mu(B_{20\tau}(q)).$$

So, using now coordinates centered at q and adapted to $\mathcal{N}$ and defining h as before, we get

$$h'(\tau) \geq -C(\delta\tau^{-1} + \sigma\tau^{-2} + 1)\frac{\mu(B_{20\tau}(p))}{\tau^2} - C\sigma^2$$

for $\tau \geq 2r' + s \geq \sigma$; note that now also the boundary term in (2.3.3) is taken into account, giving again a contribution bounded by $C\tau^{-3}\lambda(B_\tau(q)) \leq C\delta\tau^{-3}\mu(B_{20\tau}(q))$ in the previous right-hand side. Similarly to the above, assume $2r' + s \leq \overline{r}' \leq r/2$ to be the smallest radius in this interval such that $\frac{\mu(B_\tau(q))}{\tau^2} \leq \frac{\mu(B_s(p))}{s^2}$ for $\tau \in [\overline{r}', r/2]$; if such radius does not exist, then we have $\frac{\mu(B_{r/2}(q))}{(r/2)^2} \geq \frac{\mu(B_s(p))}{s^2}$ and we are done thanks to the inclusion $B_r(p) \supseteq B_{r/2}(q)$. Integrating from $8\overline{r}'$ to $r/40$ (again, we can assume $\overline{r}' \leq \frac{r}{320}$), we conclude that either

$$\frac{\mu(B_{r/40}(q))}{(r/40)^2} \geq \Big(\frac{1}{64} - C\delta \log(r/s) - 2Cc'\Big)\frac{\mu(B_s(p))}{s^2} - C\sigma^2,$$

in which case we are done since $\mu(B_r(p)) \geq \mu(B_{r/40}(q))$, or

$$\frac{\mu(B_{r/40}(q))}{(r/40)^2} \geq \frac{\mu(B_{2r'+s}(q))}{64(2r'+s)^2} - (C\delta \log(r/s) + 2Cc')\frac{\mu(B_s(p))}{s^2} - C\sigma^2.$$

In this second case, if $r' < s$ then we use the inequality $\frac{\mu(B_{2r'+s}(q))}{(2r'+s)^2} \geq \frac{\mu(B_s(p))}{(3s)^2}$ and we are done. Otherwise, if $r' \geq s$ we use the inequality $\frac{\mu(B_{2r'+s}(q))}{(2r'+s)^2} \geq \frac{\mu(B_{r'}(p))}{(3r')^2}$ and we conclude using the already obtained lower bound for this last ratio. □

Remark 2.4.4. A similar choice of test vector fields gives the following monotonicity for general free boundary stationary varifolds $\mathbf{v}$: given p in $\mathcal{M}$, one has

$$\frac{|\mathbf{v}|(B_r(p))}{r^2} \geq (1 + C(\mathcal{M}, \mathcal{N})\sqrt{r})^{-1}\frac{|\mathbf{v}|(B_s(p))}{s^2} \tag{2.4.4}$$

for $0 < s < r < \operatorname{diam}(\mathcal{M})$ if $p \in \mathcal{N}$, and for $0 < s < r < \operatorname{dist}(p, \mathcal{N})$ otherwise. Indeed, it suffices to establish (2.4.4) assuming r small, and also $s \geq \frac{r}{2}$, since for $s < \frac{r}{2}$ we can then compare dyadic radii $r, \frac{r}{2}, \dots, 2^{-k}r$ until $2^{-k-1}r \leq s$. Pick coordinates as in Proposition 2.4.3, with $|\operatorname{dist}(\cdot, p) - |\xi|| \leq C \operatorname{dist}(\cdot, p)^2$, and take now χ such that $\chi = 1$ on $[0, 1 - 2\sqrt{r}]$, $\chi = 0$ on $[1 - \sqrt{r}, \infty)$ and $|\chi'| \leq Cr^{-1/2}$, so that χ_τ is supported in $B_r(p)$ for $\tau \leq r$. Setting $h(\tau) := \tau^{-2} \int_{\mathcal{M}} \chi_\tau \, d|\mathbf{v}|$, the stationarity of $\mathbf{v}$ and (2.4.2) then give

$$h'(\tau) \geq -Cr^{-5/2} |\mathbf{v}|(B_r(p)),$$

which, integrating from s to r, implies

$$\frac{|\mathbf{v}|(B_r(p))}{r^2} - \frac{|\mathbf{v}|(B_{(1-C\sqrt{r})s}(p))}{s^2} \geq \int_s^r h'(\tau) \, d\tau \geq -Cr^{-3/2} |\mathbf{v}|(B_r(p))$$

and (2.4.4) follows easily. Hence, the density

$$\theta(\mathbf{v}, p) := \lim_{s \to 0} \frac{|\mathbf{v}|(B_r(p))}{\pi r^2}$$

exists at any $p \in \mathcal{M}$. It also follows that

$$|\mathbf{v}|(B_r(p)) \leq C(\mathcal{M}, \mathcal{N}) |\mathbf{v}|(\mathcal{M}) r^2 \tag{2.4.5}$$

for all $r > 0$: this is clear if $p \in \mathcal{N}$, while for $p \notin \mathcal{N}$ and $r \geq \operatorname{dist}(p, \mathcal{N})$ we have $B_r(p) \subseteq B_{2r}(q)$ for some $q \in \mathcal{N}$, so that $|\mathbf{v}|(B_r(p)) \leq C |\mathbf{v}|(\mathcal{M})(2r)^2$, and (2.4.5) follows also for $r < \operatorname{dist}(p, \mathcal{N})$ thanks to (2.4.4) again. In the same way, using the inclusions $B_s(p) \supseteq B_{s-d}(q)$ and $B_{2d}(q) \supseteq B_d(p)$, with $d := \operatorname{dist}(p, \mathcal{N})$ and $q \in \mathcal{N}$ a nearest point to p, we deduce that $|\mathbf{v}|(B_s(p)) \geq cs^2 \theta(\mathbf{v}, p)$ holds even for $3d < s < \operatorname{diam}(\mathcal{M})$. Thus,

$$|\mathbf{v}|(B_r(p)) \geq c(\mathcal{M}, \mathcal{N}) \theta(\mathbf{v}, p) r^2 \tag{2.4.6}$$

for all $p \in \mathcal{M}$ and all $0 < r < \operatorname{diam}(\mathcal{M})$.

Corollary 2.4.5. *Let Φ be a σ^5-critical point for E_σ, let $\delta > 0$, and let $U \subseteq \Sigma$ be an open set which intersects $\partial\Sigma$ but does not contain entirely any boundary component of Σ. Denote S_δ the set of points $p \in \mathcal{M} \setminus \Phi(\partial U)$ satisfying the maximal bound*

$$\lambda(B_s(p)) \leq \delta \mu(B_{5s}(p)) \quad \textit{for all radii } s > 0.$$

Let T be a Borel set of points having distance less than σ from $\Phi(\partial\Sigma \cap U)$, and such that their distance from $\Phi(\partial U)$ is at least 5σ. Then we have

$$\mu(S_\delta \cap T) \leq C\sigma \operatorname{length}(\Phi|_{\partial\Sigma \cap U}) \frac{\mu(\mathcal{M})}{\operatorname{dist}(T, \Phi(\partial U))^2} + C\sigma^3 \operatorname{length}(\Phi|_{\partial\Sigma \cap U}),$$

for some C depending on $\mathcal{M}$ and $\mathcal{N}$, provided $\delta \log(1/\sigma)$ is small enough.

Proof. Let $L := \mathrm{length}(\Phi|_{\partial\Sigma\cap U})$. We first note that the set of points T' in $\Phi(\partial\Sigma \cap U)$ with distance less than σ from T can be covered with at most $\sigma^{-1}L$ balls $B_\sigma(p_j)$, with $\mathrm{dist}(p_j, \Phi(\partial U)) \geq 4\sigma$. Indeed, note first that $\mathrm{dist}(T', \Phi(\partial U)) \geq 4\sigma$; we can discard the components of $\partial\Sigma \cap U$ producing an arc of length less than 2σ, since this arc is disjoint from T'; we are left with finitely many components, corresponding to curves $\gamma_i : I_i \to \mathcal{N}$ with endpoints in $\Phi(\partial U)$, where $I_i = (0, |I_i|)$ is an open interval; assuming each of them to be parametrized by arclength, we then subdivide $[\sigma, |I_i| - \sigma]$ into at most $\sigma^{-1}|I_i|$ intervals $I_{i\ell}$ of size less than σ and we pick a point $p_{i\ell}$ in $\gamma_i(I_{i\ell}) \cap T'$, discarding the intervals for which this intersection is empty. The resulting collection of balls $\{B_\sigma(p_{i\ell})\}$ is the desired one.

Hence, T is covered by a collection of balls $\{B_{2\sigma}(p_j) \mid j \in J\}$, with $|J| \leq \sigma^{-1}L$ and $\mathrm{dist}(p_j, \Phi(\partial U)) \geq 4\sigma$.

Now let $J' \subseteq J$ denote the set of indices j such that $B_{2\sigma}(p_j)$ intersects S_δ and, for $j \in J'$, choose a point $q_j \in S_\delta \cap B_{2\sigma}(p_j)$. Then we have

$$T \cap S_\delta \subseteq \bigcup_{j\in J'} B_{4\sigma}(q_j).$$

Note that $\mathrm{dist}(q_j, \Phi(\partial U)) \geq \mathrm{dist}(T, \Phi(\partial U)) - 3\sigma$, which is comparable with $\mathrm{dist}(T, \Phi(\partial U))$, so that Proposition 2.4.2 gives

$$\frac{\mu(\mathcal{M})}{\mathrm{dist}(T, \Phi(\partial U))^2} \geq (c - C\delta\log(1/\sigma))\frac{\mu(B_{4\sigma}(q_j))}{(4\sigma)^2} - C\sigma^2$$

for constants c, C depending solely on $\mathcal{M}, \mathcal{N}$. Summing over $j \in J'$, we obtain

$$\mu(T) \leq \sum_{j\in J'} \mu(B_{4\sigma}(q_j)) \leq \sigma^{-1}L\frac{C}{1 - C\delta\log(1/\sigma)}\left(\sigma^2\frac{\mu(\mathcal{M})}{\mathrm{dist}(T, \Phi(\partial U))^2} + \sigma^4\right)$$

and the statement follows. □

Corollary 2.4.6. *Under the same assumptions as in Proposition 2.4.2, if $B_\sigma(p) \cap \mathcal{N} = \emptyset$ then*

$$\frac{\mu(B_r(p))}{r^2} \geq c - C\delta\log(r/\sigma) - C\sigma^2,$$

provided δ and σ are small enough.

Proof. We first claim that

$$\mu(B_\sigma(p)) > c'\sigma^2 \tag{2.4.7}$$

for some universal $c' > 0$.

The second fundamental form of the immersed surface Φ in $\mathbb{R}^Q$ is bounded by $|\mathrm{II}^\Phi| + C(\mathcal{M})$, so the monotonicity formula in the ball $\widetilde{B}_t(p) := B_t^{\mathbb{R}^Q}(p)$ (see, e.g., [98, eq. (17.4)], whose proof carries over to the setting of immersed surfaces) and Hölder's inequality give

$$\begin{aligned}\frac{\mu(\widetilde{B}_t(p))}{t^2} - \frac{\mu(\widetilde{B}_{t/2}(p))}{(t/2)^2} &\geq -Ct^{-1}(\sigma^{-1}\lambda(\widetilde{B}_t(p))^{1/4}\mu(\widetilde{B}_t(p))^{3/4} + \mu(\widetilde{B}_t(p))) \\ &\geq -Ct^{-1}(\sigma^{-1}\delta^{1/4} + 1)\mu(\widetilde{B}_{20t}(p))\end{aligned} \tag{2.4.8}$$

for $t \le \sigma$ small enough. Let $\bar{t} \le \frac{\sigma}{2}$ be the biggest radius such that $\mu(\widetilde{B}_{\bar{t}}(p)) \ge \frac{\pi}{2}\bar{t}^2$; note that $\bar{t}$ exists since $\lim_{t\to 0} \frac{\mu(\widetilde{B}_t(p))}{\pi t^2} \ge 1$. If $\bar{t} \ge \frac{\sigma}{80}$ then we are done, thanks to the inclusion $B_{2\bar{t}}(p) \supseteq \mathcal{M} \cap \widetilde{B}_{\bar{t}}(p)$. Otherwise, (2.4.8) gives

$$\frac{\mu(\widetilde{B}_t(p))}{t^2} - \frac{\mu(\widetilde{B}_{t/2}(p))}{(t/2)^2} \ge -Ct(\sigma^{-1}\delta^{1/4} + 1)$$

for $\bar{t} \le t \le \frac{\sigma}{40}$. Setting $t := 2^{-k}(\sigma/40)$ in the last inequality and summing on $k = 0, \dots, k_0 - 1$, where k_0 is the biggest integer such that $t \ge \bar{t}$, we get

$$\frac{\mu(\widetilde{B}_{\sigma/40}(p))}{(\sigma/40)^2} \ge \frac{\mu(\widetilde{B}_{\bar{t}}(p))}{4\bar{t}^2} - C\delta^{1/4} - C\sigma$$

and claim (2.4.7) follows again, for δ and σ small enough.

The statement now follows by applying Proposition 2.4.2 with $s := \sigma$. $\square$

2.5 Asymptotic behavior of the area, in Σ and in $\mathcal{M}$

We now investigate the asymptotic behavior of the maps Φ_k introduced in Section 2.2. Recall that ν_k is the area measure of Φ_k on Σ, meaning that $\nu_k(U)$ is the area of the immersion $\Phi_k|_U$ for any open set $U \subseteq \Sigma$. Also, let $\mu_k := (\Phi_k)_*\nu_k$ be the corresponding measure on $\mathcal{M}$, and recall that $\mathbf{v}_k$ is the 2-varifold induced by Φ_k, namely $\mathbf{v}_k := (\Phi_k)_*(\Sigma)$, the varifold pushforward of the canonical multiplicity one 2-varifold on Σ.

Up to subsequences, we can assume that μ_k, ν_k and $\mathbf{v}_k$ converge weakly to limits μ_∞, ν_∞ and $\mathbf{v}_\infty$, in the sense of Radon measures and varifolds.

In this section we show structure theorems for the limit measures ν_∞, μ_∞ and for the limit varifold $\mathbf{v}_\infty$, namely Theorem 2.5.2, Theorem 2.5.3 and Theorem 2.5.11. The regularity of $\mathbf{v}_\infty$ will be studied in Section 2.7.

We will assume for simplicity that the maps Φ_k induce the same conformal structure on Σ; we will discuss the general case later, in Section 2.6.

Given a reference metric g_0 (on Σ) compatible with this structure, vol_{g_0} will denote either the corresponding volume form or the associated measure.

Note that $\nu_k = \frac{1}{2}|d\Phi_k|_{g_0}^2 \mathrm{vol}_{g_0}$. Hence, viewing $\mathcal{M} \subset \mathbb{R}^Q$, the maps Φ_k are bounded in $W^{1,2}(\Sigma, \mathbb{R}^Q)$ and, up to subsequences, we can extract a weak limit Φ_∞. Note that we have the strong convergence in L^2 for the maps $\Phi_k \to \Phi_\infty$ and the traces $\Phi_k|_{\partial\Sigma} \to \Phi_\infty|_{\partial\Sigma}$; hence, Φ_∞ and its trace $\Phi_\infty|_{\partial\Sigma}$ take values into $\mathcal{M}$ and $\mathcal{N}$, respectively.

Proposition 2.5.1. *Given $x \in \Sigma$, fix a local conformal chart centered at x such that the chart domain corresponds to $U' := B_1^2$ if $x \notin \partial\Sigma$, or to $U' := B_1^2 \cap \{\Im(z) \ge 0\}$ if $x \in \partial\Sigma$. Given $0 < r < 1$, assume that $\Phi_k|_{\partial B_r^2 \cap U'}$ converges to the trace $\Phi_\infty|_{\partial B_r^2 \cap U'}$ in C^0, and that $s := \operatorname{diam} \Phi_\infty(\partial B_r^2 \cap U') < c_V$, with c_V the constant appearing in Lemma A.10.*

Then either $\limsup_{k\to\infty} \nu_k(B_r^2 \cap U') \ge c_Q$, with a constant $c_Q > 0$ depending only on $\mathcal{M}$ and $\mathcal{N}$, or $\operatorname{spt}(\mu)$ is included in a $2s$-neighborhood of $\Phi_\infty(\partial B_r^2 \cap U')$, for any weak limit μ of $(\Phi_k|_{B_r^2 \cap U'})_*\nu_k$.

Here the letter Q in c_Q stands for *quantization*; it is not related to the dimension of the Euclidean space $\mathbb{R}^Q$.

Proof. Assume $\limsup_{k\to\infty}\nu_k(B^2_r\cap U')<c_Q$, for c_Q to be specified below, and let μ be the weak limit of $(\Phi_k|_{B^2_r\cap U'})_*\nu_k$ along a subsequence (not relabeled). The maps $\Phi_k|_{B^2_r\cap U'}$ induce varifolds $\widetilde{\mathbf{v}}_k$.

If $x\in\partial\Sigma$, then we can repeat the proof of Theorem 2.3.4 with vector fields X supported outside $\Gamma:=\Phi_\infty(\partial B^2_r\cap U')$, with the corresponding variation w_k given by $w_k=X(\Phi_k)$ on $B^2_r\cap U'$ and $w_k=0$ on the complement (in Σ). We deduce that the limit (up to further subsequences) $\widetilde{\mathbf{v}}_\infty$ is a free boundary stationary varifold outside Γ. If $x\notin\partial\Sigma$, then $\widetilde{\mathbf{v}}_\infty$ is actually stationary outside Γ, since any vector field supported outside Γ produces a variation which does not change Φ_k outside B^2_r.

Also, if $x\in\partial\Sigma$ we let $p_k\in\Phi_k(\partial B^2_r\cap\{\Im(z)=0\})\in\mathcal{N}$ and call p any limit point; we then have $p\in\Gamma\cap\mathcal{N}$ and $\Gamma\subseteq\overline{B}_s(p)$. If $x\notin\partial\Sigma$, we just take any $p\in\Gamma$ and again we have $\Gamma\subseteq\overline{B}_s(p)$.

Observing that $(\Phi_k|_{B^2_r\cap U'})_*\nu_k=|\widetilde{\mathbf{v}}_k|$ converges both to μ and to $|\widetilde{\mathbf{v}}_\infty|$, we deduce $\mu=|\widetilde{\mathbf{v}}_\infty|$. Also, $\widetilde{\mathbf{v}}_\infty$ has density bounded below by a certain constant c, on $\mathcal{M}\setminus\Gamma$. To show this, fix a compact set $K\subset\mathcal{M}\setminus\Gamma$; it suffices to prove that

$$\limsup_{k\to\infty}|\widetilde{\mathbf{v}}_k|(B_s(q))\geq cs^2 \tag{2.5.1}$$

for all $s<\operatorname{dist}(K,\Gamma)$ and all $q\in K$ outside a set F_k, with $|\widetilde{\mathbf{v}}_k|(F_k)\to 0$. This can be obtained with Proposition 2.4.2, Corollary 2.4.5, Corollary 2.4.6 and a covering argument: let

$$\lambda_k:=(\Phi_k|_{B^2_r\cap U'})_*(\sigma_k^4|\mathrm{I\!I}^{\Phi_k}|^4\,\nu_k),$$

so that by hypothesis $\lambda_k(\mathcal{M})=\frac{\delta_k^2}{\log\sigma_k^{-1}}$ for some sequence $\delta_k\to 0$. Let $F'_k\subseteq K$ be the set of points q such that

$$\lambda_k(B_s(q))>\frac{\delta_k}{\log\sigma_k^{-1}}|\widetilde{\mathbf{v}}_k|(B_{5s}(q)),\quad\text{for some } s>0.$$

Then, by Vitali's covering lemma, we can find a subcollection $\{B_{s_i}(q_i)\}$ of disjoint balls such that $F'_k\subseteq\bigcup_i B_{5s_i}(q_i)$. This gives $|\widetilde{\mathbf{v}}_k|(F'_k)\leq\frac{\log\sigma_k^{-1}}{\delta_k}\lambda_k(\mathcal{M})=\delta_k$, which is infinitesimal.

On the other hand, let F''_k be the σ_k-neighborhood of $\Phi(\partial\Sigma\cap U)$ intersected with K. Then eventually Corollary 2.4.5 is satisfied, with $B^2_r\cap U'$, F''_k, $\mathcal{M}\setminus F'_k$ and $\frac{\delta_k}{\log\sigma_k^{-1}}$ in place of U, T, S_δ and δ, and we obtain

$$\mu_k(F''_k\setminus F'_k)\leq C(K)\sigma_k\operatorname{length}(\Phi_k|_{\partial\Sigma}),$$

which is infinitesimal. Hence, we can set $F_k:=F'_k\cup F''_k$ and, for $q\notin F_k\cup\Gamma$, Corollary 2.4.6 eventually gives

$$\frac{|\widetilde{\mathbf{v}}_k|(B_s(q))}{s^2}\geq c-C\frac{\delta_k}{\log\sigma_k^{-1}}\log(s/\sigma_k)-C\sigma_k^2.$$

The right-hand side converges to $c > 0$ as $k \to \infty$, giving (2.5.1).

Hence, if $x \in \partial\Sigma$, then $\widetilde{\mathbf{v}}_\infty$ satisfies the assumption of Lemma A.10, and the statement follows. Otherwise, we can conclude using Remark A.11. $\square$

Theorem 2.5.2. *The limiting measure ν_∞ has finitely many atoms (possibly none), with weight at least c_Q. On the complement $\widetilde{\Sigma}$ of this finite set of atoms, ν_∞ is absolutely continuous with respect to vol_{g_0} and Φ_∞ has a continuous representative. Moreover, for every open subset $\omega \subset\subset \widetilde{\Sigma}$ with $\nu_\infty(\partial\omega) = 0$, we have $(\Phi_k|_\omega)_*\nu_k \rightharpoonup (\Phi_\infty|_\omega)_*\nu_\infty$.*

Proof. Given an atom $\{x\}$, we fix a local conformal chart centered at x, identifying a neighborhood U of x with the unit disk $U' := B_1^2$ if $x \notin \partial\Sigma$, or with $U' := B_1^2 \cap \{\Im(z) \geq 0\}$ if $x \in \partial\Sigma$.

For all $0 < r < 1$ we can select $\frac{r}{2} < t < r$ such that $\int_{\partial B_t^2 \cap U'} |d\Phi_\infty|^2 \leq \frac{2}{r} \int_{B_r^2 \cap U'} |d\Phi_\infty|^2$ and such that the trace $\Phi_\infty|_{\partial B_t^2 \cap U'}$ has a $W^{1,2}$ representative, with weak derivative given by the restriction of $d\Phi_\infty$ and $\Phi_k|_{\partial B_t^2 \cap U'} \to \Phi_\infty|_{\partial B_t^2 \cap U'}$ in C^0 along a subsequence, which we do not relabel (see, e.g., Lemma A.3 and Lemma A.5).

Then, by Cauchy–Schwarz, $s := \mathrm{diam}(\Phi_\infty(\partial B_t^2 \cap U')) \leq C\Big(\int_{B_r^2 \cap U'} |d\Phi_\infty|^2 \Big)^{1/2}$ and hence Proposition 2.5.1 is satisfied, if r is small enough. Identifying $\nu_k|_U$ with measures on U', we deduce that either $\nu_\infty(\overline{B}_t^2 \cap U') \geq c_Q$ or, for some $p \in \mathcal{M}$,

$$\begin{aligned}
\nu_\infty(B_t^2 \cap U') &\leq \liminf_{k\to\infty} \nu_k(B_t^2 \cap U') = \liminf_{k\to\infty} (\Phi_k|_{B_t^2 \cap U'})_*\nu_k(\mathcal{M}) \leq \liminf_{k\to\infty} \mu_k(B_{3s}(p)) \\
&\leq \mu_\infty(\overline{B}_{3s}(p)) \leq Cs^2 \leq C \int_{B_r^2} |d\Phi_\infty|^2.
\end{aligned}$$

The penultimate inequality follows from (2.4.5). For r small enough this second possibility cannot happen, since $\nu_\infty(\{0\}) > 0$. Hence we deduce $\nu_\infty(\{x\}) \geq c_Q$ and thus there are finitely many atoms.

Assume now that K is a compact set containing no atoms. Assume that $K \subset U$ for a chart domain U; we identify K with a compact subset of the unit ball or half unit ball U' as above.

We deal with the half-ball case, whose proof covers also the case $U' = B_1^2$. We denote $\partial U' := \{z \in U' : \Im(z) = 0\}$. Fix an intermediate set $K \subset V \subset\subset U'$ open in U' (hence, V is allowed to contain points in $\partial U'$). Since ν_∞ has no atoms on U, we can find a radius $r > 0$ such that $B_{5r}^2(y) \subseteq B_1^2$, $B_{5r}^2(y) \cap U' \subseteq V$ and $\nu_\infty(B_{5r}^2(y) \cap U') < c_Q$, for all $y \in K$.

Taking a maximal subset of centers $\{y_i'\} \subseteq K$ with pairwise distances at least $\frac{r}{2}$, we can cover K with a finite collection of balls $\{B_{r/2}^2(y_i')\}$ with $\sum_i \mathbf{1}_{B_{5r}^2(y_i')} \leq C$. If $B_r^2(y_i') \subseteq U'$ then we set $y_i := y_i'$ and $r_i := r$; otherwise we choose y_i to be a point in $B_r^2(y_i') \cap \partial U'$, and we set $r_i := 4r$. Note that $B_{r/2}^2(y_i') \subseteq B_{r_i/2}^2(y_i)$ and $B_{r_i}^2(y_i) \subseteq B_{5r}^2(y_i')$, so the collection of balls $B_{r_i/2}^2(y_i)$ still covers K and has $\sum_i \mathbf{1}_{B_{r_i}^2(y_i)} \leq C$. Moreover, either $B_{r_i}^2(y_i) \subseteq V$ or $y_i \in \partial U'$, with $B_{r_i}^2(y_i) \cap U' \subseteq V$. Also, $\nu_\infty(B_{r_i}^2(y_i) \cap U') < c_Q$.

We can fix $t_i \in (\frac{r_i}{2}, r_i)$ such that $\Phi_k|_{\partial B^2_{t_i}(y_i)\cap U'} \to \Phi_\infty|_{\partial B^2_{t_i}(y_i)\cap U'}$ in C^0 along a subsequence independent of i, and such that the diameter s_i of $\Phi_\infty(\partial B^2_{t_i}(y_i)\cap U')$ satisfies

$$s_i^2 \le C\int_{B^2_{r_i}(y_i)\cap U'} |d\Phi_\infty|^2.$$

We now work along this subsequence, which we do not relabel. By Proposition 2.5.1, if r was chosen small enough, any weak limit of the measures $(\Phi_k|_{B^2_{t_i}(y_i)\cap U'})_*\nu_k$ is supported in $B_{3s_i}(p_i)$ for some $p_i \in \mathcal{M}$. Hence,

$$\lim_{k\to\infty}(\Phi_k|_{B^2_{t_i}(y_i)\cap U'})_*\nu_k(\mathcal{M}\setminus B_{3s_i}(p_i)) = 0. \tag{2.5.2}$$

Since $\nu_k = \frac{1}{2}|d\Phi_k|^2\,\mathcal{L}^2$ on U', setting $h_i := (\mathrm{dist}(\cdot,p_i)-3s_i)^+$ we deduce that $h_i\circ\Phi_k \to 0$ in $W^{1,2}(B^2_{t_i}(y_i)\cap U')$. Hence, the essential image of $\Phi_\infty|_{B^2_{t_i}(y_i)\cap U'}$ is included in $\overline{B}_{3s_i}(p_i)$. We deduce

$$\begin{aligned}
\int_K \mathrm{dist}(\Phi_k,\Phi_\infty)\,d\nu_k &\le \sum_i \int_{B^2_{t_i}(y_i)\cap U'} \mathrm{dist}(\Phi_k,\Phi_\infty)\,d\nu_k\\
&\le \sum_i 6s_i\nu_k(B^2_{t_i}(y_i)\cap U')\\
&\quad + \mathrm{diam}(\mathcal{M})\sum_i (\Phi_k|_{B^2_{t_i}(y_i)\cap U'})_*\nu_k(\mathcal{M}\setminus B_{3s_i}(p_i))\\
&\le C(\sup s_i)\nu_k(V) + C\sum_i (\Phi_k|_{B^2_{t_i}(y_i)\cap U'})_*\nu_k(\mathcal{M}\setminus B_{3s_i}(p_i)).
\end{aligned}$$

In the limit $k\to\infty$, using (2.5.2), we get

$$\limsup_{k\to\infty}\int_K \mathrm{dist}(\Phi_k,\Phi_\infty)\,d\nu_k \le C(\sup_i s_i)\nu_\infty(\overline{V}).$$

Since we could arrange that $\sup_i s_i$ is arbitrarily small, we arrive at

$$\lim_{k\to\infty}\int_K \mathrm{dist}(\Phi_k,\Phi_\infty)\,d\nu_k = 0. \tag{2.5.3}$$

Also, choosing η so small that any ball $B^2_\eta(y)$ is included in some $B^2_{t_i}(y_i)$, for all $y\in K$, the essential oscillation of $\Phi_\infty|_{B^2_\eta(y)\cap U'}$ is then bounded by $\sup_i s_i$. Since the latter is arbitrarily small, it follows that Φ_∞ has a continuous representative on K, hence on $\widetilde{\Sigma}$.

Finally, if $\mathcal{L}^2(K)=0$ then, arguing as in the first part of the proof, we have

$$\begin{aligned}
\nu_\infty(K) &\le \sum_i \nu_\infty(B^2_{t_i}(y_i)\cap U') \le \liminf_{k\to\infty}\sum_i \nu_k(B^2_{t_i}(y_i)\cap U') \le C\sum_i s_i^2\\
&\le C\sum_i\int_{B^2_{r_i}(y_i)\cap U'}|d\Phi_\infty|^2 \le C\int_V |d\Phi_\infty|^2.
\end{aligned}$$

Since V is an arbitrary neighborhood of K, we deduce $\nu_\infty(K)\le C\int_K|d\Phi_\infty|^2 = 0$. The absolute continuity of ν_∞ with respect to vol_{g_0} on $\widetilde{\Sigma}$ follows.

Finally, given ω as in the statement and covering $\overline{\omega}$ with finitely many charts, it follows from (2.5.3) that $\lim_{k\to\infty}\int_{\overline{\omega}}\operatorname{dist}(\Phi_k,\Phi_\infty)\,d\nu_k=0$. Hence, for any $\psi\in C^0(\mathcal{M})$,

$$\lim_{k\to\infty}\int_\omega \psi\circ\Phi_k\,d\nu_k=\lim_{k\to\infty}\int_\omega \psi\circ\Phi_\infty\,d\nu_k=\int_\omega \psi\circ\Phi_\infty\,d\nu_\infty$$

the last equality coming from the continuity of $\psi\circ\Phi_\infty$ near $\overline{\omega}$ and the assumption $\nu_\infty(\partial\omega)=0$. The weak convergence $(\Phi_k|_\omega)_*\nu_k\rightharpoonup(\Phi_\infty|_\omega)_*\nu_\infty$ follows. □

Theorem 2.5.3. *The absolutely continuous part of ν_∞, which we denote $m\operatorname{vol}_{g_0}$, has $m=0$ a.e. on the set of points where $d\Phi_\infty$ does not have rank 2. Moreover, $m=NJ(d\Phi_\infty)$ for a bounded, integer valued function $N\geq 1$.*

In the statement $J(d\Phi_\infty)$ denotes the Jacobian of Φ_∞ with respect to the volume form vol_{g_0}. Hence, in a conformal chart, we are asserting that the absolutely continuous part of ν_∞ is $N|\partial_1\Phi_\infty\wedge\partial_2\Phi_\infty|\,\mathcal{L}^2$.

Proof. Working in a conformal chart for $\operatorname{int}(\Sigma)$, we fix a point x which is Lebesgue for $d\Phi_\infty$, and such that $\nu_\infty(\{x\})=0$. We have to show that $\frac{\nu_\infty(B_r^2(x))}{\pi r^2}\to N|\partial_1\Phi_\infty\wedge\partial_2\Phi_\infty|(x)$ for some bounded integer $N\geq 1$, as $r\to 0$ along some sequence.

We can assume $x=0$. For all $r>0$ small enough, call $\mathbf{v}_{k,r}$ the varifold induced by $\Phi_k|_{B_r^2}$. We can select an arbitrarily small r such that the trace $\Phi_\infty|_{\partial B_r^2}$ has

$$\Phi_\infty(ry)=\Phi_\infty(0)+rd\Phi_\infty(0)[y]+o(r)\quad\text{for }|y|=1$$

and such that the traces $\Phi_k|_{\partial B_r^2}$ converge subsequentially to $\Phi_\infty|_{\partial B_r^2}$ in C^0 (see Lemma A.4 and Lemma A.5). By Proposition 2.5.1, any (subsequential) weak limit of $|\mathbf{v}_{k,r}|$ is supported in a ball $B_{Cr}(p)$, with $p:=\Phi_\infty(0)$ and C depending also on $|d\Phi_\infty(0)|$.

Moreover, any (subsequential) limit $\mathbf{v}=\lim_{k\to\infty}\mathbf{v}_{k,r}$ is stationary in $\mathcal{M}\setminus\Phi_\infty(\partial B_r^2)$ and satisfies $|\mathbf{v}|(B_s(q))\leq Cs^2$ for all $q\in\mathcal{M}$, since the varifolds $\mathbf{v}_k$ induced by Φ_k (from the full domain) have trivially $|\mathbf{v}_k|\geq|\mathbf{v}_{k,r}|$ and, by Theorem 2.3.4, they converge subsequentially to a free boundary stationary varifold $\mathbf{v}_\infty$, for which (2.4.5) gives the desired bound.

Hence, with a diagonal argument, we may find a subsequence of k's (not relabeled) and a sequence of radii $r_k\to 0$ such that the dilated varifolds $\mathbf{v}'_k:=(r_k^{-1}(\cdot-p))_*\mathbf{v}_{k,r_k}$ in $\mathbb{R}^Q$ form a tight sequence, converging to a varifold $\mathbf{v}'_\infty$ which has

$$|\mathbf{v}'_\infty|(B_s^Q(q))\leq Cs^2\quad\text{for all }q\in\mathbb{R}^Q\text{ and all }s>0 \tag{2.5.4}$$

with a constant C independent of x, has compact support and is stationary in $\mathbb{R}^Q\setminus\mathcal{C}$, with

$$\mathcal{C}=\lim_{k\to\infty}(r_k^{-1}\Phi_\infty(\partial B_{r_k}^2)-p)=\{d\Phi_\infty(0)[y]\mid y\in\partial B_1^2\}.$$

We can also assume that

$$r_k^{-2}\sigma_k^4\int_{B_{r_k}^2}f_k^2\,d\nu_k\to 0,\quad r_k^{-1}\sigma_k\to 0, \tag{2.5.5}$$

and that

$$|\mathbf{v}'_\infty|(\mathbb{R}^Q) = \lim_{k\to\infty} |\mathbf{v}'_k|(\mathbb{R}^Q) = \lim_{k\to\infty} \frac{\nu_\infty(B^2_{r_k})}{r_k^2} = \lim_{k\to\infty} \frac{\nu_k(B^2_{r_k})}{r_k^2};$$

since the convex hull $\mathrm{co}(\mathcal{C})$ of $\mathcal{C}$ has area $\pi|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|(0)$, we are left to show that

$$|\mathbf{v}'_\infty|(\mathbb{R}^Q) = N\mathcal{H}^2(\mathrm{co}(\mathcal{C}))$$

for some bounded integer $N \geq 1$. By [98, Theorem 19.2], which holds for general varifolds, $|\mathbf{v}'_\infty|$ is supported in the convex hull of $\mathcal{C}$. If $d\Phi_\infty(0)$ has rank less than 2, then $\mathcal{C}$ is either a segment or a point. Hence, we can cover it with $O(s^{-1})$ balls of radius s; recalling (2.5.4), we deduce $|\mathbf{v}'_\infty|(\mathcal{C}) = 0$ and hence $\mathbf{v}'_\infty = 0$. Thus the claim follows in this case.

If instead $d\Phi_\infty(0)$ has rank 2, we first observe that the area of the map $\Psi_k := r_k^{-1}(\Phi_k|_{B^2_{r_k}} - p)$ is, up to an infinitesimal error, at least the area of $\mathrm{co}(\mathcal{C})$ in the plane Π containing it: this follows immediately considering the composition $\overline{\Psi}_k$ of this map with the projection onto Π, and noting that any compact subset $K \subset \mathrm{co}(\mathcal{C}) \setminus \mathcal{C}$ belongs eventually to the image of $\overline{\Psi}_k$, since $\overline{\Psi}_k$ has (eventually) nontrivial degree relative to the points in K. Hence,

$$\mathcal{H}^2(\mathrm{co}(\mathcal{C})) \leq \lim_{k\to\infty} \frac{\nu_k(B^2_{r_k})}{r_k^2} = |\mathbf{v}'_\infty|(\mathbb{R}^Q). \tag{2.5.6}$$

Up to rotations, we can assume $\Pi = \mathbb{R}^2 \times \{0\}$. Since $\mathcal{C}$ is a smooth curve, we have $|\mathbf{v}'_\infty|(\mathcal{C}) = 0$. Also, $\mathbf{v}'_\infty$ is stationary on $\mathbb{R}^Q \setminus \mathcal{C}$ and supported on Π. By the constancy theorem [98, Theorem 41.1], it follows that $\mathbf{v}'_\infty$ is rectifiable and equals a multiple N of $\mathrm{co}(\mathcal{C})$. By (2.5.6) we have $N \geq 1$, while from (2.5.4) it follows that $N \leq C$. We are left to show $N \in \mathbb{N}$.

Note that $\mathbf{v}'_k$ is the varifold induced by Ψ_k; hence, the varifold convergence $\mathbf{v}'_k \rightharpoonup \mathbf{v}'_\infty$ implies that

$$\int_{B^2_{r_k}} |d\Psi^j_k|^2 \to 0 \quad \text{for } j = 3, \dots, Q, \tag{2.5.7}$$

where we write $\Psi_k = (\Psi^1_k, \dots, \Psi^Q_k)$.

Fix $\alpha > 0$ such that $\mathcal{C}$ encloses a ball $B^2_{2\alpha}$ in the plane Π. Consider a family (ρ_τ) of mollifiers in $\mathbb{R}^Q$, namely nonnegative smooth functions supported in B^Q_τ with $\int_{\mathbb{R}^Q} \rho_\tau = 1$ and $|d\rho_\tau| \leq C\tau^{-Q-1}$. For any vector field $X \in C^\infty_c(B^2_\alpha, \mathbb{R}^2)$, viewing X as a vector field on $\mathbb{R}^Q$, constant in the last $Q-2$ variables, we define the vector fields X_k and Y_k on $\mathcal{M}_k := r_k^{-1}(\mathcal{M} - p)$ given pointwise by the projection of X and $\rho_{\tau_k} * X$ onto the tangent space to $\mathcal{M}_k$, respectively, with $\tau_k := r_k^{-1}\sigma_k$.

Since $\mathcal{M}_k$ converges to an m-plane graphically (in any neighborhood of 0), we have

$$|\nabla^{\mathcal{M}_k}_v X_k - \nabla^{\mathbb{R}^Q}_v X| \leq \delta_k \|dX\|_{L^\infty}|v| \tag{2.5.8}$$

for some sequence $\delta_k \to 0$ and any $v \in T\mathcal{M}_k$. Also, we have

$$|(\nabla^{\mathcal{M}_k})^2 Y_k| \le C\|\rho_{\tau_k} * X\|_{C^2} \le C\tau_k^{-1}\|dX\|_{L^\infty}.$$

Note that Ψ_k, when extended to Σ with the same formula $r_k^{-1}(\Phi_k - p)$, is τ_k^5-critical for E_{τ_k} for the manifold $\mathcal{M}_k$ and the corresponding Finsler manifold $\mathfrak{M}_k$: indeed, identifying $T_{\Phi_k}\mathfrak{M}$ and $T_{\Psi_k}\mathfrak{M}_k$ with subsets of $W^{2,4}(\Sigma, \mathbb{R}^Q)$, for all $w \in T_{\Psi_k}\mathfrak{M}_k$ we have

$$|dE_{\tau_k}(\Psi_k)[w]| = r_k^{-2}|dE_{\sigma_k}(\Phi_k)[r_k w]| \le r_k^{-1}\sigma_k^5\|w\|_{\Phi_k} \le r_k^2\tau_k^5\|w\|_{\Phi_k}$$

and it is immediate to check that $\|w\|_{\Phi_k} \le r_k^{-3/2}\|w\|_{\Psi_k} \le r_k^{-2}\|w\|_{\Psi_k}$ (assuming $r_k \le 1$).

For the vector field Y_k, recalling (2.3.4), the term $f\langle\nabla n, \nabla\omega\rangle$ in (2.3.3) is bounded by

$$\begin{aligned}
&C|\mathrm{II}^{\Psi_k}|^4\|\nabla^{\mathcal{M}_k}Y_k\|_{L^\infty} + C|\mathrm{II}^{\Psi_k}|^3\|(\nabla^{\mathcal{M}_k})^2 Y_k\|_{L^\infty} \\
&\le C(|\mathrm{II}^{\Psi_k}|^4 + \tau_k^{-1}|\mathrm{II}^{\Psi_k}|^3)\|dX\|_{L^\infty}.
\end{aligned}$$

We now use the almost criticality of Ψ_k with the infinitesimal variation $Y_k(\Psi_k)$, or more precisely $Y_k(\Psi_k)\mathbf{1}_{B^2_{r_k}}$ for the extension $r_k^{-1}(\Phi_k - p) \in \mathfrak{M}_k$ of Ψ_k. For k large enough, $\Psi_k(\partial B^2_{r_k})$ does not intersect $B^2_{2\alpha} \times \mathbb{R}^{Q-2}$, where Y_k is supported, and hence this is an admissible variation. As in the proof of Theorem 2.3.4, since the immersions Ψ_k have bounded area we obtain

$$\begin{aligned}
\|Y_k(\Psi_k)\|_{\Psi_k} &\le C\|dX\|_{L^\infty} + C\|(\nabla^{\mathcal{M}_k})^2 Y_k\|_{L^\infty} + \|dX\|\Big(\int_{B^2_{r_k}} |\mathrm{II}^{\Psi_k}|^4 \operatorname{vol}_{\Psi_k}\Big)^{1/4} \\
&\le C\tau_k^{-1}\|dX\|_{L^\infty}\Big(1 + \Big(\int_{B^2_{r_k}} \tau_k^4|\mathrm{II}^{\Psi_k}|^4 \operatorname{vol}_{\Psi_k}\Big)^{1/4}\Big)
\end{aligned}$$

for some C independent of X. Hence, (2.3.3) and the τ_k^5-criticality of Ψ_k, together with Young's inequality, give

$$\begin{aligned}
\Big|\int_{B^2_{r_k}} \langle\partial_i\Psi_k, \nabla Y_k(\Psi_k)[\partial_i\Psi_k]\rangle\Big| &\le C\|dX\|_{L^\infty}\int_{B^2_{r_k}} (\tau_k^4|\mathrm{II}^{\Psi_k}|^4 + \tau_k^3|\mathrm{II}^{\Psi_k}|^3) \operatorname{vol}_{\Psi_k} \\
&\quad + C\tau_k^4\|dX\|_{L^\infty}.
\end{aligned}$$

Since $\tau_k^4|\mathrm{II}^{\Psi_k}|^4 = \sigma_k^4|\mathrm{II}_k^{\Phi}|^4$, Hölder's inequality gives the upper bound

$$C\Big(r_k^{-2}\sigma_k^4\int_{B^2_{r_k}} |\mathrm{II}^{\Phi_k}|^4 \operatorname{vol}_{\Phi_k} + \Big(r_k^{-2}\sigma_k^4\int_{B^2_{r_k}} |\mathrm{II}^{\Phi_k}|^4 \operatorname{vol}_{\Phi_k}\Big)^{3/4} + \tau_k^4\Big)\|dX\|_{L^\infty}$$

for the previous right-hand side. In view of (2.5.5), it follows that

$$\Big|\int_{B^2_{r_k}} \langle\partial_i\Psi_k, \nabla Y_k(\Psi_k)[\partial_i\Psi_k]\rangle\Big| \le \delta_k'\|dX\|_{L^\infty}$$

for some $\delta_k' \to 0$ independent of X. Also, replacing Y_k with X_k is not harmful, since the last integral is the first variation of the area and thus

$$\begin{aligned}
&\Big| \int_{B^2_{r_k}} \langle \partial_i \Psi_k, \nabla Y_k(\Psi_k)[\partial_i \Psi_k] \rangle - \int_{B^2_{r_k}} \langle \partial_i \Psi_k, \nabla X_k(\Psi_k)[\partial_i \Psi_k] \rangle \Big| \\
&\le 2 \int_{B^2_{r_k}} |H^{\Psi_k}| |\rho_{\tau_k} * X - X| \operatorname{vol}_{\Psi_k} \\
&\le C\tau_k \|dX\|_{L^\infty} \int_{B^2_{r_k}} |\mathbb{I}^{\Psi_k}| \operatorname{vol}_{\Psi_k}
\end{aligned}$$

is infinitesimal with respect to $\|dX\|_{L^\infty}$. Choose now $X := \varphi(x_1, x_2) e_1$. Writing $\Psi_k = (\Psi_k^1, \dots, \Psi_k^m)$, in view of (2.5.8) the previous integral (with X_k replacing Y_k) is just

$$\int_{B^2_{r_k}} (\partial_1 \varphi(\Psi_k) \partial_i \Psi_k^1 \partial_i \Psi_k^1 + \partial_2 \varphi(\Psi_k) \partial_i \Psi_k^1 \partial_i \Psi_k^2),$$

up to another infinitesimal error. Let $J_k := |\partial_1 \Psi_k^1 \partial_2 \Psi_k^2 - \partial_2 \Psi_k^1 \partial_1 \Psi_k^2|$ denote the Jacobian of the composition of Ψ with the projection onto Π. Using (2.5.7), this integral equals

$$\int_{B^2_{r_k}} J_k \partial_1 \varphi(\Psi_k) \operatorname{vol}_{\Psi_k}$$

plus an error which is infinitesimal with respect to $\|d\varphi\|_{L^\infty}$ (see also Lemma A.6). Hence, by the area formula, the projection $\mathbf{v}_k''$ of $\mathbf{v}_k'$ onto Π has an integer multiplicity N_k satisfying

$$\Big| \int_{B^2_\alpha} N_k \partial_1 \varphi \, d\mathcal{L}^2 \Big| \le \delta_k'' \|dX\|_{L^\infty} \quad \text{with } \delta_k'' \to 0$$

and, using the vector field $\varphi(x_1, x_2) e_2$, the same holds replacing $\partial_1 \varphi$ with $\partial_2 \varphi$. So, by Allard's strong constancy lemma [4, Theorem 1.(4)], it follows that N_k is close in L^1 to a constant $\overline{N}_k$ on the ball $B^2_{\alpha/2}$, with a distance $O(\delta_k'')$. As N_k is integer valued, it follows that $\operatorname{dist}(\overline{N}_k, \mathbb{N}) \to 0$. Finally, since $\mathbf{v}_k''$ converges to $\mathbf{v}_\infty'$, we have

$$\pi(\alpha/2)^2 N = \lim_{k \to \infty} \int_{B^2_{\alpha/2}} N_k \, d\mathcal{L}^2 = \lim_{k \to \infty} \pi(\alpha/2)^2 \overline{N}_k$$

and we deduce $N \in \mathbb{N}$. □

Remark 2.5.4. Note that, in the previous proof, testing immediately the stationarity of Ψ_k against X_k would have run into trouble, since we would have got a bound for $\int_{B^2_\alpha} N_k \partial_i \varphi$ depending also on the Hessian of X, making it impossible to apply Allard's strong constancy lemma.

Definition 2.5.5. Given an open set $\omega \subseteq \widetilde{\Sigma}$, we define the subset $\mathcal{G}_\omega \subseteq \omega \setminus \partial\Sigma$ of Lebesgue points for $d\Phi_\infty$ where this differential has rank 2. We equip the image $\Phi_\infty(\mathcal{G}_\omega)$ with the multiplicity

$$\theta_\omega(p) := \sum_{x \in \mathcal{G}_\omega \cap \Phi_\infty^{-1}(p)} N(x).$$

Note that, by the area formula (see, e.g., Lemma A.2), $\Phi_\infty(\mathcal{G}_\omega)$ is 2-rectifiable and θ_ω is well defined as a function in $L^1(\mathcal{H}^2 \llcorner \Phi_\infty(\mathcal{G}_\omega))$. We can then view this set, with multiplicity θ_ω, as a rectifiable varifold in $\mathcal{M}$, which we call $\mathbf{v}_\omega$.

Note that, by Theorem 2.5.2, Theorem 2.5.3 and the area formula, the weight $|\mathbf{v}_\omega|$ coincides with $(\Phi_\infty|_\omega)_*\nu_\infty$.

Proposition 2.5.6. *Given an open subset $\omega \subset\subset \widetilde{\Sigma}$ with $\nu_\infty(\partial\omega) = 0$, the immersions $\Phi_k|_\omega$ converge to the varifold $\mathbf{v}_\omega$.*

Proof. By splitting ω into finitely many pieces with ν_∞-negligible boundary, we can reduce to the case that ω is contained in a local chart; in the sequel, we identify ω with a subset of $\mathbb{C} = \mathbb{R}^2$.

Let $\mathbf{v}_{k,\omega} := (\Phi_k)_*(\omega)$ be the varifold induced by $\Phi_k|_\omega$. Viewing $\mathbf{v}_{k,\omega}$ (for $k \le \infty$) as a varifold in $\mathbb{R}^Q$, by the area formula and Theorem 2.5.3 it suffices to show that

$$\int_\omega \varphi(\Phi_k(x), d\Phi_k(x)[T_x\Sigma])\, d\nu_k(x) \to \int_\omega \varphi(\Phi_\infty(x), d\Phi_\infty(x)[T_x\Sigma])\, d\nu_\infty(x) \tag{2.5.9}$$

for any $\varphi \in C^1_c(\mathrm{Gr}_2(\mathbb{R}^Q))$. The last integrand is meant to be zero at points where $d\Phi_\infty$ does not have full rank. In order to simplify the notation, we indicate the plane $d\Phi_k(x)[T_x\Sigma] = d\Phi_k(x)[\mathbb{R}^2]$ by $\Pi_k(x)$.

Let $\mathcal{G}'_\omega$ be the subset of $\mathcal{G}_\omega$ consisting of the points x where additionally $\int_{B^2_r(x)} |d\Phi_\infty - d\Phi_\infty(x)|^2 = o(r^2)$. For any point $x \in \mathcal{G}'_\omega$, pick a sequence of radii r satisfying

$$|\Phi_\infty(x + ry) - \Phi_\infty(x) - d\Phi_\infty(x)[ry]| = o(r) \quad \text{for } |y| = 1.$$

Given any $\varepsilon > 0$, we claim that

$$\limsup_{k\to\infty} \int_{B^2_r(x)} |\varphi(\Phi_k, \Pi_k) - \varphi(\Phi_\infty(x), \Pi_\infty(x))|\, d\nu_k \le \varepsilon r^2 \tag{2.5.10}$$

for r small enough in this sequence. If this is not true, then using a diagonal argument as in the proof of Theorem 2.5.3 we may find a subsequence of k's (not relabeled) and radii $r_k \to 0$ such that

$$\int_{B^2_{r_k}(x)} |\varphi(\Phi_k, \Pi_k) - \varphi(\Phi_\infty(x), \Pi_\infty(x))|\, d\nu_k \ge \varepsilon r_k^2 \tag{2.5.11}$$

as well as

$$r_k^{-2}|\nu_k(B^2_{r_k}(x)) - \nu_\infty(B^2_{r_k}(x))| \to 0,$$

and such that the varifolds induced by $\Psi_k := r_k^{-1}(\Phi_k|_{B^2_{r_k}(x)} - \Phi_\infty(x))$ converge tightly to a rectifiable varifold $\mathbf{v}'$ supported in a bounded subset of $\Pi_\infty(x)$. In particular, the ν_k-measure of the set of points in $B^2_{r_k}(x)$ where $|\Psi_k| > r_k^{-1/2}$ is $o(r_k^2)$. Since $\Phi_k = \Phi_\infty(x) + r_k\Psi_k$, we deduce that

$$\int_{B^2_{r_k}(x)} |\varphi(\Phi_k,\Pi_k) - \varphi(\Phi_\infty(x),\Pi_k)|\, d\nu_k \le r_k^{1/2}\|d\varphi\|_{L^\infty}\nu_k(B^2_{r_k}(x)) + o(r_k^2) = o(r_k^2) \tag{2.5.12}$$

as $k \to \infty$. Also, testing the tight varifold convergence of Ψ_k to $\mathbf{v}'$ against the function $|\varphi(\Phi_\infty(x),\cdot) - \varphi(\Phi_\infty(x),\Pi_\infty(x))|$, we get

$$\int_{B^2_{r_k}(x)} |\varphi(\Phi_\infty(x),\Pi_k) - \varphi(\Phi_\infty(x),\Pi_\infty(x))|\, d\nu_k = o(r_k^2). \tag{2.5.13}$$

Combining (2.5.12) with (2.5.13) we get a contradiction to (2.5.11). By the Besicovitch covering lemma, we can then cover any fixed compact set $K \subseteq \mathcal{G}'_\omega$ with finitely many balls $\{B_j\}$ included in ω such that (2.5.10) holds, for $B_j = B^2_{r_j}(x_j)$ in place of $B^2_r(x)$, as well as

$$\int_{B_j} |\varphi(\Phi_\infty,\Pi_\infty) - \varphi(\Phi_\infty(x_j),\Pi_\infty(x_j))|\, d\nu_\infty \le \varepsilon r_j^2,$$

using the approximate continuity of $d\Phi_\infty$ at points in K, and such that $\sum_j \mathbf{1}_{B_j} \le C$.

Let $\mathcal{B}_j := B_j \setminus \bigcup_{\ell<j} B_\ell$ and $U := \bigcup_j B_j = \bigcup_j \mathcal{B}_j \subseteq \omega$. Since $\sum_j r_j^2 \le C$, we deduce that

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int_U \varphi(\Phi_k,\Pi_k)\, d\nu_k - \int_U \varphi(\Phi_\infty,\Pi_\infty)\, d\nu_\infty\right| \\
&\le \sum_j \limsup_{k\to\infty}\left|\int_{\mathcal{B}_j} \varphi(\Phi_\infty(x_j),\Pi_\infty(x_j))\, d\nu_k - \int_{\mathcal{B}_j} \varphi(\Phi_\infty(x_j),\Pi_\infty(x_j))\, d\nu_\infty\right| + C\varepsilon.
\end{aligned}$$

The sum vanishes, since $\nu_\infty(\partial\mathcal{B}_j) = 0$. Also, since $\nu_\infty(\partial\omega) = 0$, we have

$$\limsup_{k\to\infty} \nu_k(\omega\setminus U) \le \nu_\infty(\omega\setminus U) \le \nu_\infty(\omega\setminus K)$$

and this quantity can be made arbitrarily small, proving (2.5.9). □

Definition 2.5.7. We say that a property holds *for a.e.* $\omega \subseteq \Sigma$ if, for every nonnegative $\rho \in C^\infty(\Sigma)$, it holds for a.e. superlevel set $\{\rho > \lambda\}$ with $\lambda > 0$. Similarly, we say that it holds *for a.e.* $\omega \subset\subset \mathrm{int}(\Sigma)$ if we have the same for every nonnegative $\rho \in C^\infty_c(\mathrm{int}(\Sigma))$.

Definition 2.5.8. A map $\Phi \in W^{1,2}(\Sigma)$ is *weakly conformal* if, for a.e. $x \in \Sigma$, its differential at x is zero or a linear conformal map $T_x\Sigma \to T_{\Phi(x)}\mathcal{M}$.

Definition 2.5.9. Let Σ be a compact Riemann surface, possibly with boundary, $\Phi \in W^{1,2}(\Sigma,\mathcal{M})$ weakly conformal with $\Phi(\partial\Sigma) \subseteq \mathcal{N}$, and $N \in L^\infty(\Sigma,\{1,2,\dots\})$. The triple (Σ,Φ,N) is a *parametrized free boundary stationary varifold* if, for almost every $\omega \subseteq \Sigma$, the varifold $\Phi_*(N\omega)$ is free boundary stationary (for $\mathcal{M},\mathcal{N}$) outside $\Phi_\infty(\partial\omega)$ (see Definition 2.3.1) and if, for almost every $\omega \subset\subset \mathrm{int}(\Sigma)$, $\Phi_*(N\omega)$ is stationary outside $\Phi_\infty(\partial\omega)$.

The pushforward $\Phi_*(N\omega)$ in this definition has to be interpreted as the varifold $\mathbf{v}_\omega$ in Definition 2.5.5, using the subset of ω made of Lebesgue points, for Φ and $d\Phi$, where $d\Phi$ has rank 2.

Remark 2.5.10. In this definition, $N\omega$ is viewed as a 2-varifold in the surface Σ, equipped with a metric compatible with the conformal structure; however, $\Phi_*(N\omega)$ is independent of the choice of the metric. Note that $\partial\omega$ is a compact one-dimensional submanifold and the trace $\Phi|_{\partial\omega}$ has a continuous representative for a.e. ω: this follows, e.g., from [35, Theorems 4.21 and 5.7] applied to the regular level sets of f; $\Phi(\partial\omega)$ implicitly refers to the (compact) image by means of this continuous map. Note also that the definition does not depend on the representatives of Φ and N.

Theorem 2.5.11. *There exists a compact Riemann surface Σ' and a quasiconformal homeomorphism $\varphi : \Sigma' \to \Sigma$ such that $(\Sigma', \Phi_\infty \circ \varphi, N \circ \varphi)$ is a free boundary parametrized stationary varifold for $(\mathcal{M}, \mathcal{N})$.*

We refer the reader to [56, Chapter 4] for basic properties of quasiconformal homeomorphisms.

Proof. For a.e. $\omega \subseteq \Sigma$, $\Phi_k|_{\partial\omega}$ converges in C^0 to $\Phi_\infty|_{\partial\omega}$ (up to subsequences) and $\partial\omega \cap \mathcal{A} = \emptyset$, with $\mathcal{A}$ the finite set of atoms for ν_∞.

With respect to the fixed metric g_0 on Σ, we can find an arbitrarily small radius $r > 0$ such that for any $x \in \omega \cap \mathcal{A}$ we have $B_r(x) \subset\subset \omega$ and $\Phi_k|_{\partial B_r(x)}$ also converges in C^0 to $\Phi_\infty|_{\partial B_r(x)}$ (up to subsequences). Let $\widetilde{\omega} := \omega \setminus \bigcup_{x\in\omega\cap\mathcal{A}} \overline{B}_r(x)$.

Repeating the proof of Theorem 2.3.4 with vector fields in $\mathcal{X}_{fb}$ supported outside $\Phi_\infty(\partial\widetilde{\omega})$, we deduce that the varifold limit of $\Phi_k|_{\widetilde{\omega}}$ is free boundary stationary outside this set; by Proposition 2.5.6, this limit is $\mathbf{v}_{\widetilde{\omega}}$. Since the images $\Phi_\infty(\partial B_r(x))$, for $x \in \omega \cap \mathcal{A}$, have arbitrarily small diameter (see, e.g., Lemma A.3), we deduce that $\mathbf{v}_\omega$ is free boundary stationary outside ($\Phi_\infty(\partial\omega)$ and) a finite set F. However, since Φ_k also converges to the free boundary stationary varifold $\mathbf{v}_\infty \geq \mathbf{v}_\omega$, by (2.4.5) we get $|\mathbf{v}_\omega|(B_s(p)) \leq Cs^2$ for $p \in F$. Hence, given $X \in \mathcal{X}_{fb}$ supported outside $\Phi_\infty(\partial\omega)$, we can multiply it by the product $\Pi_{p\in F}\varphi_p$ of cut-off functions φ_p, with $\varphi_p = 0$ on $B_{s/2}(p)$, $\varphi_p = 1$ outside $B_s(p)$ and $|d\varphi_p| \leq Cs^{-1}$. It is then straightforward to check that the stationarity with respect to the cut-off vector field gives the one for X, as $s \to 0$. The proof that $\mathbf{v}_\omega$ is stationary for a.e. $\omega \subset\subset \operatorname{int}(\Sigma)$ is analogous.

Finally, we show how to obtain a weakly conformal reparametrization. Note that, by Theorem 2.5.3,

$$N|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \geq \frac{1}{2}|d\Phi_\infty|^2$$

a.e. in a local conformal chart $h : U \to U'$ (with $U \subseteq \Sigma$), since the left-hand side is the density of ν_∞ (in U') and thus, for any open set $V \subset\subset U \cap \widetilde{\Sigma}$,

$$\int_V \frac{1}{2}|d\Phi_\infty|^2 \, d\mathcal{L}^2 \le \liminf_{k\to\infty} \nu_k(V) \le \nu_\infty(\overline{V}) = \int_{\overline{V}} N|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \, d\mathcal{L}^2,$$

from which the previous claim follows. Call $\overline{C}$ an upper bound for N and assume that the image U' of the chart is either the ball or the half-ball. Arguing as in the first part of the proof of Theorem 3.4.1, we can find a $\frac{\overline{C}^2-1}{\overline{C}^2+1}$-quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ such that $\Phi_\infty \circ h^{-1} \circ \psi^{-1}$ is weakly conformal on $\psi(U')$; using the Riemann mapping theorem and Carathéodory's theorem, by composing ψ with a conformal map, we can replace ψ with a homeomorphism $U' \to U'$, with the additional property that it preserves $U' \cap \{\Im(z) = 0\}$ (as a set) in the half-ball case. Recall that ψ^{-1} is quasiconformal, as well (see, e.g., [56, Theorem 4.10 and Proposition 4.2]).

Set $\theta := h^{-1} \circ \psi^{-1} : U' \to U$. Note that, given two overlapping charts U_1, U_2, the corresponding homeomorphisms θ_1 and θ_2 differ by a conformal map, namely $\theta_1^{-1} \circ \theta_2$ is conformal on $\theta_2^{-1}(U_1 \cap U_2)$. This holds since a.e. the differential $d\Phi_\infty$ either vanishes or has rank 2 and, by construction, θ_i is weakly conformal at a.e. x_i such that $d\Phi_\infty(\theta_i(x_i)) = 0$; on the other hand, the two maps $d(\Psi_\infty \circ \theta_i)(x_i) = d\Psi_\infty(\theta_i(x_i)) \circ d\theta_i(x_i)$, with $x_1 := \theta_1^{-1} \circ \theta_2(x_2)$, are both nontrivial linear conformal maps for a.e. x_2 such that $d\Phi_\infty(\theta_2(x_2)) \ne 0$, so that $d\theta_1(x_1)^{-1} \circ d\theta_2(x_2)$ is conformal at these points.

In this argument we used the facts that a quasiconformal homeomorphism carries negligible sets to negligible sets [56, Lemma 4.12] and satisfies the chain rule [65, Lemma III.6.4].

Note that the Cauchy–Riemann equations satisfied by $\theta_1^{-1} \circ \theta_2$ give its smoothness away from the boundary and, by the Schwarz reflection principle, this map is smooth up to $\theta_2^{-1}(\partial\Sigma \cap U_1 \cap U_2)$. Thus, the maps θ^{-1} define an atlas for a new smooth and conformal structure on Σ; calling Σ' a copy of Σ with this structure, we can just take φ to be the identity $\Sigma' \to \Sigma$.

Finally, as explained in Proposition 2.7.2 (whose proof does not use that Φ is weakly conformal), the stationarity property holds on Σ for all domains; the same then holds for Σ'. □

2.6 Degeneration of the conformal structure and bubbling

In this section we describe how to recover all the area in the limit as a sum of masses of parametrized (free boundary) stationary varifolds, without the assumption that the maps Φ_k induce a fixed conformal structure on Σ.

Namely, denoting $\mathbf{v}_k$ the varifold induced by Φ_k as in Section 2.2, we show that the limit varifold $\mathbf{v}_\infty$ is the superposition of finitely many parametrized free boundary stationary varifolds.

Before dealing with possible concentration points, we focus on how to remove the assumption of the fixed conformal structure.

First of all, recall that on the oriented surface Σ, which can be assumed to be connected, there exists a metric g_k which is conformal to the induced metric $g_{\Phi_k} = \Phi_k^* g$, has constant Gaussian curvature either 1, 0, or -1, and makes the boundary $\partial\Sigma$ geodesic. Precisely, the curvature is 1 if Σ is (diffeomorphic to) a sphere or a disk, 0 if Σ is a torus or an annulus, and -1 otherwise. This is in agreement with Gauss–Bonnet, which says that the sign of this constant curvature agrees with the sign of the Euler characteristic of Σ, given by $\chi(\Sigma) = 2 - 2g - b$, with g the genus and b the number of boundary components.

We also recall that (Σ, g_k), up to a change of orientation, is conformal to a surface Σ_k which is the standard sphere, hemisphere, a torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\lambda_k)$ (where we can assume $|\lambda_k| \geq 1$, $|\Re(\lambda_k)| \leq \frac{1}{2}$), an annulus $S^1 \times [0, \ell_k]$, when Σ is (diffeomorphic to) the sphere, the disk, the torus, the annulus, respectively.[1]

Hence, when Σ is the sphere, up to precomposing Φ_k with a diffeomorphism we can assume that Φ_k induces the standard conformal structure on $\Sigma = S^2$; note that this leaves the diffeomorphism invariant conditions (2.2.5) and (2.2.6) unchanged. The same holds for the disk case.

Before discussing the other situations, let us state a useful modification of Proposition 2.5.1.

Proposition 2.6.1. *Consider open domains $U_k \subseteq \Sigma$ whose boundary ∂U_k is contained in the union of two compact curves $\alpha_{k,1}$ and $\alpha_{k,2}$, and call $d_{k,i}$ the diameter of $\Phi_k(\alpha_{k,i})$. Assume that U_k does not contain any entire boundary component of Σ. Then*

$$\limsup_{k\to\infty} \nu_k(U_k) \leq \delta\Big(\limsup_{k\to\infty} \max\{d_{k,1}, d_{k,2}\}, C\Big),$$

unless the left-hand side is at least c_Q, the same constant appearing in Proposition 2.5.1. In the last inequality, C is a constant depending only on (Φ_k) and the function δ is given by Lemma A.12.

In this statement, U_k may contain points in $\partial\Sigma$ and $\partial U_k = \overline{U}_k \setminus U_k$ denotes its topological boundary in Σ.

Proof. Note that $\Phi_k(\alpha_{k,i})$ is contained in a ball $\overline{B}_{d_{k,i}}(p_{k,i})$. After extracting a subsequence realizing $\limsup_{k\to\infty} \nu_k(U_k)$, we can also assume that $p_{k,i}$ and $d_{k,i}$ converge to p_i and d_i.

The proof is now analogous to the one of Proposition 2.5.1: the maps $\Phi_k|_{U_k}$ induce varifolds whose (subsequential) limit is free boundary stationary on the complement of $\overline{B}_{d_1}(p_1) \cup \overline{B}_{d_2}(p_2)$, has mass at most Cr^2 on balls of radius r, and has density bounded below by a constant $c < 1$ (the same as in that proof). The claim follows from Lemma A.12. □

Flat case

We now treat the torus case in detail, deferring the other cases to a later discussion.

[1] This is an easy consequence of the Riemannian uniformization theorem, applied to (Σ, g_k) if $\partial\Sigma = \emptyset$, or to the doubled surface obtained by gluing two copies of Σ along $\partial\Sigma$.

If $\Sigma_k = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\lambda_k)$, setting $\ell_k := |\lambda_k| \geq 1$, we can also assume that $\ell_k \to \ell_\infty \in [1, \infty]$ up to a subsequence. If $\ell_\infty < \infty$, assuming $\lambda_k \to \lambda_\infty$ and defining $\Sigma_\infty := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\lambda_\infty)$, we can find diffeomorphisms $\varphi_k : \Sigma_\infty \to \Sigma$ such that the pullback of the conformal structure $[g_{\Phi_k}]$ converges smoothly to the flat one.

Since the area of Φ_k is bounded, the sequence $\Phi_k \circ \varphi_k$ is then bounded in $W^{1,2}(\Sigma_\infty)$ and we can extract a subsequence converging to a weak limit Φ_∞. Defining the area measure ν_k on Σ_∞ as in the previous section, note that again their limit in the sense of Radon measures (up to subsequences) is also equal to the limit of $\frac{1}{2}|d\Phi_k|^2 \operatorname{vol}_{\Sigma_\infty}$.

All the proofs in Section 2.5 carry over, just replacing Φ_k with $\Phi_k \circ \varphi_k$ and (Σ, g_0) with $(\Sigma_\infty, g_{\Sigma_\infty})$. Assume in the sequel $\ell_k \to \ell_\infty = \infty$.

Remark 2.6.2. Actually, in the proof of Theorem 2.5.3 we used the conformality of the maps Φ_k; since the proof was local in $\operatorname{int}(\Sigma)$, we can precompose Φ_k with a conformal map $h_k : B_1^2 \to (\Sigma, g_k)$ which is a diffeomorphism with the image and converges smoothly to the inverse of a conformal chart for $\Sigma = \Sigma_\infty$. The statement for the sequence (Φ_k) then follows from its validity for the conformal maps $\Phi_k \circ h_k$.

Note that, since $|\Re(\lambda_k)| \leq \frac{1}{2}$, we can use instead $S^1 \times \ell_k S^1$ as a domain for Φ_k, with the induced conformal structure becoming asymptotically the flat one. Given a big parameter L, we can subdivide the circle $\ell_k S^1$ into N_k arcs $I_{k,1}, \ldots, I_{k,N_k}$ with $L \leq |I_{k,j}| \leq 2L$. Note that the boundedness of the area of Φ_k gives

$$\int_{S^1 \times \ell_k S^1} |d\Phi_k|^2 \leq C$$

for some constant C independent of k.

Hence, for each k, there is only a bounded amount of indices j such that $\frac{1}{2} \int_{S^1 \times I_{k,j}} |d\Phi_k|^2 \geq \frac{c_Q}{8}$, for the constant c_Q from Proposition 2.6.1. Up to subsequences, we can then find a nonempty collection of arcs $J_{k,1}, \ldots, J_{k,h}$ which are unions of the previous intervals, in such a way that

$$L < \lim_{k \to \infty} |J_{k,j}| < \infty, \quad \operatorname{dist}(J_{k,j}, J_{k,j'}) \to \infty \text{ for } j \neq j'$$

and $\frac{1}{2} \int_{S^1 \times I_{k,j}} |d\Phi_k|^2 < \frac{c_Q}{8}$ whenever $I_{k,j}$ is not included in one of the arcs $J_{k,1}, \ldots, J_{k,h}$. We now claim that

$$\limsup_{k \to \infty} \int_{S^1 \times (\ell_k S^1 \setminus \bigcup_{j=1}^h R J_{k,j})} |d\Phi_k|^2 \to 0 \quad \text{as } R \to \infty, \tag{2.6.1}$$

provided L was chosen big enough. Here $RJ_{k,j} \subseteq \ell_k S^1$ is the arc dilated by a factor R, with the same center.

Once this is proved, we can fix $j \in \{1, \ldots, h\}$ and, shifting $J_{k,j}$ to be centered at 0, we obtain a (local) weak limit $\Phi_{\infty,j} : S^1 \times \mathbb{R} \to \mathcal{M}$ of the maps Φ_k, viewing these as maps defined on bigger and bigger subsets of $S^1 \times \mathbb{R}$. We can again repeat the analysis which was done in the previous section.

Note that in the limit we get a map with domain $S^1 \times \mathbb{R}$. Since this cylinder is conformally the same as the sphere minus two points, we can see the domain as the sphere: note that replacing the cylinder with the sphere preserves stationarity, by the same argument used in the proof of Theorem 2.5.11 to remove the set of atoms.

By (2.6.1), the sum of the masses of the limit varifolds for $j = 1, \dots, h$ is equal to the limit of the area of Φ_k, up to the contribution of concentration points in the h copies of $S^1 \times \mathbb{R}$. We will discuss later how to recover the area which gets concentrated at these points.

In order to prove (2.6.1), fix k and j, and let $I_{k,s}, \dots, I_{k,s+t}$ be the intervals lying between two consecutive arcs $J_{k,j}$ and $J_{k,j+1}$ (with indices modulo N_k and modulo h). We claim that eventually we cannot have $\sum_{i=2}^{t'} \int_{S^1 \times I_{k,s+i}} \frac{1}{2}|d\Phi_k|^2 \geq \frac{c_Q}{2}$ for any $1 < t' < t$. If t' is the minimum such index, since the energy carried by each $S^1 \times I_{k,s+i}$ is at most $\frac{c_Q}{8}$ we deduce that the sum is less than $\frac{5}{8}c_0$.

Since $|I_{k,i}| \geq L$, we can select $a \in I_{k,s+1}$ and $b \in I_{k,s+t'+1}$ such that $\int_{S^1 \times \{a,b\}} |d\Phi_k| \leq CL^{-1/2}$; we can apply Proposition 2.6.1 with $U_k := S^1 \times [a,b]$ and deduce that eventually $\int_{S^1 \times [a,b]} \frac{1}{2}|d\Phi_k|^2$ is either at least $\frac{7}{8}c_Q$ or at most $2\delta(CL^{-1/2}, C)$. Since the first possibility cannot happen, we are in the second case. Hence, we get $\frac{c_Q}{2} \leq 2\delta(CL^{-1/2}, C)$, which is a contradiction for L big enough, since $\delta(CL^{-1/2}, C) \to 0$ as $L \to \infty$.

But then we can repeat the argument selecting a' in the part of $RJ_{k,j} \setminus (R/2)J_{k,j}$ following $J_{k,j}$ and b' in the part of $RJ_{k,j+1} \setminus (R/2)J_{k,j+1}$ preceding $J_{k,j+1}$, with $\int_{S^1 \times \{a',b'\}} |d\Phi_k| \leq CR^{-1/2}$. We already know that the area carried by the region $S^1 \times [a', b']$ is eventually less than $\frac{c_Q}{2}$, so we deduce that it is bounded by $2\delta(CR^{-1/2}, C)$, and (2.6.1) follows.

In the case of the annulus, namely $\Sigma_k = S^1 \times [0, \ell_k]$, up to subsequences either we are in the easy case that ℓ_k has a limit in $(0, \infty)$, or $\ell_k \to \infty$, or $\ell_k \to 0$. The second case can be dealt with in the same way as before, by subdividing the interval $[0, \ell_k]$ and making sure that $I_{k,1} \subseteq J_{k,1}$ and $I_{k,N_k} \subseteq J_{k,h}$. In this case, $J_{k,j}$ produces again infinite cylinders, or equivalently spheres, in the limit for $1 < j < h$. On the other hand, $J_{k,1}$ and $J_{k,h}$ produce (possibly constant) limit maps whose domain is $S^1 \times [0, \infty)$, which is conformally the disk minus the origin. We can thus view their domain as the full disk.

In the last case $\ell_k \to 0$, we can replace $S^1 \times [0, \ell_k]$ with the conformally equivalent surface $[0, 1] \times \ell_k^{-1} S^1$. We then subdivide the circle and argue in the same way as before. In the limit we get maps with domain $[0, 1] \times \mathbb{R}$, which is conformally a disk (minus two boundary points which can be ignored).

Hyperbolic case

Finally, we explain how to deal with the hyperbolic case $\chi(\Sigma) < 0$. In this case there is no straightforward description of all the possible conformal classes of surfaces. In case Σ has no boundary, by Bers' theorem we can decompose (Σ, g_k) into hyperbolic pairs of pants, with lengths of their boundaries bounded above in terms of the topology of Σ: see [54, Theorem IV.3.7] for a self-contained proof. We call $\{\beta_i\}$ the collection of closed geodesics, depending on k but with fixed cardinality, which bound the pairs of pants. Up to subsequences, we can assume that the combinatorial configuration of the decomposition does not depend on k, with a consistent labeling for the curves β_i, and that the length of β_i converges to a finite number as $k \to \infty$.

Then we can apply [54, Proposition IV.5.1] to the connected components of the surface (Σ, g_k), cut open along those geodesics $\{\beta_i\}_{i\in I}$ whose length converges to 0. We get a possibly disconnected limit surface Σ_∞, which equals a closed Riemann surface minus finitely many points (two for each degenerating β_i), and diffeomorphisms $\psi_k : \Sigma_\infty \to \Sigma \setminus \bigcup_{i\in I} \beta_i$ such that the pullback metric $\psi_k^* g_k$ converges locally to the metric of Σ_∞. Then we repeat the analysis with the maps $\Phi_k \circ \psi_k$ and obtain a limit parametrized varifold, whose domain Σ_∞ can be replaced with a (possibly disconnected) closed surface. Apart from concentration points, part of the area of Φ_k could be concentrating in collar neighborhoods of the geodesics β_i, for $i \in I$. These neighborhoods can be conformally identified with cylinders $S^1 \times [0, L_{k,i}]$, with $L_{k,i} \to \infty$ as $k \to \infty$, and one can recover the missing part of the area as in the degenerating cylinder case; note that the pieces $S^1 \times J_{k,1}$ and $S^1 \times J_{k,h}$ from that analysis have to be discarded, since their contribution is already given by Σ_∞, while all the other pieces produce varifolds parametrized by spheres.

If $\partial\Sigma \neq \emptyset$, let us call $\gamma_1, \dots, \gamma_b$ the boundary components of Σ. We cannot directly decompose Σ into pairs of pants whose boundary curves have bounded length, since the length of some γ_i with respect to g_k could fail to stay bounded as $k \to \infty$.

Instead, we first glue two copies of (Σ, g_k) along the geodesic boundary $\partial\Sigma = \bigcup_{i=1}^b \gamma_i$, obtaining a hyperbolic surface $\widetilde{\Sigma}_k$. This surface comes equipped with a canonical involution i_k, which flips the two glued copies.

For a decomposition for $\widetilde{\Sigma}_k$ as in the closed case, we can assume that all the simple closed geodesics of length less than $2\sinh^{-1}(1)$ appear in the collection $\{\beta_i\}$: see [54, Lemma IV.4.1] and the proof of Bers' theorem.

The thin part $T_k := \{x \in \widetilde{\Sigma}_k : \mathrm{inj}(x) \leq \lambda\}$ is invariant under i_k, since i_k is an isometry. For λ small enough, it consists of finitely many disjoint annuli containing a (simple closed) geodesic of length at most 2λ, which is then in $\{\beta_i\}$: see the proof of [54, Proposition IV.4.2], which also shows that the curves β_j with length bigger than 2λ are disjoint from T_k. Hence, choosing λ small enough, we can assume $\beta_j \cap T_k = \emptyset$ for the indices $j \notin I$ corresponding to non-degenerating curves.

The boundary of T_k has a constant geodesic curvature $\kappa = \kappa(\lambda)$. Let $S_k := \widetilde{\Sigma}_k \setminus \mathrm{int}(T_k)$. Taking a limit $\widetilde{\Sigma}_\infty$ as in the previous discussion, the proof of [54, Proposition IV.5.1] shows that we can assume $\psi_k^{-1}(S_k)$ to be a constant domain S_∞, whose complement is the union of finitely many cusps $\{C_j\}_{j\in J}$. Namely, each $\overline{C}_j$ is isometric to the quotient of $\{\Im(z) \geq \Lambda\} \subset \mathbb{H}$ by the standard parabolic isometry $z \mapsto z + 1$, for some $\Lambda > 0$ depending on λ.

The maps $\psi_k^{-1} \circ i_k \circ \psi_k$ converge locally smoothly to an isometry $i_\infty : \widetilde{\Sigma}_\infty \to \widetilde{\Sigma}_\infty$, since i_k is an isometry for $\widetilde{\Sigma}_k$. The components of ∂T_k meeting $\partial\Sigma \subset \widetilde{\Sigma}_k$ are necessarily invariant sets for i_k, so that $\partial\Sigma$ meets ∂T_k orthogonally on $\partial\Sigma \cap \partial T_k$. Also, we have a lower bound on the injectivity radius on S_k; this implies that a shortest path α joining a point in $S_k \cap \gamma_i$ to another curve $\gamma_{i'}$ has length bounded below by λ, since the geodesic $i_k \circ \alpha$ has the same endpoints; similarly, a shortest path between two close points in $S_k \cap \gamma_i$ must be γ_i itself. Also, the length of a geodesic γ_i intersecting S_k cannot be smaller than 2λ. These remarks imply that on S_∞ the one-dimensional submanifold $\psi_k^{-1}(\partial\Sigma)$ converges graphically to a limit $\Gamma_\infty \subseteq \{x \in S_\infty : i_\infty(x) = x\}$, which meets $\partial S_\infty = \partial T_\infty$ orthogonally.

Thus, the domains $\psi_k^{-1}(\Sigma)$ converge graphically on S_∞ to a domain S'_∞ bounded by Γ_∞. If C is an i_k-invariant component of T_k, either i_k interchanges the two circles in ∂C or it preserves them (as sets). In the former case, the core geodesic of C appears in both collections $\{\gamma_i\}$ and $\{\beta_j\}$, and equals $\partial\Sigma \cap C$. In the latter case, there are just two diametrically opposite fixed points of i_k on each circle, so $\partial\Sigma$ splits C into two isometric pieces; we can thus assume that $\psi_k^{-1}(\Sigma \cap C)$ equals two half-cusps in this case.

Hence, $T'_\infty := \psi_k^{-1}(\Sigma \cap T_k)$ is a constant union of cusps and half-cusps. The union $S'_\infty \cup T'_\infty$ is the desired limit surface, which is a compact Riemann surface Σ_∞ minus finitely many points (in the interior or on the boundary). The area contribution which gets lost because of degenerating geodesics can be recovered as in the case of degenerating tori or annuli.

Note that Σ_∞ has at least $b(\Sigma) - |I|$ boundary components. Also, the Euler characteristic of its double is

$$\begin{aligned} 2(2 - 2g(\Sigma_\infty) - b(\Sigma_\infty)) &= 2\chi(\Sigma_\infty) = \chi(\widetilde{\Sigma}_j) + 2|I| = 2\chi(\Sigma) + 2|I| \\ &= 2(2 - 2g(\Sigma) - b(\Sigma)) + 2|I| \end{aligned}$$

and we deduce $\chi(\Sigma_\infty) \geq \chi(\Sigma)$, $g(\Sigma_\infty) \leq g(\Sigma)$.

Note, however, that the number of boundary components could increase in principle: for instance, if Σ has genus one and one boundary component, (Σ, g_k) could degenerate conformally into an annulus.

Concentration points

We finally deal with concentration points for the area, or equivalently for the Dirichlet energy. The problem is local; since there can be only finitely many concentration points, we can deal with just a single one. Let U' denote the ball or the half-ball. Up to precomposing the maps Φ_k with suitable diffeomorphisms $U' \to U \subset \Sigma$, we can assume that the induced conformal classes converge smoothly to the standard one, and that we have the tight convergence

$$\nu'_k := \frac{1}{2}|d\Phi_k|^2\,\mathcal{L}^2 \rightharpoonup m\,\mathcal{L}^2 + \alpha\delta_0$$

of measures on U'. Looking at a sufficiently small neighborhood of the concentration point, we can assume that $\int_{U'} m < \frac{c_Q}{2}$, while from Theorem 2.5.2 we have the lower bound $\alpha \geq c_Q$.

Let $B^2_{r_k}(x_k)$ be a ball of minimal radius such that $\int_{B^2_{r_k}(x_k)\cap U'} \frac{1}{2}|d\Phi_k|^2 \geq \alpha - \frac{c_Q}{2}$, so that the integral is exactly $\alpha - \frac{c_Q}{2}$ and necessarily $r_k \to 0$, $x_k \to 0$. It suffices to show that

$$\limsup_{k\to\infty} \nu'_k((B^2_{R^{-1}}(x_k) \setminus B^2_{Rr_k}(x_k)) \cap U') \to 0 \quad \text{as } R \to \infty. \tag{2.6.2}$$

Once this is done, we deduce that the area (or Dirichlet energy) measures of $\Psi_k := \Phi_k(x_k + r_k\cdot)$ converge subsequentially to a measure ν (on the plane or a upper half-plane) of total mass α. There could be further concentration points for this new sequence of maps, but their masses are at most $\alpha - \frac{c_Q}{2}$: this is obvious if there are at least two such points; if there is only one point $\overline{x}$ of mass bigger than $\alpha - \frac{c_Q}{2}$, then eventually

$$\int_{B_{r_k/2}(x_k + r_k\overline{x})\cap U'} \frac{1}{2}|d\Phi_k|^2 = \int_{B^2_{1/2}(\overline{x})\cap U'_k} \frac{1}{2}|d\Psi_k|^2 > \alpha - \frac{c_Q}{2},$$

where $U'_k := r_k^{-1}(U' - x_k)$, contradicting the minimality of r_k. Thus, this blow-up process has to be iterated only a finite amount of times.

The proof of (2.6.2) is similar to the one of (2.6.1). Select radii $R^{1/2}r_k < a < Rr_k$ and $R^{-1} < b < R^{-1/2}$ such that

$$\int_{\partial B^2_a(x_k)\cap U'} |d\Phi_k|^2 \leq \frac{2C}{a\log R}, \qquad \int_{\partial B^2_b(x_k)\cap U'} |d\Phi_k|^2 \leq \frac{2C}{b\log R},$$

where C is an upper bound for $\int_{U'} |d\Phi_k|^2$; this can be done since the right-hand sides integrate to C on the two intervals. Now the length of $\Phi_k|_{\partial B^2_a\cap U'}$ and $\Phi_k|_{\partial B^2_b\cap U'}$ is bounded by $\frac{C}{\sqrt{\log R}}$, for a different constant C. Since the area of Φ_k between the two radii is bounded by $\alpha + \int_{U'} m - (\alpha - \frac{c_Q}{2}) + o(1)$, whose limit is less than c_Q, for R big enough we can apply Proposition 2.6.1 and deduce that

$$\limsup_{k\to\infty} \nu'_k((B^2_b(x_k) \setminus B^2_a(x_k)) \cap U') \leq \delta\Big(\frac{C}{\sqrt{\log R}}, C\Big).$$

Since $B^2_{R^{-1}}(x_k) \setminus B^2_{Rr_k}(x_k) \subseteq B^2_b(x_k) \setminus B^2_a(x_k)$, this proves (2.6.2).

The limit maps produced by concentration points have domains which are the plane or a half-plane, hence conformally the sphere or the disk (minus one point).

Proof of Theorem 2.1.1. Thanks to the arguments from this section and the previous one, we obtain disjoint domains $U_{k,1}, \dots, U_{k,N} \subseteq \Sigma$ such that the varifold induced by $\Phi_k|_{U_{k,i}}$ converges to a parametrized free boundary stationary varifold, as $k \to \infty$, and $\int_{\Sigma\setminus\bigcup_i U_{k,i}} \mathrm{vol}_{\Phi_k} \to 0$.

Since we can merge the domains of these parametrized varifolds into a (possibly disconnected) compact Riemann surface, the statement follows. □

2.7 Regularity

From the previous section we know that the limit varifold $\mathbf{v}_\infty$ is a parametrized free boundary stationary varifold (Σ', Φ, N'), for some weakly conformal map $\Phi : \Sigma' \to \mathcal{M}$ with

$\Phi(\partial\Sigma') \subseteq \mathcal{N}$ and $N' \in L^\infty(\Sigma', \{1, 2, \dots\})$. This parametrized varifold gives rise, in local charts for Σ', to a local parametrized stationary varifold, as defined in Definition 3.2.9 (see also Remark 3.2.3). The main result of the next chapter, namely Theorem 3.5.7, tells us that N' is locally constant and Φ is a branched minimal immersion, on (the interior of) the components of Σ' where Φ is not (a.e.) constant.

Hence, in order to study the regularity of Φ, we can discard these trivial components and replace N' with 1, without affecting the stationarity property enjoyed by the parametrized varifold (recall Definition 2.5.9).

For simplicity, since we will not need to refer back to the original setting, we will write Σ in place of Σ' in the rest of this section. In order to prove Theorem 2.1.4, we wish to show the following result. The fact that $\Phi|_{\Sigma\setminus\partial\Sigma}$ is a branched minimal immersion then follows as discussed in the last step of the proof of Theorem 3.5.7.

Theorem 2.7.1. *The map $\Phi : \Sigma \to \mathcal{M}$ is C^∞-smooth up to the boundary $\partial\Sigma$ and has $\partial_\nu\Phi \perp T\mathcal{N}$ at $\partial\Sigma$.*

We first show a simple strenghtening of Theorem 2.5.11. In the sequel, given $\omega \subseteq \Sigma$ open, we let $\mathbf{v}_\omega := \Phi_*(\omega)$.

Proposition 2.7.2. *The map Φ is continuous and the stationarity (respectively, free boundary stationarity) of $\mathbf{v}_\omega$ in Definition 2.5.9 holds for any domain $\omega \subset\subset \Sigma \setminus \partial\Sigma$ (respectively, $\omega \subseteq \Sigma$).*

Proof. The continuity of Φ can be obtained by the same arguments used in the proof of Theorem 2.5.2.

As for the second statement, given $\omega \subseteq \Sigma$ and a vector field $X \in \mathcal{X}_{fb}$ supported outside $\Phi(\partial\omega)$, we can find a nonnegative smooth function $\rho \in C_c^\infty(\omega)$ such that $\rho = 1$ near the compact set $\Phi^{-1}(\mathrm{spt}(X)) \cap \omega$. The stationarity of $\mathbf{v}_\omega$ against the vector field X then follows from the same property for the varifolds $\mathbf{v}_{\{\rho>\lambda\}}$, for $0 < \lambda < 1$, each of which agrees with $\mathbf{v}_\omega$ near $\mathrm{spt}(X)$. The proof in the case $\omega \subset\subset \Sigma \setminus \partial\Sigma$ is analogous. $\square$

Let us fix a metric on Σ, compatible with the conformal structure. As in [92], we first show that

$$\Phi \text{ is smooth near } \mathcal{G}', \tag{2.7.1}$$

with $\mathcal{G}' \subseteq \Sigma \setminus \partial\Sigma$ defined to be the set of points x such that $d\Phi(x)$ has rank 2 and, in a chart centered at x, $\int_{B_r^2} |d\Phi - d\Phi(0)|^2 \, d\mathcal{L}^2 = o(r^2)$.

Before proving this, let us set $\mathcal{B}' := \Sigma \setminus \mathcal{G}'$, $\mathcal{B} := \Phi^{-1}(\Phi(\mathcal{B}'))$ and $\mathcal{G} := \Sigma \setminus \mathcal{B}$.

Remark 2.7.3. Note that $\mathcal{B}$ and $\mathcal{G} \subseteq \mathcal{G}'$ are both Φ-saturated: this means that whenever $\Phi(x) = \Phi(y)$ and $x \in \mathcal{B}$, the same holds for y, and similarly for $\mathcal{G}$.

Arguing as in the proof of Theorem 2.5.2, we have

$$|\mathbf{v}_\Sigma|(\Phi(\mathcal{B}')) \le C \int_{\mathcal{B}'} |d\Phi|^2 \ \mathrm{vol}_\Sigma = 0.$$

Hence, as $|\mathbf{v}_\Sigma| = \Phi_*(\frac{1}{2}|d\Phi|^2 \ \mathrm{vol}_\Sigma)$, we get $d\Phi = 0$ a.e. on $\mathcal{B}$.

Proof of (2.7.1). Given $x \in \mathcal{G}'$, we can choose a conformal chart centered at x, mapping a neighborhood U of x to B^2_1. Viewing $\mathcal{M} \subset \mathbb{R}^Q$, we can then select an arbitrarily small radius $r > 0$ such that $\Phi(ry) = \Phi(0) + d\Phi(0)[ry] + o(r)$, for $|y| = 1$ (see, e.g., [84, Lemma A.4]).

Moreover, $\int_{B^2_r} \frac{1}{2}|d\Phi|^2 \, d\mathcal{L}^2 = \pi s^2 + o(r^2)$, with $s := |\partial_1\Phi|(0)r = |\partial_2\Phi|(0)r$. Hence, assuming that the above error $o(r)$ is less that δr, for a fixed δ small enough, we can apply Allard's regularity result [3, p. 466] (see also [98, Theorem 23.1]) on the ball $B^Q_{(1-\delta)s}(\Phi(0))$, where the varifold $\mathbf{v}_{B^2_r}$ has generalized mean curvature bounded in L^∞, small excess (for r small), and total mass $\pi(1-\delta)^2 s^2 + O(\delta)s^2$.

We deduce that on some ball $B^Q_\theta(\Phi(0))$ the varifold $\mathbf{v}_{B^2_r}$ agrees with the graph S of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}^{Q-2}$, with multiplicity one, up to rotating the coordinates.[2]

Selecting a new radius r' much smaller than θ, such that $\Phi(r'y) = \Phi(0) + d\Phi(0)[r'y] + o(r')$, from the continuity of Φ we deduce that $|\mathbf{v}_{B^2_{r'}}|$ is supported in S. Hence, viewing $\mathcal{G} \cap U$ as a subset of B^2_1 and setting $\widetilde{\mathcal{G}} := \mathcal{G} \cap \overline{B}^2_{r'}$, from $|\mathbf{v}_{B^2_{r'}}| = (\Phi|_{B^2_{r'}})_*(\frac{1}{2}|d\Phi|^2 \, \mathcal{L}^2)$ we deduce $\Phi(y) \in S$ for all $y \in \widetilde{\mathcal{G}}$.

Thus, the map $\mathrm{dist}(\Phi, S)$ is $W^{1,2}$ on $B^2_{r'}$ and vanishes on $\widetilde{\mathcal{G}}$, and hence its differential vanishes a.e. here. But $d\Phi = 0$ a.e. on $\mathcal{B}$; it follows that this function is constant, giving $\Phi(\overline{B}^2_{r'}) \subseteq S$. Thus, $\Phi|_{\overline{B}^2_{r'}}$ factors as $(\mathrm{id} \times f) \circ \Psi$ for a suitable map $\Psi \in C^0 \cap W^{1,2}(\overline{B}^2_{r'}, \mathbb{R}^2)$. By the chain rule, any point $y \in \widetilde{\mathcal{G}}$ is necessarily Lebesgue for $d\Psi$, with $d\Psi(y)$ invertible.

For any $y \in \widetilde{\mathcal{G}}$ there exist arbitrarily small radii s such that $\mathbf{v}_{B^2_s(y)}$ is supported in S and has density at least one at $\Phi(y)$. As $\mathbf{v}_{B^2_{r'}}$ has multiplicity one on B^Q_θ, this implies that Φ is injective on $\widetilde{\mathcal{G}}$.

But then, recalling Remark 2.7.3, it follows that $\Phi(y)$ is disjoint from $\Phi(\overline{B}^2_{r'} \setminus \{y\})$ for all $y \in \widetilde{\mathcal{G}}$, and the same follows for Ψ. Given $y \in \widetilde{\mathcal{G}}$ close to 0 and choosing a homotopy in $\overline{B}^2_{r'} \setminus \{y\}$ between the circles $\partial B^2_{r'}(0)$ and $\partial B^2_s(y)$, with their canonical orientation, we deduce that the maps $\Psi|_{\partial B^2_{r'}} - \Psi(y)$ and $\Psi|_{\partial B^2_s(y)} - \Psi(y)$ determine the same element in $\pi_1(\mathbb{R}^2 \setminus \{0\})$.

But the first map is homotopic to $\Psi|_{\partial B^2_{r'}} - \Psi(0)$, provided $\Psi(y)$ is close enough to $\Psi(0)$, while the second is homotopic to $d\Phi(y)|_{S^1}$ if s is selected in the same way as r. We deduce that $d\Psi$ is either always orientation preserving or always orientation reversing on $\widetilde{\mathcal{G}}$, near 0. Thus Φ, in local coordinates for S, solves the Cauchy–Riemann equations (up to conjugation) near 0, establishing (2.7.1). □

[2]The smoothness of f can be assumed by standard Schauder theory, since f satisfies an elliptic equation on a small ball.

Remark 2.7.4. We implicitly ask that the chosen representative of $d\Phi$ agrees with the classical differential on the regular set of Φ. Hence, by what we just proved, $\mathcal{G}'$ is open. It follows that $\mathcal{B}'$ and $\mathcal{B}$ are closed, so that $\mathcal{G}$ is open again.

Remark 2.7.5. Given $x \in \mathcal{G}$ and a neighborhood $U \subset\subset \mathrm{int}(\Sigma)$ such that $\Phi|_{\overline{U}}$ is a diffeomorphism with the image, we can express any section $w \in C_c^\infty(U)$ of $\Phi^*T\mathcal{M}$ as $w = X(\Phi)$, where X is a (smooth) vector field on $\mathcal{M}$ vanishing near $\Phi(\partial U)$. Hence, using Proposition 2.7.2, we get

$$\int_U \langle \nabla w, d\Phi \rangle \ \mathrm{vol}_\Sigma = 0,$$

so that Φ solves the harmonic map equation $\nabla^* d\Phi = 0$ on $\mathcal{G}$.

In order to show Theorem 2.7.1, let $y \in \Sigma$ and pick a conformal chart $U \to U'$ centered at y, with image equal to B_1^2 if $y \notin \partial\Sigma$ and to $B_1^2 \cap \{\Im(z) \geq 0\}$ otherwise. By continuity of Φ we can assume that $\overline{\Phi(U')}$ is contained in a coordinate chart for $\mathcal{M}$. We call $\{x^1, \dots, x^m\}$ the coordinates and we let $\Phi^i := x^i \circ \Phi$. We can also require that $\mathcal{N}$ corresponds to $\{x^{n+1} = \cdots = x^m = 0\}$ if $y \in \partial\Sigma$, with $g_{ij} = 0$ for $i \leq k$ and $j > k$ on this set.

Then, writing $e_k := \frac{\partial}{\partial x^k}$, it suffices to show that

$$\int_{U'} \langle \nabla(f e_k), d\Phi \rangle \, d\mathcal{L}^2 = 0 \tag{2.7.2}$$

for all $k = 1, \dots, m$ and all nonnegative $f \in C_c^\infty(U')$, with the additional constraint $f \in C_c^\infty(U' \setminus \partial U')$ if $k > n$ and $y \in \partial\Sigma$, where we write $\partial U' := U' \cap \{\Im(z) = 0\}$.

Indeed, once this is done, if $y \notin \partial\Sigma$ then $\Phi = (\Phi^1, \dots, \Phi^m)$ is a weak solution of the system

$$-\partial_i(g_{jk}(\Phi)\partial_i\Phi^j) + \Gamma_{pk}^j(\Phi) g_{jq}(\Phi) \partial_i\Phi^p \partial_i\Phi^q,$$

where Γ_{pk}^j is defined by the relation $\nabla_{e_p} e_k = \Gamma_{pk}^j e_j$. The smoothness of Φ then follows from Proposition A.7 and Remark A.9.

If instead $y \in \partial\Sigma$, we get a weak solution to the system

$$\begin{cases} -\partial_i(g_{jk}(\Phi)\partial_i\Phi^j) + \Gamma_{pk}^j(\Phi) g_{jq}(\Phi) \partial_i\Phi^p \partial_i\Phi^q = 0, \\ \partial_\nu \Phi_k = 0 \quad \text{on } \partial U', \text{ for } k \leq n, \\ \Phi_k = 0 \quad \text{on } \partial U', \text{ for } k > n, \end{cases}$$

in the sense specified in Remark A.8, and regularity follows again from Proposition A.7 and Remark A.9.

By the coarea formula, (2.7.2) is equivalent to

$$\int_0^\infty \left(-\int_{\partial\{f>\lambda\}} \langle e_k(\Phi), \partial_\nu \Phi \rangle + \int_{\{f>\lambda\}} \langle \nabla(e_k \circ \Phi), d\Phi \rangle \right) d\lambda = 0.$$

In order to conclude, we will show that the quantity between brackets vanishes for a.e. λ.

Proposition 2.7.6. *For almost every value of $\lambda > 0$, for $\omega := \{f > \lambda\} \subset\subset U'$ it holds*

$$-\int_{\partial\omega} \langle e_k(\Phi), \partial_\nu \Phi\rangle + \int_\omega \langle \nabla(e_k \circ \Phi), d\Phi\rangle = 0.$$

Proof. Fix λ such that ω has smooth boundary, transverse to $\partial U'$ if $y \in \partial\Sigma$, and such that the trace $\Phi|_{\partial\omega}$ is $W^{1,2}$, with differential given by the restriction of $d\Phi$ and vanishing a.e. on $\partial\omega \cap \mathcal{B}$. For all $\varepsilon > 0$, we call $\mathcal{B}_\varepsilon$ the closed ε-neighborhood of $\mathcal{B}$ in U'.

Take a smooth function ρ vanishing near $\Phi(\partial\omega \cap \mathcal{B}_\varepsilon)$. Then Φ is a smooth immersion in a neighborhood of $S \cap \partial\omega$, with $S := \mathrm{spt}(\rho \circ \Phi)$, since $S \cap \partial\omega \subseteq \mathcal{G}$.

We can cover $S \cap \partial\omega$ with finitely many disjoint closed arcs $\{\gamma_j\} \subseteq \mathcal{G}$, with endpoints in $\partial U' \cup \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon$, so that Φ is an immersion near each of them. Fix now a smooth unit vector field $\widetilde{\nu}$ on $\partial\omega$ which points towards ω, with $\widetilde{\nu} \in T\partial U'$ on the finite set $\partial\omega \cap \partial U'$. We can find functions $f_j : \gamma_j \to [0,1)$ such that the curves

$$\widetilde{\gamma}_j := \{x + f_j(x)\widetilde{\nu}(x) \mid x \in \gamma_j\}$$

are disjoint, included in $\mathcal{G}$, have endpoints in $U' \setminus S$, and have images $\Gamma_j := \Phi(\widetilde{\gamma}_j)$ transverse to each other (meaning also self-transverse). Note that all f_j's can be chosen arbitrarily close to 0 in the C^∞ topology.

We now consider the domain

$$\Omega := \omega \setminus \bigcup_j \{x + s f_j(x)\widetilde{\nu}(x) \mid 0 \le s \le 1,\, x \in \gamma_j\}.$$

Note also that we can assume the sets in the last union to be disjoint and

$$\rho = 0 \text{ near } \Phi(\{x + s f_j(x)\widetilde{\nu}(x) \mid 0 \le s \le 1\}) \tag{2.7.3}$$

whenever $x \in \mathcal{B}_\varepsilon$ is an endpoint of one of the curves γ_j. This implies

$$\partial\Omega \cap S \subseteq \bigcup_j \mathrm{int}(\widetilde{\gamma}_j), \tag{2.7.4}$$

where $\mathrm{int}(\widetilde{\gamma}_j)$ denotes $\widetilde{\gamma}_j$ minus the endpoints.

Fix a smooth function $\chi : [0,\infty) \to [0,1]$ with $\chi = 1$ on $[1,\infty)$ and $\chi = 0$ on $[0, \frac{1}{2}]$. Let $\Gamma := \bigcup_j \Gamma_j$ and $\chi_\eta := \chi(\frac{\mathrm{dist}(\cdot,\Gamma)}{\eta})$.

Let F denote the closure of $\bigcup_j \Phi^{-1}(\Gamma_j) \setminus \bigcup_j \widetilde{\gamma}_j$, together with all the endpoints of the curves $\widetilde{\gamma}_j$. By transversality and conformality of Φ, for each $x \in \bigcup_j \widetilde{\gamma}_j \setminus F$ we have $\mathrm{dist}(\Phi(x - s\nu(x)), \Gamma) = s|\partial_\nu \Phi(x)| + o(s)$, where ν is the outward unit normal for Ω, and the gradient of $\mathrm{dist}(\Phi(\cdot), \Gamma)$ at $x - s\nu(x)$ is $-|\partial_\nu \Phi(x)|\nu(x) + o(1)$, where $o(1)$ is infinitesimal as $s \to 0$ ($s > 0$). These estimates hold uniformly on compact subsets of $\bigcup_j \widetilde{\gamma}_j \setminus F$.

Moreover, by transversality again, for any fixed small $r > 0$ the support of $\chi_\eta \circ \Phi$ intersects the r-neighborhood U_r of $\bigcup_j \widetilde{\gamma}_j$ in the union of an $O(\eta)$-neighborhood of $\bigcup_j \widetilde{\gamma}_j$, plus a set of measure $O(r\eta)$. In view of these remarks,

$$\begin{aligned}
&\lim_{\eta\to 0}\int_{\Omega\cap U_r}\rho(\Phi)\langle e_k(\Phi)\otimes d(\chi_\eta\circ\Phi),d\Phi\rangle\\
&=-\lim_{\eta\to 0}\sum_j\int_{\widetilde{\gamma}_j}\int_0^1\chi'\Big(\frac{s|\partial_\nu\Phi(x)|}{\eta}\Big)\frac{|\partial_\nu\Phi(x)|}{\eta}\langle(\rho e_k)(\Phi),\partial_\nu\Phi\rangle(x)\,ds\,dx+O(r)\\
&=-\int_{\widetilde{\gamma}_j}\langle(\rho e_k)(\Phi),\partial_\nu\Phi\rangle+O(r).
\end{aligned}$$

Also, note that $\Phi(\mathcal{B})\cap\Gamma=\emptyset$ by Remark 2.7.3; hence, for η small, $\chi_\eta=1$ near $\Phi(\mathcal{B})$ and we deduce that $\operatorname{spt}((1-\chi_\eta)\circ\Phi)\subseteq\mathcal{G}$. Recalling also (2.7.4), we can integrate by parts as follows:

$$\begin{aligned}
&\int_{\Omega\setminus U_r}\rho(\Phi)\langle e_k(\Phi)\otimes d(\chi_\eta\circ\Phi),d\Phi\rangle\\
&=\int_{\Omega\setminus U_r}(1-\chi_\eta)(\Phi)\langle e_k(\Phi)\otimes d(\rho\circ\Phi),d\Phi\rangle+\int_{\Omega\setminus U_r}(\rho(1-\chi_\eta))(\Phi)\langle\nabla(e_k(\Phi)),d\Phi\rangle\\
&\quad+\int_{\Omega\cap\partial U_r}(\rho(1-\chi_\eta))(\Phi)\langle e_k(\Phi),\partial_\nu\Phi\rangle,
\end{aligned}$$

where we used the harmonicity of Φ on $\mathcal{G}$. The convergence $(1-\chi_\eta)(\Phi)\to 0$ a.e. on $\Omega\setminus U_r$ and on ∂U_r (for r small enough) implies that the right-hand side is infinitesimal as $\eta\to 0$.

But, by the stationarity property of $\mathbf{v}_\Omega$, setting $X_\eta:=\rho\chi_\eta e_k$ we have

$$\int_\Omega\langle\nabla(X_\eta\circ\Phi),d\Phi\rangle=0,$$

since X_η vanishes near $\Phi(\partial\Omega)$ by the choice of χ_η and (2.7.3). Hence, from the previous computations we deduce

$$-\sum_j\int_{\widetilde{\gamma}_j}\rho(\Phi)\langle e_k(\Phi),\partial_\nu\Phi\rangle+\int_\Omega\langle e_k(\Phi)\otimes d(\rho\circ\Phi),d\Phi\rangle+\int_\Omega\rho(\Phi)\langle\nabla e_k(\Phi)[d\Phi],d\Phi\rangle=0.$$

Letting $f_j\to 0$ we deduce our claim, provided we can replace ρ with 1. This is achieved as follows: the compact set $T:=\Phi(\partial\omega\cap\mathcal{B}_\varepsilon)$ has

$$\mathcal{H}^1(T)\le\int_{\partial\omega\cap\mathcal{B}_\varepsilon}|d\Phi|.$$

Hence, can cover T with finitely many balls $B_{r_i}(p_i)$ intersecting T, such that

$$2\sum_i r_i\le\mathcal{H}^1(T)+\varepsilon \tag{2.7.5}$$

and $r_i<\varepsilon$. Take now cut-off functions $0\le\rho_i\le 1$ which equal 0 on $B_{r_i}(p_i)$ and 1 on $\mathcal{M}\setminus B_{2r_i}(p_i)$, with $|d\rho_i|\le Cr_i^{-1}$. Then the function $\rho:=\prod_i\rho_i$ satisfies

$$\int_\omega|d\rho|(\Phi)|d\Phi|^2\le C\sum_i r_i^{-1}\int_{\omega\cap\Phi^{-1}(B_{2r_i}(p_i))}|d\Phi|^2\le Cr_i,$$

because $(\Phi)_*(\frac{1}{2}|d\Phi|^2) \le \mathbf{v}_\Sigma$ and $\mathbf{v}_\Sigma(B_{2r_i}(p_i)) \le Cr_i^2$ (see (2.4.5)). Note that the right-hand side of (2.7.5) becomes infinitesimal as $\varepsilon \to 0$, as $\int_{\partial\omega\cap\mathcal{B}} |d\Phi| = 0$.

Finally, writing T_ε and ρ_ε in place of T and ρ to emphasize the dependence on ε, we have $\rho_\varepsilon(\Phi) \to 1$ pointwise on $\mathcal{G}$: indeed, since $T_\varepsilon \to \Phi(\partial\omega \cap \mathcal{B})$ in the Hausdorff topology, if $\rho_\varepsilon(\Phi(x))$ does not converge to 1 then $\Phi(x) \in \Phi(\partial\omega \cap \mathcal{B})$ and thus, by Remark 2.7.3, $x \in \mathcal{B}$. Hence,

$$\begin{aligned}
0 &= -\int_{\partial\omega} \rho_\varepsilon(\Phi)\langle e_k(\Phi), \partial_\nu\Phi\rangle + \int_\omega \langle e_k(\Phi)\, d(\rho_\varepsilon \circ \Phi), d\Phi\rangle + \int_\omega \rho_\varepsilon(\Phi)\langle \nabla e_k(\Phi)[d\Phi], d\Phi\rangle \\
&\to -\int_{\partial\omega} \langle e_k(\Phi), \partial_\nu\Phi\rangle + \int_\omega \langle \nabla e_k(\Phi)[d\Phi], d\Phi\rangle,
\end{aligned}$$

as desired. □

Appendix

Lemma A.1. *Let $\Omega \subseteq \mathbb{C}$ be an open connected set. Let $\Psi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^Q)$ and assume that any point in a nonempty measurable set $E \subseteq \Omega$ is a Lebesgue point for Ψ, as well as $\nabla\Psi = 0$ a.e. on $\Omega \setminus E$. Then the essential image of Ψ equals $\overline{\Psi(E)}$.*

We recall that the essential image of Ψ is the (closed) set of values $p \in \mathbb{R}^Q$ such that $\mathcal{L}^2(\Psi^{-1}(B^Q_s(p))) > 0$ for all $s > 0$, or equivalently it is the support of the positive measure $\Psi_*(\mathbf{1}_\Omega \mathcal{L}^2)$.

Proof. Fix $x \in E$, $s > 0$ and let $p := \Psi(x)$. By Chebyshev's inequality

$$\mathcal{L}^2(B^2_r(x) \setminus \Psi^{-1}(B^Q_s(p))) \le s^{-1} \int_{B^2_r(x)} |\Psi - \Psi(x)|\, d\mathcal{L}^2 = o(r^2),$$

so p belongs to the essential image. We deduce that $\overline{\Psi(E)}$ is included in the essential image. Conversely, assume that $B^Q_s(p)$ is disjoint from $\overline{\Psi(E)}$. We can find a function $\rho \in C^\infty(\mathbb{R}^Q, \mathbb{R}^Q)$ with $\rho = \mathrm{id}$ on $\mathbb{R}^Q \setminus B^Q_s(p)$ and $\rho = \mathrm{id} + e_1$ on $B^Q_{s/2}(p)$. By the chain rule,

$$\nabla(\rho \circ \Psi) = (\nabla\rho \circ \Psi)\nabla\Psi = \nabla\Psi$$

a.e. on Ω, since $\nabla\Psi = 0$ a.e. on $\{\nabla\rho \circ \Psi \ne \mathrm{id}\}$. Thus, $\rho \circ \Psi - \Psi$ is constant a.e. But this function vanishes on E, hence 0 belongs to its essential image (by the same argument used above). Thus $\rho \circ \Psi = \Psi$ a.e. and we infer that $\Psi^{-1}(B^Q_{s/2}(p)) \subseteq \{\rho \circ \Psi \ne \Psi\}$ is negligible. This shows that p does not belong to the essential image. □

Lemma A.2. *Let $\Omega \subseteq \mathbb{C}$ be open. If $\Psi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^Q)$ and $\mathcal{G}$ denotes the set of Lebesgue points for both Ψ and $\nabla\Psi$, then there exist Lebesgue measurable sets $E_1, E_2, \ldots$ such that $\mathcal{G} = \bigcup_i E_i$ and $\Psi\big|_{E_i}$ is Lipschitz.*

Proof. We set $F_j := \left\{x \in \mathcal{G} \cap \Omega_{1/j} : \fint_{B^2_r(x)} |\Psi(y) - \Psi(x)|\, dy \le jr \text{ for } 0 < r \le j^{-1}\right\}$ (where $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$), which is a Lebesgue measurable set. By [35, Theorem 6.1] (and its proof), for all $x \in \mathcal{G}$

$$\begin{aligned}\fint_{B^2_r(x)} |\Psi(y) - \Psi(x)|\, dy &\le r|\nabla\Psi(x)| + \fint_{B^2_r(x)} |\Psi(y) - \Psi(x) - \langle \nabla\Psi(x), y - x\rangle|\, dy \\ &= r|\nabla\Psi(x)| + o(r).\end{aligned}$$

We infer that $\mathcal{G} = \bigcup_{j\ge1} F_j$. Assume now that $x, x' \in F_j$ are distinct and $|x - x'| \le \frac{1}{2j}$. Let $r := |x - x'|$ and notice that $B^2_r(x) \subseteq B^2_{2r}(x')$. We can estimate

$$\begin{aligned}\big|\Psi(x) - \Psi(x')\big| &\le \fint_{B^2_r(x)} |\Psi(y) - \Psi(x)|\, dy + \fint_{B^2_r(x)} \big|\Psi(y) - \Psi(x')\big|\, dy \\ &\le \fint_{B^2_r(x)} |\Psi(y) - \Psi(x)|\, dy + 4\fint_{B^2_{2r}(x')} |\Psi(y) - \Psi(x)|\, dy \\ &\le jr + 8jr = 9j\big|x - x'\big|\end{aligned}$$

(since $2r \le j^{-1}$). The result follows once we split each F_j into countably many subsets of diameter at most $\frac{1}{2j}$. □

Lemma A.3. *Assume $\Psi \in W^{1,2}(B^2_{(1+\tau)r}(0), \mathbb{R}^Q)$, with $r, \tau > 0$. Then there is a measurable subset $E \subseteq (r, (1+\tau)r)$ of positive measure such that, for all $r' \in E$, $\Psi\big|_{\partial B^2_{r'}(0)}$ has an absolutely continuous representative whose image satisfies*

$$\mathrm{diam}\, \Psi(\partial B^2_{r'}(0)) \le \left(\frac{2\pi(1+\tau)}{\tau} \int_{B^2_{(1+\tau)r}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2\right)^{1/2}.$$

Proof. For a.e. $r' \in (r, (1+\tau)r)$ the function $\theta \mapsto \Psi(r'\cos\theta, r'\sin\theta)$ has an absolutely continuous representative, whose weak derivative is $r'\langle \nabla\Psi(r'\cos\theta, r'\sin\theta), (-\sin\theta, \cos\theta)\rangle$ (see [35, Theorem 4.21]). Moreover,

$$\fint_r^{r+\tau r} \int_{\partial B^2_t(0)} |\nabla\Psi|^2\, d\mathcal{H}^1\, dt \le \frac{1}{\tau r} \int_{B^2_{(1+\tau)r}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2.$$

Hence, for a set of radii r' of positive measure, the inner integral is less or equal than the right-hand side, giving

$$\left(\int_{\partial B^2_{r'}(0)} |\nabla\Psi|\, d\mathcal{H}^1\right)^2 \le 2\pi r' \int_{\partial B^2_{r'}(0)} |\nabla\Psi|^2\, d\mathcal{H}^1 \le \frac{2\pi(1+\tau)}{\tau} \int_{B^2_{(1+\tau)r}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2. \quad □$$

Lemma A.4. *Assume that 0 is a Lebesgue point for $\Psi \in W^{1,2}(B^2_R(0), \mathbb{R}^Q)$ and $\nabla\Psi$, as well as $\Psi(0) = \nabla\Psi(0) = 0$. Then, for any $\tau, \varepsilon > 0$ and any $0 < r \le \overline{r}$, we can select $r' \in (r, (1+\tau)r)$ such that $\Psi\big|_{\partial B^2_{r'}(0)}$ has an absolutely continuous representative with*

$$\max_{y\in S^1} \big|\Psi(r'y)\big| \le \varepsilon r,$$

for a suitable $\overline{r} = \overline{r}(\Psi, \tau, \varepsilon)$.

Proof. We can assume $\varepsilon < 1$. By [35, Theorem 6.1] and its proof we have

$$\int_{B^2_r(0)} |\Psi|^2 \, d\mathcal{L}^2 = o(r^4) \tag{A.1}$$

(since $1^* = 2$) and, as a consequence,

$$\begin{aligned}
&\fint_r^{r+\tau r} \int_{\partial B^2_t(0)} ((2\pi t)^{-1}|\Psi|^2 + r|\nabla\Psi|) \, d\mathcal{H}^1 \, dt \\
&\leq \frac{1}{2\pi\tau r^2} \int_{B^2_{(1+\tau)r}(0)} |\Psi|^2 \, d\mathcal{L}^2 + \frac{1}{\tau} \int_{B^2_{(1+\tau)r}(0)} |\nabla\Psi| \, d\mathcal{L}^2 = o(r^2).
\end{aligned}$$

Hence, if r is small enough, there exists some $r' \in (r, r+\tau r)$ such that

$$(2\pi r')^{-1} \|\Psi\|^2_{L^2(\partial B^2_{r'}(0))} + r\|\nabla\Psi\|_{L^1(\partial B^2_{r'}(0))} \leq \frac{\varepsilon^2 r^2}{4}$$

and $\Psi\big|_{\partial B^2_{r'}(0)}$ is absolutely continuous (a.e.), with derivative given by the chain rule. The elementary inequality

$$\|\Psi\|_{L^\infty(\partial B^2_{r'}(0))} \leq \left| \fint_{\partial B^2_{r'}(0)} \Psi \, d\mathcal{H}^1 \right| + \int_{\partial B^2_{r'}(0)} |\nabla\Psi| \, d\mathcal{H}^1 \leq \frac{\|\Psi\|_{L^2(\partial B^2_{r'}(0))}}{\sqrt{2\pi r'}} + \|\nabla\Psi\|_{L^1(\partial B^2_{r'}(0))}$$

gives the desired result. $\square$

Lemma A.5. *If $\Psi_k \rightharpoonup \Psi_\infty$ in $W^{1,2}(B^2_R(0), \mathbb{R}^Q)$, then for a.e. $r' \in (0, R)$ there exists a subsequence (k_i) such that $\Psi_\infty\big|_{\partial B^2_{r'}(0)}$ and $\Psi_{k_i}\big|_{\partial B^2_{r'}(0)}$ have absolutely continuous representatives and*

$$\Psi_{k_i}\big|_{\partial B^2_{r'}(0)} \to \Psi_\infty\big|_{\partial B^2_{r'}(0)} \quad \textit{in } L^\infty.$$

Proof. We observe that $\Psi_\infty\big|_{\partial B^2_{r'}(0)}$ coincides with the trace of Ψ_∞ on $\partial B^2_{r'}(0)$ for a.e. $r' \in (0, R)$: actually, this happens whenever $\mathcal{H}^1$-a.e. point on $\partial B^2_{r'}(0)$ is a Lebesgue point of Ψ_∞ (see [35, Theorem 5.7]); the same holds for Ψ_k. Now, by Fatou's lemma,

$$\begin{aligned}
&\int_0^R \liminf_{k\to\infty} \int_{\partial B^2_{r'}(0)} (|\Psi_k|^2 + |\nabla\Psi_k|^2) \, d\mathcal{H}^1 \, dr' \\
&\leq \liminf_{k\to\infty} \int_0^R \int_{\partial B^2_{r'}(0)} (|\Psi_k|^2 + |\nabla\Psi_k|^2) \, d\mathcal{H}^1 \, dr' \\
&= \liminf_{k\to\infty} \|\Psi_k\|^2_{W^{1,2}} < \infty.
\end{aligned}$$

Hence, for a.e. $r' \in (0, R)$,

$$\liminf_{k\to\infty} \int_{\partial B^2_{r'}(0)} (|\Psi_k|^2 + |\nabla\Psi_k|^2) \, d\mathcal{H}^1 < \infty,$$

which means that there exists a subsequence (k_i) such that

$$\sup_i \int_{\partial B^2_{r'}(0)} (|\Psi_{k_i}|^2 + |\nabla\Psi_{k_i}|^2) \, d\mathcal{H}^1 < \infty.$$

We can also assume that $\Psi_\infty|_{\partial B^2_{r'}(0)}$ equals the trace, that it has a $W^{1,2}$ representative with weak derivative given by the chain rule and that the analogous statements hold also for Ψ_{k_i}.

In particular, the sequence $\left(\Psi_{k_i}|_{\partial B^2_{r'}(0)}\right)$ is bounded in $W^{1,2}(\partial B^2_{r'}(0), \mathbb{R}^Q)$: by the compact embedding into $C^0(\partial B^2_{r'}(0), \mathbb{R}^Q)$ we deduce that (up to subsequences) $\Psi_{k_i}|_{\partial B^2_{r'}(0)} \to f$ in L^∞, for some $f \in C^0(\partial B^2_{r'}(0), \mathbb{R}^Q)$. By the weak continuity of the trace, we have $\Psi_{k_i}|_{\partial B^2_{r'}(0)} \rightharpoonup \Psi_\infty|_{\partial B^2_{r'}(0)}$ in $L^2(\partial B^2_{r'}(0), \mathbb{R}^Q)$, hence $f = \Psi_\infty|_{\partial B^2_{r'}(0)}$ $\mathcal{H}^1$-a.e. □

Lemma A.6. *Let $\mathcal{C} \subseteq \mathbb{R}^{Q\times 2}$ denote the set of matrices M with $\sum_{i=1}^Q M_{ij}M_{ik} = \frac{|M|^2}{2}\delta_{jk}$, where $|M|$ is the Hilbert–Schmidt norm of M ($\mathcal{C}$ can be identified with the set of linear weakly conformal maps $\mathbb{R}^2 \to \mathbb{R}^Q$). For any $\tau > 0$ there exists $C = C(\tau, Q)$ such that*

$$|M_{11}M_{21} + M_{12}M_{22}| + |M_{11}^2 + M_{12}^2 - J(M)| + |M_{21}^2 + M_{22}^2 - J(M)| \le \tau|M|^2 + C\sum_{i=3}^Q\sum_{j=1}^2 M_{ij}^2$$

for all $M \in \mathcal{C}$, where $J(M) := |M_{11}M_{22} - M_{12}M_{21}|$.

Proof. Assume by contradiction that, for a sequence $(M^k) \subseteq \mathcal{C}$, we have

$$\begin{aligned}&|M_{11}^kM_{21}^k + M_{12}^kM_{22}^k| + |(M_{11}^k)^2 + (M_{12}^k)^2 - J(M^k)| + |(M_{21}^k)^2 + (M_{22}^k)^2 - J(M^k)| \\ &> \tau|M^k|^2 + k\sum_{i=3}^Q\sum_{j=1}^2 (M_{ij}^k)^2.\end{aligned}$$

By homogeneity we can assume $|M^k| = 1$ for all k. As a consequence, $\sum_{i=3}^Q\sum_{j=1}^2 (M_{ij}^k)^2 \to 0$. So the sequence has a limit point $M^\infty \in \mathbb{R}^{Q\times 2}$ satisfying

$$M^\infty \in \mathcal{C}, \qquad M_{ij}^\infty = 0 \quad \text{for } i = 3, \dots, Q \text{ and } j = 1, 2, \tag{A.2}$$

$$|M_{11}^\infty M_{21}^\infty + M_{12}^\infty M_{22}^\infty| + |(M_{11}^\infty)^2 + (M_{12}^\infty)^2 - J(M^\infty)| + |(M_{21}^\infty)^2 + (M_{22}^\infty)^2 - J(M^\infty)| \ge \tau.$$

But conditions (A.2) force $\begin{pmatrix} M_{11}^\infty & M_{12}^\infty \\ M_{21}^\infty & M_{22}^\infty \end{pmatrix} = \begin{pmatrix} a & \mp b \\ b & \pm a \end{pmatrix}$ for some $a, b \in \mathbb{R}$, so the left-hand side of the last inequality vanishes, yielding the desired contradiction. □

Proposition A.7. *A continuous, $W^{1,2}$ map $u : B_1^2 \to \mathbb{R}^m$ solving a linear system of the form*

$$-\partial_i(g_{jk}\partial_i u^j) + b_{kpq}\partial_i u^p \partial_i u^q = 0,$$

with $g \ge \lambda > 0$ symmetric and continuous and b bounded, is $W^{1,r}_{loc}$ for all $r < \infty$.

The same holds for u defined on the half-ball $U' := B_1^2 \cap \{\Im(z) \ge 0\}$, if in addition we have

$$\partial_\nu u^k = 0 \text{ for } k \le n, \quad u^k = 0 \text{ for } k > n,$$

as well as $g_{ij} = 0$ for $i \le n$, $j > n$, on the boundary $\partial U'$, for some $0 \le n \le m$.

Remark A.8. The condition $\partial_\nu u^k = 0$ could be written more faithfully as $g_{jk}\partial_\nu \Phi^j = 0$ and is of course meant in a weak sense, coupled with the equation: namely, we require $\int_{U'}(g_{jk}\partial_i f \partial_i u^j + b_{kpq} f \partial_i u^p \partial_i u^q) = 0$ for all $f \in C_c^\infty(U')$ and $k \le n$, allowing f to be nonzero on $\partial U'$.

Proof. Assume u is a solution on the unit ball. Then, for any ball $B^2_{2r}(x) \subseteq B^2_1$, we can integrate the equation against $\eta^2(u - (u)_{B^2_{2r}(x)})$, where $\eta \in C_c^\infty(B^2_{2r}(x))$ is a cut-off function satisfying $\eta = 1$ on $B^2_r(x)$ and $|d\eta| \le \frac{2}{r}$. Recall that the notation $(u)_S$ indicates the average of u on a set S. This gives

$$\lambda \int \eta^2 |du|^2 \le C \int \eta |du|\, |d\eta|\, |u - (u)_{B^2_{2r}(x)}| + C \int \eta^2 |du|^2 \operatorname{osc}(u, B^2_{2r}(x))$$

and, applying Young's inequality, it follows that

$$\int_{B^2_r(x)} |du|^2 \le C r^{-2} \int_{B^2_{2r}(x)} |u - (u)_{B^2_{2r}(x)}|^2 \le C r^{-2} \Big(\int_{B^2_{2r}(x)} |du| \Big)^2$$

whenever $\operatorname{osc}(u, B^2_{2r}(x))$ is small enough. The classical Gehring's lemma (see, e.g., [43, Theorem V.1.2]) then implies that $du \in L^r(B)$ for some $r > 2$ and any fixed ball $B \subset\subset B^2_1$ (with r depending on B). Then the nonlinear term $b_{kpq}\partial_i u^p \partial_i u^q$ is $L^{r/2}(B)$ and standard elliptic regularity theory gives $du \in L^s_{loc}(B)$, with $\frac{1}{s} = \frac{2}{r} - \frac{1}{2}$, so that $s > r$; iterating, we get $du \in L^t_{loc}$ for any $t < \infty$.

If we are in the half-ball case, then we can reduce to the previous case by reflection. We extend g and u to $\widetilde{g}$ and $\widetilde{u}$ on the ball B^2_1, by means of the formula

$$g(s,-t) := U g(s,t) U, \qquad \begin{pmatrix} \widetilde{u}^1 \\ \vdots \\ \widetilde{u}^m \end{pmatrix}(s,-t) := U \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}(s,t)$$

for $(s,-t)$ in the lower half-ball, with $U := \begin{pmatrix} I_n & \\ & -I_{m-n} \end{pmatrix}$. Note that, by our hypotheses on g, $\widetilde{g}$ is still continuous. Also, it is straightforward to check that $\widetilde{u}$ solves

$$-\partial_i(\widetilde{g}_{jk}\partial_i \widetilde{u}^j) + \widetilde{b}_{kpq}\partial_i \widetilde{u}^p \partial_i \widetilde{u}^q = 0,$$

with $\widetilde{b}_{kpq}$ extending b_{kpq} according to the following rule: if $k \le n$ then $\widetilde{b}_{kpq}(s,-t) := b_{kpq}(s,t)$ if p and q belong to the same set in the partition $\{\{1,\dots,n\},\{n+1,\dots,m\}\}$, and $\widetilde{b}_{kpq}(s,-t) := -b_{kpq}(s,t)$ otherwise; if $k > n$ then the opposite holds. Then from the case of the full ball we deduce $d\widetilde{u} \in L^t_{loc}$ for any $t < \infty$. $\square$

Remark A.9. If the coefficients are smooth functions of u, then u is smooth. To check this, note that in the full ball case u is $C^{0,\alpha}_{loc}$ for any $\alpha < 1$. The same is then true for the coefficients $g_{jk}(u)$. Since the nonlinearity $b_{kpq}\partial_i u^p \partial_i u^q$ belongs to L^r_{loc} for all $r < \infty$, classical Schauder theory then gives $du \in C^{0,\alpha}_{loc}$ for all $\alpha < 1$ and bootstrapping we reach $u \in C^\infty$.

In the half-ball case, we can still argue in the same way that $d\widetilde{u} \in C^{0,\alpha}_{loc}$ for all $\alpha < 1$. So $\widetilde{g}$ is locally Lipschitz and we deduce $\widetilde{u} \in W^{2,r}_{loc}$ for all $r < \infty$. Differentiating the original equation in the first variable preserves the boundary conditions and leads to an equation of the form

$$\partial_i(g_{jk}\partial_i(\partial_1 u^j)) + f_k = 0$$

with $f_k \in L^r_{loc}$ for all $r < \infty$, and the same reflection trick (applied to $w := \partial_1 u$) gives $\partial_1 u \in W^{2,r}_{loc}$ for all $r < \infty$. Iterating we get the same for all derivatives $\partial_1^k u$. Now the equation allows to deduce inductively that $u \in W^{k,r}_{loc}$ for all k, since $g_{jk}(u)\Delta u^j = -\partial_i(g_{jk}(u))\partial_i u^j + b_{kpq}(u)\partial_i u^p \partial_i u^q$; this expresses $\partial_{22}u$ in terms of $\partial_{11}u$ and lower order derivatives and hence, for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geq 2$, we deduce that $\partial^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2} u \in L^r_{loc}$ for all $r < \infty$ from the same property enjoyed by $\partial_1^{\alpha_1+2}\partial_2^{\alpha_2-2}u$ and lower order derivatives of u.

The following statements deal with general varifolds. It is clear that we can assume the smallness constant c_V appearing in all of them to be always the same.

Lemma A.10. *There exists $c_V(\mathcal{M}, \mathcal{N}) > 0$ with the following property. Given $p \in \mathcal{N}$ and $0 < s < c_V$, for any 2-varifold $\mathbf{v}$ on $\mathcal{M}$ which is free boundary stationary outside $\overline{B}_s(p)$ and has density $\theta \geq \overline{\theta}$ on $\operatorname{spt}(|\mathbf{v}|) \setminus \overline{B}_s(p)$, either $\operatorname{spt}(|\mathbf{v}|) \subseteq B_{2s}(p)$ or $|\mathbf{v}|(\mathcal{M} \setminus \overline{B}_s(p)) \geq c_V\overline{\theta}$.*

Proof. Pick $\gamma > 0$ small, to be fixed along the proof; we will choose $c_V \leq \gamma$, so that the varifold is free boundary stationary outside $\overline{B}_\gamma(p)$ Possibly multiplying $\mathbf{v}$ by $\overline{\theta}^{-1}$, we can assume $\overline{\theta} = 1$. Note that if $q \in \operatorname{spt}(|\mathbf{v}|) \setminus B_{2\gamma}(p)$ then by (2.4.6) we have

$$|\mathbf{v}|(B_\gamma(q)) \geq c(\mathcal{M}, \mathcal{N})\gamma^2\theta(|\mathbf{v}|, q) \geq c(\mathcal{M}, \mathcal{N})\gamma^2. \tag{A.3}$$

Otherwise, $|\mathbf{v}|$ is supported in $B_{2\gamma}(p)$. Assume we are in this second case and pick a set of coordinates $(x_1, \dots, x_m) : B_{5\gamma}(p) \to \mathbb{R}^m$ centered at p, with $\mathcal{N}$ corresponding to $\{x_{n+1} = \cdots = x_m = 0\}$. We can impose that $\|g_{ij} - \delta_{ij}\|_{C^1} \leq \gamma$ (in coordinates), for γ small, independently of $p \in \mathcal{N}$.

On this ball, we define the vector field X to be $X(x) := \chi(|x|)x_i\frac{\partial}{\partial x_i}$, where $\chi : [0, \infty) \to [0, 1]$ is smooth and such that $\chi' \geq 0$ on $[0, 3\gamma]$, $\chi = 1$ on $[\frac{5}{3}s, 3\gamma]$, $\chi = 0$ on $[0, \frac{4}{3}s] \cup [4\gamma, \infty)$. Assuming $\{|x| \leq 4\gamma\} \subset\subset B_{5\gamma}(p)$, we can smoothly extend X to all of $\mathcal{M}$, with $X = 0$ outside the ball. For γ small enough (independently of p and $s < \gamma$), the C^1 closeness of g_{ij} to δ_{ij} guarantees

$$\operatorname{div}_\Pi X \geq 0$$

for all $(p, \Pi) \in \mathrm{Gr}_2(\mathcal{M})$ in the support of $\mathbf{v}$, since we can assume $\operatorname{spt}(|\mathbf{v}|) \subseteq \{|x| < 3\gamma\}$: indeed, here the contribution of χ' is nonnegative, while the one of the position vector $x_i\frac{\partial}{\partial x_i}$ is close to 2 (multiplied by $\chi(|x|)$). Also, the inequality is strict if $|x(p)| \geq \frac{5}{3}s$. Moreover, X

is tangent to $\mathcal{N}$. We can also assume that $\overline{B}_s(p) \subset\subset \{|x| \leq \frac{4}{3}s\}$; hence, we can test the stationarity of $\mathbf{v}$ against X and reach the contradiction

$$0 = \int_{(p,\Pi)\in \mathrm{Gr}_2(\mathcal{M})} \mathrm{div}_\Pi X \, d\mathbf{v}(p,\Pi) > 0$$

unless $\mathrm{spt}(|\mathbf{v}|)$ is contained in $\{|x| \leq \frac{5}{3}s\}$. Since the latter can be assumed to be included in $B_{2s}(p)$, the statement follows from (A.3). □

Remark A.11. The same statement holds if $\mathbf{v}$ is stationary, without the assumption $p \notin \mathcal{N}$. The proof is analogous (but simpler, in that we do not need coordinates adapted to $\mathcal{N}$).

Lemma A.12. *There exist $c_V > 0$ and $\delta : (0,\infty)^2 \to (0,\infty)$, with $\lim_{s\to 0} \delta(s,t) = 0$ for every t, satisfying the following property. Given two points $p_1, p_2 \in \mathcal{M}$ and a radius $s > 0$, let $B := \overline{B}_s(p_1) \cup \overline{B}_s(p_2)$; if a 2-varifold $\mathbf{v}$ on $\mathcal{M}$ is free boundary stationary outside B, has density $\theta \geq \overline{\theta}$ on $\mathrm{spt}(|\mathbf{v}|) \setminus B$ and satisfies the bound*

$$|\mathbf{v}|(B_r(q)) \leq c' r^2 \quad \textit{for all } q \in \mathcal{M},\ r > 0,$$

then either $|\mathbf{v}|(\mathcal{M}) \leq \overline{\theta}\delta(s, c'/\overline{\theta})$ or $|\mathbf{v}|(\mathcal{M}) \geq c_V \overline{\theta}$. The constant c_V and the function δ depend only on $\mathcal{M}$ and $\mathcal{N}$.

Proof. We can assume $\overline{\theta} = 1$. From (2.4.6) it follows that any nontrivial free boundary stationary varifold $\mathbf{v}'$ with density at least 1 on $\mathrm{spt}(|\mathbf{v}|)$ has $|\mathbf{v}'|(\mathcal{M}) \geq \lambda(\mathcal{M},\mathcal{N})$. Let $\delta(s,c')$ be the supremum of all possible masses $|\mathbf{v}|(\mathcal{M})$ which are smaller than c_V, for $\mathbf{v}$ as in the statement, with c_V to be specified below. Take a sequence $s_k \to 0$ of positive numbers and a sequence $\mathbf{v}_k$ satisfying the assumptions with $s = s_k$, as well as $\delta(s_k, c') - 2^{-k} < |\mathbf{v}_k|(\mathcal{M}) < c_V$.

Up to subsequences we get a limit varifold $\mathbf{v}_\infty$ which is free boundary stationary on the complement of two points $\overline{p}_1$ and $\overline{p}_2$. We still have $|\mathbf{v}_\infty|(B_r(q)) \leq c' r^2$ for all centers q and all radii r. This upper bound implies easily that actually $\mathbf{v}_\infty$ is free boundary stationary on the full manifold: see the proof of Theorem 2.5.11 for the details. Also, by (2.4.6) it has a lower bound $c \leq 1$ for its density on $\mathrm{spt}(|\mathbf{v}_\infty|)$. Hence, $|\mathbf{v}_\infty|(\mathcal{M}) \geq c\lambda$ unless $\mathbf{v}_\infty = 0$.

Since $|\mathbf{v}_\infty|(\mathcal{M}) = \lim_{k\to\infty} |\mathbf{v}_k|(\mathcal{M}) \leq c_V$, choosing any $c_V < c\lambda$ forces $\mathbf{v}_\infty = 0$, so that $\delta(s_k, c') \to 0$. This shows that $\delta(s,c') \to 0$ as $s \to 0$. □

3 Regularity of parametrized stationary varifolds

3.1 Introduction

In this chapter we study the regularity of *parametrized stationary varifolds*. We will deal only with closed domains for simplicity. Actually this is not restrictive, since the main result will be deduced from a local version, which is enough to obtain the full regularity also in the free boundary case (as seen in Section 2.7).

Recall that such varifolds are, essentially, integer stationary varifolds admitting a parametrization in the following sense: they are induced by a weakly conformal map $\Phi \in W^{1,2}(\Sigma, \mathcal{M})$, where Σ is a closed Riemann surface, together with a multiplicity function $N \in L^\infty(\Sigma, \mathbb{N} \setminus \{0\})$ *on the domain*. They are required to satisfy a natural stationarity property which is local in the domain: namely, we assume that, for almost all domains $\omega \subseteq \Sigma$, the varifold induced by the map $\Phi\big|_\omega$ with the multiplicity function $N\big|_\omega$ is stationary in the complement of the compact set $\Phi(\partial\omega)$ (see Definition 3.2.2 and Remark 3.2.3 for the precise definitions). In this chapter it is convenient to embed $\mathcal{M} \subset \mathbb{R}^Q$ isometrically, and to look also at the situation where $\mathcal{M}$ equals $\mathbb{R}^Q$ itself.

The main result is the following, which appears in Corollary 3.5.8 and Remark 3.5.9.

Theorem. *The triple* (Σ, Φ, N) *is a parametrized stationary varifold, in a compact manifold* $\mathcal{M} \subseteq \mathbb{R}^Q$ *or in* $\mathbb{R}^Q$ *itself, if and only if* Φ *is a smooth conformal harmonic map and* N *is a.e. constant. In this case,* Φ *is a minimal branched immersion.*

In fact we also get a local version of the result, which holds for *local parametrized stationary varifolds*: see Definition 3.2.9 and Theorem 3.5.7. Let us stress the fact that the result holds in *arbitrary codimension* and does not assume any *stability* hypothesis on the image varifold.

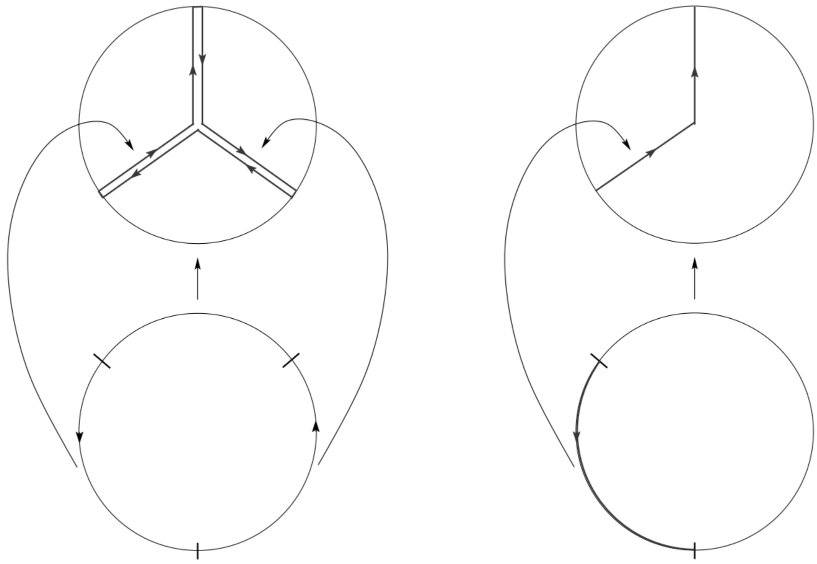
The possibility to localize the stationarity with respect to the domain is crucially used in many places in order to get our main regularity result. The weak conformality of Φ also happens to be important, since it ensures that the map Φ is holomorphic when the codomain is the plane, a fact which we establish in Section 3.3: the peculiar properties of nonconstant holomorphic functions, namely that they are branched covering maps and that they cannot vanish to infinite order, turn out to be useful several times.

Our main result relies on the previous treatment of the simpler situation where N is constant (see [92], by the second author; see also Theorem 3.2.13 below). We point out that the starting point of the strategy of [92], i.e. the observation that Lebesgue points z for Φ and $d\Phi$ (with $d\Phi(z) \neq 0$) belong to the regular set of Φ, does not apply when N is a priori not constant. This difficulty is due to the fact that we cannot invoke Allard's ε-regularity result in this more general situation.

We will actually show that N is a.e. constant and Φ satisfies the harmonic map equation $-\Delta\Phi = A(\Phi)(\nabla\Phi, \nabla\Phi)$ in any local conformal chart. These two facts are essentially equivalent to each other, in view of the preliminary work [92], which is presented also in Section 2.7; in some situations it will be easier to establish the former, while in other instances (where Lemma 3.5.1 is invoked) the latter is more convenient. The smoothness of weak solutions to the harmonic map equation was first proved in full generality by Hélein (see [51, Section 4.1]) and then obtained again by the second author in [89], as a consequence of a general result for linear elliptic systems with an antisymmetric potential. In our situation the smoothness of Φ is a classical fact, since for parametrized stationary varifolds the map Φ is easily seen to be continuous (see Proposition 3.2.4).

We want to give a simple one-dimensional example illustrating why, in spite of the difficulties surrounding integer stationary varifolds, one should expect to get much more information from the notion of parametrized stationary varifold (and eventually the regularity). The picture depicts a portion of an integer 1-rectifiable stationary varifold in S^2 (e.g. the union of three geodesic segments joining the north and south poles, with an angle $\frac{2\pi}{3}$ between any two of them), with multiplicity 2 a.e. This varifold has a nontrivial singular set.

This varifold can be parametrized by a map $\Phi : S^1 \to S^2$, as shown in the picture. However, the parametrized varifold $(S^1, \Phi, 1)$ fails to satisfy the local stationarity condition, as witnessed by the highlighted part on the right. Of course, this example is just a heuristic motivation for the hope to obtain the full regularity, as no effective structure theorem is known for two-dimensional stationary varifolds.



During the investigation of this problem, as a potential intermediate step towards the regularity, we asked ourselves whether any conformal solution $\Phi \in W^{1,2}(B_1^2(0), \mathbb{R}^Q)$ to the so-called *conductivity equation* $-\operatorname{div}(N\nabla\Phi) = 0$ (for some bounded measurable N with values in positive integers) is necessarily harmonic. Actually, we can give a positive answer to this question as a consequence of the main theorem.

Corollary. *Assume $\Phi \in W^{1,2}(B_1^2(0), \mathbb{R}^Q)$ is weakly conformal, $N \in L^\infty(B_1^2(0), \mathbb{N} \setminus \{0\})$ and*

$$-\operatorname{div}(N\nabla\Phi) = 0 \quad \textit{in } \mathcal{D}'(B_1^2(0), \mathbb{R}^Q).$$

Then $\Delta\Phi = 0$ and, if Φ is nonconstant, N is a.e. constant.

We refer to Theorem 3.6.2 in the body of the chapter. However, we are not aware of any purely analytic proof of this fact and leave it as an open problem to find such a proof. We are able to succeed in this task in the planar case $Q = 2$: see Theorem 3.6.1.

We end the introduction with a brief summary of the contents of the chapter.

- In Section 3.2 we establish some basic facts about parametrized stationary varifolds: we show the continuity of the parametrization map Φ (see Proposition 3.2.4), we define an upper semicontinuous representative $\widetilde{N}$ of the multiplicity function N satisfying $\widetilde{N} = N$ a.e. with respect to the measure $|\nabla\Phi|^2\mathcal{L}^2$ (see Definition 3.2.7 and Proposition 3.2.8) and we introduce a local notion of parametrized stationary varifold.

- In Section 3.3 we generalize the topological notion of triod (first introduced by Moore in [81]) and we show that the plane cannot contain uncountably many such disjoint generalized triods (see Lemma 3.3.2). We use this topological fact to show the regularity of parametrized stationary varifolds contained in a polyhedral cone (see Theorem 3.3.7). This special case of the problem turns out to be important in order to study a blow-up or a limiting situation obtained in later compactness arguments.

- In Section 3.4 we provide a general result (see Theorems 3.4.1 and 3.4.6) which allows to blow-up a varifold at a given point or along a sequence of points, with mild assumptions on the decay of the Dirichlet energy of Φ. We show that in the limit one still gets a parametrized stationary varifold and that the parametrization map is the blow-up of Φ, up to a quasiconformal homeomorphism.

- In Section 3.5, devoted to the regularity in the general case, we initially show a singularity removability lemma (see Lemma 3.5.1). Then we introduce the set of admissible points where the blow-up can be performed and we prove that the image of its complement has zero Hausdorff dimension (see Lemma 3.5.3). By means of delicate compactness arguments, using the results of Sections 3.3 and 3.4, we show that at any such point the speed of decay of the Dirichlet energy is controlled in a uniform way (see Lemma 3.5.4 and Corollary 3.5.6). We infer that admissible points are relatively open in a set where $\widetilde{N}$ is suitably pinched and we deduce the full regularity result by means of a final blow-up argument (see Theorem 3.5.7).

- In Section 3.6 we apply our regularity result to positively answer the aforementioned question on the conductivity equation. We also provide an independent, self-contained proof in the planar case $Q = 2$.

3.2 First properties of parametrized stationary varifolds

Let $\mathcal{M}^m$ be either a closed embedded C^∞-smooth submanifold of $\mathbb{R}^Q$ or $\mathbb{R}^Q$ itself, where $Q \geq m \geq 2$ are arbitrary integers. Let Σ be a closed connected Riemann surface.

Recall that a map $\Phi \in W^{1,2}(\Sigma, \mathbb{R}^Q)$ is *weakly conformal* if, for a.e. $x \in \Sigma$, $d\Phi(x)$ is either zero or a linear conformal map, with respect to the conformal structure of Σ. For any such map we call $\mathcal{G} \subseteq \Sigma$ the set of Lebesgue points for both Φ and $d\Phi$ and we let $\mathcal{G}^f := \{x \in \mathcal{G} : d\Phi(0) \neq 0\}$ (hence $d\Phi(x)$ is injective and conformal for $x \in \mathcal{G}^f$).

Recall the following notion, which already appeared in the previous chapter.

Definition 3.2.1. We say that a certain property holds *for almost every domain* $\omega \subseteq \Sigma$ if, for any nonnegative $\rho \in C^\infty(\Sigma)$, the property holds for $\omega = \{\rho > t\}$, for a.e. regular value $t > 0$ (so in particular it holds for Σ, as is seen by choosing $\rho \equiv 1$). Similarly, given an open set $\Omega \subseteq \mathbb{C}$, a property holds *for almost every domain* $\omega \subset\subset \Omega$ if, for any nonnegative $\rho \in C_c^\infty(\Omega)$, the property holds for $\omega = \{\rho > t\}$, for a.e. regular value $t > 0$.

In the definition below, we will implicitly restrict to the regular values $t > 0$ of ρ such that $\Phi\big|_{\partial\{\rho>t\}}$ has a continuous representative (which are a set of full measure, by Sard's theorem and [35, Theorem 4.21]) and, with a slight abuse of notation, $\Phi(\partial\omega)$ will denote the image by this continuous representative.

We can give the following alternative definition of parametrized stationary varifold.

Definition 3.2.2. A triple (Σ, Φ, N) with $\Phi \in W^{1,2}(\Sigma, \mathbb{R}^Q)$, $N \in L^\infty(\Sigma, \mathbb{N} \setminus \{0\})$ and $\Phi(\Sigma) \subseteq \mathcal{M}$ is called a *parametrized stationary varifold* (in $\mathcal{M}$) if Φ is weakly conformal and if, for almost every domain $\omega \subseteq \Sigma$,

$$\int_\omega N\Big(\langle d(F(\Phi)); d\Phi\rangle_h - F(\Phi) \cdot A(\Phi)(d\Phi, d\Phi)_h\Big)\, d\,\mathrm{vol}_h = 0 \tag{3.2.1}$$

for all $F \in C_c^\infty(\mathcal{M} \setminus \Phi(\partial\omega), \mathbb{R}^Q)$. Here h is an arbitrary Riemannian metric compatible with the conformal structure of Σ (its choice does not matter, by conformal invariance), $A(X, Y) = -\nabla_X^{\mathbb{R}^Q} Y$ denotes (minus) the second fundamental form of $\mathcal{M}$ in $\mathbb{R}^Q$ (so that $A = 0$ if $\mathcal{M} = \mathbb{R}^Q$) and $A(\Phi)(d\Phi, d\Phi)_h$ is defined in local coordinates by $\sum_{i,j} h^{ij} A(\Phi)(\partial_i\Phi, \partial_j\Phi)$.

We will usually just say that (Σ, Φ, N) is a *parametrized stationary varifold.*

Remark 3.2.3. The definition is clearly independent of the particular representatives of Φ, $d\Phi$, N. Calling $\mathcal{G}$ the set of Lebesgue points for both Φ and $d\Phi$ and applying Lemma A.2 to a finite atlas of conformal charts, we see that $\Phi(\mathcal{G})$ is $\mathcal{H}^2$-rectifiable (and $\mathcal{H}^2$-measurable). Moreover, again by Lemma A.2, the area formula applies and (3.2.1) amounts to say that for almost every $\omega \subseteq \Sigma$ the 2-rectifiable varifold

$$\mathbf{v}_\omega := (\Phi(\mathcal{G} \cap \omega), \theta_\omega), \quad \theta_\omega(p) := \sum_{x \in \mathcal{G} \cap \omega \cap \Phi^{-1}(p)} N(x)$$

is stationary in $\mathcal{M} \setminus \Phi(\partial\omega)$, as is easily seen by writing $F = \pi_{T\mathcal{M}} F + \pi_{T^\perp\mathcal{M}} F$. In particular, the generalized mean curvature of $\mathbf{v}_\omega$ in $\mathbb{R}^Q \setminus \Phi(\partial\omega)$ is bounded (in L^∞) by $\sqrt{2} \max_{\mathcal{M}} |A|$.

Proposition 3.2.4. *The map Φ has a continuous representative. This representative (still called Φ) satisfies the stationarity property for every open subset $\omega \subseteq \Sigma$, namely*

$$\int_\omega N\Big(\langle d(F \circ \Phi); d\Phi\rangle_h - F(\Phi) \cdot A(\Phi)(d\Phi, d\Phi)_h\Big)\, d\,\mathrm{vol}_h = 0$$

for all $F \in C_c^\infty(\mathcal{M} \setminus \Phi(\partial\omega), \mathbb{R}^Q)$. Moreover, if ω is connected, $\Phi(\overline{\omega}) = \mathrm{spt}\,(\|\mathbf{v}_\omega\|)$ unless Φ is constant on ω.

The first part of the statement was already proved in Section 2.7. However, we prefer to give another proof, since the estimates that we get here will be useful later on.

Proof. Let $\phi : U \to \Omega$ be a local conformal chart (with $U \subseteq \Sigma$ and $\Omega \subseteq \mathbb{C}$) and let $\Psi := \Phi \circ \phi^{-1}$, $\widetilde{\mathcal{G}} := \phi(\mathcal{G} \cap U)$ and $\widetilde{\mathcal{G}^f} := \phi(\mathcal{G}^f \cap U)$. For any $x \in \Omega$ and any $r < \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)$ we can apply Lemma A.3 (with $\tau = 1$) and obtain a radius $r' \in (r, 2r)$ such that (3.2.1) applies with $\omega = \phi^{-1}(B^2_{r'}(x))$ and such that

$$\mathrm{diam}\,\Psi(\partial B^2_{r'}(x)) \le \sqrt{4\pi}\left(\int_{B^2_{2r}(x)} |\nabla\Psi|^2\, d\mathcal{L}^2\right)^{1/2}, \quad \mathcal{H}^1(\partial B^2_{r'}(x) \setminus \widetilde{\mathcal{G}}) = 0. \tag{3.2.2}$$

Let us assume that Ψ is not (a.e.) constant on $B^2_{r'}(x)$. If $z \in \overline{B}^2_{r'}(x) \cap \widetilde{\mathcal{G}^f}$ we have $\Psi(z) \in \operatorname{spt}(\|\mathbf{v}_\omega\|)$, as

$$\|\mathbf{v}_\omega\|(B^Q_s(\Psi(z))) = \frac{1}{2}\int_{\Psi^{-1}(B^Q_s(\Psi(z)))\cap B^2_{r'}(x)} (N\circ\phi^{-1})|\nabla\Psi|^2\,d\mathcal{L}^2 > 0$$

for all $s > 0$. Hence, since $\operatorname{spt}(\|\mathbf{v}_\omega\|)$ is closed in $\mathbb{R}^Q$, by Lemma A.1 the essential image of $\Psi\big|_{B^2_{r'}(x)}$ is included in $\operatorname{spt}(\|\mathbf{v}_\omega\|)$. The converse inclusion trivially holds, as well, so we conclude that the essential image of $\Psi\big|_{B^2_{r'}(x)}$ coincides with $\operatorname{spt}(\|\mathbf{v}_\omega\|)$; in particular, the latter includes the compact set $\Gamma := \Psi(\partial B^2_{r'}(x))$ (by the second part of (3.2.2)).

Moreover, since in $\mathbb{R}^Q \setminus \Gamma$ the varifold $\mathbf{v}_\omega$ has generalized mean curvature bounded by $\sqrt{2}\|A\|_{L^\infty}$, from the monotonicity formula [98, Theorem 17.6] and [98, Remark 17.9(1)] we deduce

$$\frac{1}{2}\int_{B^2_{2r}(x)} (N\circ\phi^{-1})|\nabla\Psi|^2\,d\mathcal{L}^2 \geq \|\mathbf{v}_\omega\|(B^Q_s(p)) \geq e^{-(\sqrt{2}\|A\|_{L^\infty})s}\cdot \pi s^2$$

for all $p \in \operatorname{spt}(\|\mathbf{v}_\omega\|) \setminus \Gamma$ and all $s \leq \operatorname{dist}(p,\Gamma)$. If $\mathcal{M}$ is compact, choosing $s := \operatorname{dist}(p,\Gamma) \leq \operatorname{diam}\mathcal{M}$ and recalling (3.2.2), we conclude that

$$\begin{aligned}
&\operatorname{diam}\Psi(\widetilde{\mathcal{G}}\cap B^2_r(x)) \leq \operatorname{diam}\operatorname{spt}(\|\mathbf{v}_\omega\|) \leq \operatorname{diam}\Gamma + 2\max_{p\in\operatorname{spt}(\|\mathbf{v}_\omega\|)}\operatorname{dist}(p,\Gamma)\\
&\leq 2\left(\sqrt{\pi} + \left(e^{(\sqrt{2}\|A\|_{L^\infty})\operatorname{diam}\mathcal{M}}\frac{\|N\|_{L^\infty}}{2\pi}\right)^{1/2}\right)\left(\int_{B^2_{2r}(x)}|\nabla\Psi|^2\,d\mathcal{L}^2\right)^{1/2}.
\end{aligned} \tag{3.2.3}$$

If instead $\mathcal{M} = \mathbb{R}^Q$, then we have

$$\begin{aligned}
&\operatorname{diam}\Psi(\widetilde{\mathcal{G}}\cap B^2_r(x)) \leq \operatorname{diam}\operatorname{spt}(\|\mathbf{v}_\omega\|) \leq \operatorname{diam}\Gamma + 2\sup_{p\in\operatorname{spt}(\|\mathbf{v}_\omega\|)}\operatorname{dist}(p,\Gamma)\\
&\leq 2\left(\sqrt{\pi} + \left(\frac{\|N\|_{L^\infty}}{2\pi}\right)^{1/2}\right)\left(\int_{B^2_{2r}(x)}|\nabla\Psi|^2\,d\mathcal{L}^2\right)^{1/2}.
\end{aligned} \tag{3.2.4}$$

This estimate for $\operatorname{diam}\Psi(\widetilde{\mathcal{G}}\cap B^2_r(x))$ is trivially true also when Ψ is a.e. constant on $B^2_{r'}(x)$. The last expressions are infinitesimal as $r \to 0$, locally uniformly in x. We infer that $\Psi\big|_{\widetilde{\mathcal{G}}}$ is locally uniformly continuous on Ω and thus has a continuous representative. This shows that Φ has a continuous representative.

We record here another estimate for $\operatorname{diam}\operatorname{spt}(\|\mathbf{v}_\omega\|)$ independent of $\operatorname{diam}\mathcal{M}$, which will be useful later. All the points in $\operatorname{spt}(\|\mathbf{v}_\omega\|)$ have distance at most $2D+2$ from Γ, where $D := \frac{e^H}{\pi}\|\mathbf{v}_\omega\|(\mathbb{R}^Q)$ and H is an upper bound for the generalized mean curvature of $\mathbf{v}_\omega$ in $\mathbb{R}^Q \setminus \Gamma$: if this were not the case, we would have Φ nonconstant on ω and thus $\Gamma \subseteq \Phi(\overline{\omega}) = \operatorname{spt}(\|\mathbf{v}_\omega\|)$. By connectedness of $\Phi(\overline{\omega})$ we could find points $p_j \in \operatorname{spt}(\|\mathbf{v}_\omega\|)$ such that $\operatorname{dist}(p_j,\Gamma) = 2j$, for $1 \leq j \leq \lfloor D\rfloor + 1$, and since the balls $B^Q_1(p_j)$ are disjoint we would have

$$\|\mathbf{v}_\omega\|(\mathbb{R}^Q) \geq \sum_j \|\mathbf{v}_\omega\|(B^Q_1(p_j)) \geq (\lfloor D\rfloor + 1)e^{-H}\pi > De^{-H}\pi,$$

which is a contradiction. We deduce that

$$\operatorname{diam} \operatorname{spt}(\|\mathbf{v}_\omega\|) \le \operatorname{diam}\Gamma + \frac{4e^H}{\pi}\|\mathbf{v}_\omega\|(\mathbb{R}^Q) + 4. \tag{3.2.5}$$

We now show the statement about the stationarity property. If $F \in C_c^\infty(\mathcal{M} \setminus \Phi(\partial\omega), \mathbb{R}^Q)$, then we can find a nonnegative $\rho \in C_c^\infty(\omega)$ such that $\rho = 1$ on the compact set $\omega \cap \Phi^{-1}(\operatorname{spt}(F))$. For almost every $t \in (0,1)$ the stationarity property (3.2.1) holds in $\{\rho > t\}$, so

$$\int_{\{\rho>t\}} N\Big(\langle d(F\circ\Phi); d\Phi\rangle_h - F(\Phi)A(\Phi)(d\Phi, d\Phi)_h\Big)\, d\operatorname{vol}_h = 0$$

and clearly the left-hand side does not change if we replace $\{\rho > t\}$ with ω.

Finally, the last statement is obtained with the same argument used in the first part of the proof. $\square$

From now on, we will always assume that the map Φ is continuous, for any parametrized stationary varifold.

Proposition 3.2.5. *We have $\mathcal{H}^2(\Phi(\Sigma \setminus \mathcal{G}^f)) = 0$.*

Proof. As already observed in Remark 3.2.3, the area formula can be applied on subsets of $\mathcal{G}$. In particular, since $d\Phi = 0$ on $\mathcal{G} \setminus \mathcal{G}^f$, we get $\mathcal{H}^2(\Phi(\mathcal{G} \setminus \mathcal{G}^f)) = 0$. In order to show that $\mathcal{H}^2(\Phi(\Sigma \setminus \mathcal{G})) = 0$, we pick any local conformal chart $\phi : U(\subseteq \Sigma) \to \Omega(\subseteq \mathbb{C})$ and, as in the previous proof, we set $\Psi := \Phi \circ \phi^{-1}$ and $\widetilde{\mathcal{G}} := \phi(\mathcal{G} \cap U)$.

Fix an arbitrary $\delta > 0$ and an open set $W \subseteq \Omega$ containing $\Omega \setminus \widetilde{\mathcal{G}}$. For any $z \in W$ we can find a radius $r < \frac{1}{2}\operatorname{dist}(z, \partial W) \wedge 1$ such that $\overline{C}\Big(\int_{B^2_{2r}(z)} |\nabla\Psi|^2\, d\mathcal{L}^2\Big)^{1/2} < \delta$, where $\overline{C}$ is the constant appearing in the right-hand side of (3.2.3) (or (3.2.4) if $\mathcal{M} = \mathbb{R}^Q$), and

$$\int_{B^2_{2r}(z)} |\nabla\Psi|^2\, d\mathcal{L}^2 \le 8\int_{B^2_r(z)} |\nabla\Psi|^2\, d\mathcal{L}^2 + 4r^2 :$$

indeed, if such r did not exist, for j big enough we would have $(2^{-j})^2 \le \int_{B^2_{2^{-j}}(z)} |\nabla\Psi|^2\, d\mathcal{L}^2 \le \frac{1}{8}\int_{B^2_{2^{-j+1}}(z)} |\nabla\Psi|^2\, d\mathcal{L}^2$, hence $(2^{-j})^2 \le \int_{B^2_{2^{-j}}(z)} |\nabla\Psi|^2\, d\mathcal{L}^2 = O(2^{-3j}) = o((2^{-j})^2)$, which is a contradiction. By Besicovitch covering theorem, we can extract countably many balls $B^2_{r_i}(x_i)$ from this collection with $\mathbf{1}_W \le \sum_i \mathbf{1}_{B^2_{r_i}(x_i)} \le \mathfrak{N}\mathbf{1}_W$, for some universal constant $\mathfrak{N}$. By inequality (3.2.3) (or (3.2.4)) we have $\operatorname{diam}\Phi(B^2_{r_i}(x_i)) < \delta$ and

$$\begin{aligned}
\sum_i \frac{\pi}{4}(\operatorname{diam}\Phi(B^2_{r_i}(x_i)))^2 &\le \frac{\pi\overline{C}^2}{4}\sum_i \int_{B^2_{2r_i}(x_i)} |\nabla\Psi|^2\, d\mathcal{L}^2 \\
&\le 2\pi\overline{C}^2 \sum_i \int_{B^2_{r_i}(x_i)} |\nabla\Psi|^2\, d\mathcal{L}^2 + \overline{C}^2\mathcal{L}^2(B^2_{r_i}(x_i)) \\
&\le 2\pi\overline{C}^2\mathfrak{N}\int_W |\nabla\Psi|^2\, d\mathcal{L}^2 + \overline{C}^2\mathfrak{N}\mathcal{L}^2(W).
\end{aligned}$$

Since δ was arbitrary, we get $\mathcal{H}^2(\Phi(\Omega \setminus \widetilde{\mathcal{G}})) \le 2\pi\overline{C}^2\mathfrak{N}\int_W |\nabla\Psi|^2\, d\mathcal{L}^2 + \overline{C}^2\mathfrak{N}\mathcal{L}^2(W)$. Since $\mathcal{L}^2(\Omega \setminus \widetilde{\mathcal{G}}) = 0$ and W was arbitrary as well, we arrive at $\mathcal{H}^2(\Phi(\Omega \setminus \widetilde{\mathcal{G}})) = 0$. $\square$

Proposition 3.2.6. *For any $p \in \mathcal{M}$ the compact set $\Phi^{-1}(p)$ has finitely many connected components. If $x \in \mathcal{G}^f$ then x is isolated in $\Phi^{-1}(\Phi(x))$.*

Proof. Assume without loss of generality that Φ is not constant; recall that in this chapter we assume Σ to be connected. Since the varifold $\mathbf{v}_\Sigma$ is stationary, the limit

$$M := \lim_{s \to 0} \frac{\|\mathbf{v}_\Sigma\|(B_s^Q(p))}{\pi s^2}$$

exists. We claim that the number of connected components of $\Phi^{-1}(p)$ is not greater than M. If this were not the case, we could split $\Phi^{-1}(p) = \bigsqcup_{j=1}^J K_j$, where the subsets K_j are disjoint and compact and $J > M$ is a finite integer (if the maximum value of J such that this can be done were at most M, then one of the subsets would be disconnected, contradicting the maximality). We could then find disjoint open neighborhoods $\omega_j \supseteq K_j$ and we would have

$$M = \lim_{s \to 0} \frac{\|\mathbf{v}_\Sigma\|(B_s^Q(p))}{\pi s^2} \geq \sum_{j=1}^J \lim_{s \to 0} \frac{\|\mathbf{v}_{\omega_j}\|(B_s^Q(p))}{\pi s^2} \geq J$$

(by [98, Remark 17.9(1)]: notice that Φ must be nonconstant on any connected component of ω_j intersecting K_j, hence by Proposition 3.2.4 $p \in \operatorname{spt}\left(\|\mathbf{v}_{\omega_j}\|\right)$). This is a contradiction.

Assume now $x \in \mathcal{G}^f$ and call K_x the connected component of $\Phi^{-1}(\Phi(x))$ containing x. It suffices to show that $K_x = \{x\}$, since we already know that $\Phi^{-1}(\Phi(x))$ is a finite union of compact connected sets. If ϕ is a local conformal chart centered at x and $\Psi := \Phi \circ \phi^{-1}$, as in the proof of Proposition 3.2.8 below we can find a radius $r' > 0$ such that $\Phi(x) = \Psi(0) \notin \Psi(\partial B_{r'}^2(0))$. Hence $K_x \subseteq \phi^{-1}(B_{r'}^2(0))$ and, since r' is arbitrarily small, we deduce $K_x = \phi^{-1}(\{0\}) = \{x\}$. ☐

In the remainder of the section we assume that Φ is not constant. We now define a more robust representative $\widetilde{N}$ of the multiplicity function N, which is canonically defined everywhere and is upper semicontinuous. We point out that (a priori) $\widetilde{N}$ could take values in $[1, \infty)$ instead of $\mathbb{N} \setminus \{0\}$.

Definition 3.2.7. Given $x \in \Sigma$, we call K_x the connected component of $\Phi^{-1}(\Phi(x))$ containing x and we let

$$\widetilde{N}(x) := \inf_{\substack{\omega \supseteq K_x, \\ \Phi(x) \notin \Phi(\partial\omega)}} \lim_{s \to 0} \frac{\|\mathbf{v}_\omega\|(B_s^Q(\Phi(x)))}{\pi s^2}.$$

The limit exists and is at least 1, by the stationarity of $\mathbf{v}_\omega$ in $\mathcal{M} \setminus \Phi(\partial\omega)$ (which contains $\Phi(x)$) and the fact that $\Phi(x) \in \operatorname{spt}(\|\mathbf{v}_\omega\|)$ (by Proposition 3.2.4, since Φ is necessarily nonconstant on the connected component of ω containing x). Notice that $\widetilde{N} = \widetilde{N}(x)$ on K_x. Moreover, the infimum is actually a minimum and is achieved whenever $\overline{\omega}$ is disjoint from the compact set $\Phi^{-1}(\Phi(x)) \setminus K_x$.

Proposition 3.2.8. *The function $\widetilde{N}$ is upper semicontinuous and $\widetilde{N} \geq 1$. Moreover, $\widetilde{N} = N$ a.e. on $\mathcal{G}^f$.*

Proof. We already observed that $\widetilde{N} \geq 1$ everywhere. Let $\lambda > 1$ and $x \in \Sigma$ such that $\widetilde{N}(x) < \lambda$. Choose any open set $\omega \supseteq K_x$ with $\overline{\omega}$ disjoint from $\Phi^{-1}(\Phi(x)) \setminus K_x$, so that $\lim_{s \to 0} \frac{\|\mathbf{v}_\omega\|(B_s^Q(\Phi(x)))}{\pi s^2} < \lambda$. Whenever $z \in \omega$ is close enough to x we have $\Phi(z) \notin \Phi(\partial\omega)$, so $K_z \subseteq \omega$ and by definition

$$\widetilde{N}(z) \leq \lim_{s \to 0} \frac{\|\mathbf{v}_\omega\|(B_s^Q(\Phi(z)))}{\pi s^2}.$$

But as $z \to x$ we have $\Phi(z) \to \Phi(x)$. Hence, eventually $\lim_{s \to 0} \frac{\|\mathbf{v}_\omega\|(B_s^Q(\Phi(z)))}{\pi s^2} < \lambda$ (see [98, Corollary 17.8]) and so $\widetilde{N}(z) < \lambda$.

Assume now $x \in \mathcal{G}^f$ and $\int_{B_r^2(x)} |N - N(x)|\, d\mathcal{L}^2 = o(r^2)$, $\int_{B_r^2(x)} |\nabla\Phi - \nabla\Phi(x)|^2\, d\mathcal{L}^2 = o(r^2)$. Fix any open set ω containing x. Let ϕ be a local conformal chart centered at x and set $\Psi := \Phi \circ \phi^{-1}$, $\alpha := \left(\frac{|\nabla\Psi(0)|}{\sqrt{2}}\right)^{-1}$. For any $s > 0$ small enough we have

$$\begin{aligned}
\|\mathbf{v}_\omega\|(B_s^Q(\Phi(x))) &\geq \frac{1}{2} \int_{B^2_{\alpha s}(0) \cap \Psi^{-1}(B_s^Q(\Psi(0)))} (N \circ \phi^{-1}) |\nabla\Psi|^2\, d\mathcal{L}^2 \\
&\geq \frac{1}{2} N(x) |\nabla\Psi(0)|^2 \mathcal{L}^2\Big(B^2_{\alpha s}(0) \cap \Psi^{-1}(B_s^Q(\Psi(0)))\Big) \\
&\quad - \frac{1}{2} \int_{B^2_{\alpha s}(0)} \Big| (N \circ \phi^{-1}) |\nabla\Psi|^2 - N(x) |\nabla\Psi(0)|^2 \Big|\, d\mathcal{L}^2.
\end{aligned}$$

By [35, Theorem 6.1], the function $s^{-1}(\Psi(\alpha s\, \cdot) - \Psi(0))$ converges to $\alpha\langle\nabla\Psi(0), \cdot\rangle$ (which is a linear isometry) in measure on $B_1^2(0)$, hence the first term in the right-hand side is $\pi N(x) s^2 + o(s^2)$. Moreover, the function $N \circ \phi^{-1}(\alpha s\, \cdot)$ converges to $N(x)$ in measure and is bounded by $\|N\|_{L^\infty}$, while $|\nabla\Psi|^2(\alpha s\, \cdot) \to |\nabla\Psi(0)|^2$ in $L^1(B_1^2(0))$. So the last term in the right-hand side is $o(s^2)$. This shows that $\widetilde{N}(x) \geq N(x)$.

Fix now any $0 < \varepsilon < \alpha^{-1}$. By Lemma A.4, applied to $y \mapsto \Psi(y) - \Psi(0) - \langle\nabla\Psi(0), y\rangle$, we can find a radius r' such that

$$\big|\Psi(r'y) - \Psi(0) - \big\langle\nabla\Psi(0), r'y\big\rangle\big| \leq \varepsilon r'$$

for all $y \in S^1$. Thus, choosing $\omega := \phi^{-1}(B_{r'}^2(0))$ and applying the monotonicity formula,

$$\widetilde{N}(x) \leq e^{(\sqrt{2}\|A\|_{L^\infty})(\beta - \varepsilon) r'} \frac{\|\mathbf{v}_\omega\|(B^Q_{(\beta-\varepsilon)r'}(\Phi(x)))}{\pi(\beta - \varepsilon)^2 (r')^2} \leq (1 + O(r')) \frac{\int_{B^2_{r'}(0)} (N \circ \phi^{-1}) |\nabla\Psi|^2\, d\mathcal{L}^2}{2\pi(\beta - \varepsilon)^2 (r')^2},$$

where $\beta := \frac{|\nabla\Psi(0)|}{\sqrt{2}} = \alpha^{-1}$. Since r' is arbitrarily small, we get $\widetilde{N}(x) \leq \frac{N(x)\beta^2}{(\beta - \varepsilon)^2}$. Letting $\varepsilon \to 0$ we get the converse inequality $\widetilde{N}(x) \leq N(x)$. $\square$

It is useful to introduce the following local notion of parametrized stationary varifold.

Definition 3.2.9. Let $\Omega \subseteq \mathbb{C}$ be open. A triple (Ω, Φ, N) with $\Phi \in W^{1,2}_{loc}(\Omega, \mathbb{R}^Q)$, $N \in L^\infty(\Omega, \mathbb{N} \setminus \{0\})$ and $\Phi(\Omega) \subseteq \mathcal{M}$ is called a *local parametrized stationary varifold* (in $\mathcal{M}$) if Φ is weakly conformal and if, for almost every domain $\omega \subset\subset \Omega$,

$$\int_\omega N\Big(\langle\nabla(F(\Phi)); \nabla\Phi\rangle - F(\Phi) \cdot A(\Phi)(\nabla\Phi, \nabla\Phi)\Big)\, d\mathcal{L}^2 = 0$$

for all $F \in C_c^\infty(\mathcal{M} \setminus \Phi(\partial\omega), \mathbb{R}^Q)$, with $A(\Phi)(\nabla\Phi, \nabla\Phi) := A(\Phi)(\partial_1\Phi, \partial_1\Phi) + A(\Phi)(\partial_2\Phi, \partial_2\Phi)$. We also require that

$$\|\mathbf{v}_\Omega\|(B_s^Q(p)) = \frac{1}{2}\int_{\Phi^{-1}(B_s^Q(p))} N|\nabla\Phi|^2\, d\mathcal{L}^2 = O(s^2), \tag{3.2.6}$$

uniformly in $p \in \mathbb{R}^Q$.

Notice that, in the last definition, the map Φ is allowed to be constant. The technical assumption (3.2.6) will be used only in the proof of Lemma 3.5.1, which in turn is used in Sections 3.3 and 3.5.

Remark 3.2.10. If (Σ, Φ, N) is a parametrized stationary varifold and $\phi : U(\subseteq \Sigma) \to \Omega(\subseteq \mathbb{C})$ is a local conformal chart, then $(\Omega, \Phi \circ \phi^{-1}, N \circ \phi^{-1})$ is a local parametrized stationary varifold: assumption (3.2.6) holds thanks to the monotonicity formula satisfied by $\mathbf{v}_\Sigma$.

Remark 3.2.11. Proposition 3.2.4 applies to the local case as well (with $\omega \subset\subset \Omega$ in the statement), with the same proof: hence, we will tacitly assume that Φ is continuous for all local parametrized stationary varifolds. The same is true for Proposition 3.2.5.

The first part of Proposition 3.2.6 holds whenever $\Phi^{-1}(p)$ is compact (with the same proof with a neighborhood $\Phi^{-1}(p) \subseteq \omega \subset\subset \Omega$ in place of Σ), while the second part holds in general (since, in its proof, we can apply the first part to the domain $\phi^{-1}(B_{r'}^2(0))$).

The domain of definition of the function $\widetilde{N}$, i.e. the set $\{\widetilde{N} < \infty\}$, is an open subset of Ω. The same argument of Proposition 3.2.6 shows that it consists of all points x such that K_x is compact and disjoint from the closure of $\Phi^{-1}(\Phi(x)) \setminus K_x$. Proposition 3.2.8 always holds in the local case, with the same proof.

Remark 3.2.12. A useful fact which will be used in Sections 3.3 and 3.5 is the following: if (Ω, Φ, N) is a local parametrized stationary varifold and $x \in \Omega$ satisfies $\widetilde{N}(x) < \infty$, then for any $0 < \varepsilon < 1$ we can find a neighborhood $K_x \subseteq \omega \subset\subset \Omega$ with $\Phi(\omega) \cap \Phi(\partial\omega) = \emptyset$ and such that $\omega \cap \Phi^{-1}(\Phi(y)) = K_y$ whenever $y \in \omega$ has $\widetilde{N}(y) \geq \widetilde{N}(x) - \varepsilon$.

Indeed, let $K_x \subseteq \omega \subset\subset \Omega$ with $\overline{\omega}$ disjoint from $\Phi^{-1}(\Phi(x)) \setminus K_x$, so that $\widetilde{N}(x)$ is the density of $\mathbf{v}_\omega$ at $\Phi(x)$. If $y_1, y_2 \in \omega$ have the same image and are close enough to K_x, then $K_{y_1}, K_{y_2} \subseteq \omega$ (since $\Phi(y_1) = \Phi(y_2) \notin \Phi(\partial\omega)$) and the density of $\mathbf{v}_\omega$ at $\Phi(y_1) = \Phi(y_2)$ is less than $\widetilde{N}(x) + 1 - \varepsilon$ (by upper semicontinuity of the density). Hence, if $K_{y_1} \neq K_{y_2}$, calling $K_{y_i} \subseteq \omega_i \subseteq \omega$ two disjoint neighborhoods with $\Phi(y_i) \notin \Phi(\partial\omega_i)$ we get

$$\begin{aligned}\widetilde{N}(y_1) + 1 \leq \widetilde{N}(y_1) + \widetilde{N}(y_2) &\leq \lim_{s\to 0} \frac{\|\mathbf{v}_{\omega_1}\|(B_s^Q(\Phi(y_1)))}{\pi s^2} + \lim_{s\to 0} \frac{\|\mathbf{v}_{\omega_2}\|(B_s^Q(\Phi(y_2)))}{\pi s^2} \\ &\leq \lim_{s\to 0} \frac{\|\mathbf{v}_\omega\|(B_s^Q(\Phi(y_1)))}{\pi s^2} < \widetilde{N}(x) + 1 - \varepsilon.\end{aligned}$$

The claim is established by shrinking ω and by replacing it with $\omega \setminus \Phi^{-1}(\Phi(\partial\omega))$.

The following theorem is the main result of [92]. Its proof has already been given in Section 2.7; note that the proof carries over to the local case, and to the situation $\mathcal{M} = \mathbb{R}^Q$, as well.

Theorem 3.2.13. *Let (Σ, Φ, N) be a parametrized stationary varifold in $\mathcal{M}$. If N is a.e. constant (and hence can be changed to 1 without affecting the stationarity), then $\Phi \in C^\infty(\Sigma, \mathcal{M})$ and $-\Delta_h \Phi = A(\Phi)(d\Phi, d\Phi)_h$. The same holds for local parametrized stationary varifolds.*

3.3 Regularity of parametrized stationary varifolds in a polyhedral cone

This section addresses the regularity problem for local parametrized stationary varifolds in $\mathbb{R}^Q$, under the additional constraint that they are contained in a finite union of 2-dimensional planes through the origin. We will need a nontrivial result in planar topology, which we state and prove below.

A topological lemma about triods

Definition 3.3.1. A *generalized triod* in S^2 is a quadruple $T = (K, \gamma_1, \gamma_2, \gamma_3)$ such that:

- $\emptyset \neq K \subseteq S^2$ is compact and connected;
- $\gamma_i \in C^\infty([0,1], S^2)$ are injective regular curves (i.e. $\dot\gamma(t) \neq 0$ for all $t \in [0,1]$);
- K, $\gamma_1([0,1))$, $\gamma_2([0,1))$, $\gamma_3([0,1))$ are pairwise disjoint;
- $\gamma_i(1) \in K$.

We will denote $\operatorname{spt}(T) := K \sqcup \gamma_1([0,1)) \sqcup \gamma_2([0,1)) \sqcup \gamma_3([0,1))$.

The proof of the following lemma is inspired by the proof of a simpler statement which appears in [2, Lemma 2.15].

Lemma 3.3.2. *Let $(T_j)_{j\in J}$ be a collection of generalized triods in S^2 such that $\operatorname{spt}(T_j) \cap \operatorname{spt}\left(T_{j'}\right) = \emptyset$ for any $j \neq j'$. Then J is at most countable.*

Proof. We equip the set $\mathcal{T}$ of all generalized triods in S^2 with the following metric: given $T = (K, \gamma_1, \gamma_2, \gamma_3)$, $T' = (K', \gamma_1', \gamma_2', \gamma_3') \in \mathcal{T}$ we set

$$d(T, T') := d_H(K, K') + \sum_i \max_{t\in[0,1]} d_{S^2}(\gamma_i(t), \gamma_i'(t)),$$

where d_{S^2} denotes the spherical distance on S^2 and d_H is the corresponding Hausdorff distance on the set of all nonempty compact subsets of S^2.

Since the metric space $(\mathcal{T}, d)$ is separable, it suffices to show that any triod T_{j_0} is isolated in $\{T_j \mid j \in J\} \subseteq \mathcal{T}$. Let $T_{j_0} = (K, \gamma_1, \gamma_2, \gamma_3)$.

Case 1: $\gamma_1([0,1)), \gamma_2([0,1)), \gamma_3([0,1))$ do not belong to the same connected component of $S^2 \setminus K$. Assume for instance that $\gamma_1([0,1))$ and $\gamma_2([0,1))$ belong to different connected components: then, letting

$$\varepsilon := \min \{d_{S^2}(\gamma_1(0), K), d_{S^2}(\gamma_2(0), K)\} > 0,$$

any different triod $T_j = (K', \gamma_1', \gamma_2', \gamma_3')$ satisfies $d(T_{j_0}, T_j) \geq \varepsilon$. Indeed, if this were not the case, $\gamma_1'(0)$ would lie in the same component of $S^2 \setminus K$ as $\gamma_1(0)$ (since the spherical ball $B_\varepsilon^{S^2}(\gamma_1(0))$ is a connected subset of $S^2 \setminus K$) and similarly for $\gamma_2'(0)$. But this contradicts the fact that $\operatorname{spt}(T_j)$ is a connected subset of $S^2 \setminus K$.

Case 2: $\gamma_1([0,1)), \gamma_2([0,1)), \gamma_3([0,1))$ belong to the same connected component U of $S^2 \setminus K$. Since K is connected, there exists a diffeomorphism

$$\upsilon : U \to B_1^2(0) \subseteq \mathbb{C}$$

(indeed, $S^2 \setminus U$ is connected and we can apply [94, Theorems 13.11 and 14.8]). For $t \in [0,1)$ let $\alpha_i(t) := \upsilon \circ \gamma_i(t)$. Notice that $\lim_{t \to 1} |\alpha_i(t)| = 1$. Up to applying another diffeomorphism, we can assume that $|\alpha_i(0)| = \frac{1}{2}$ and $|\alpha_i(t)| > \frac{1}{2}$ for $t \in (0,1)$ and $i = 1, 2, 3$ (e.g. by adapting the argument in [63, Theorem II.5.2]).

Let $s_i := \min \left\{t : |\alpha_i(t)| = \frac{3}{4}\right\} > 0$. Moreover, for any $\tau \in \left(\frac{3}{4}, 1\right)$ let

$$r_i(\tau) := \min \{t : |\alpha_i(t)| = \tau\} > s_i.$$

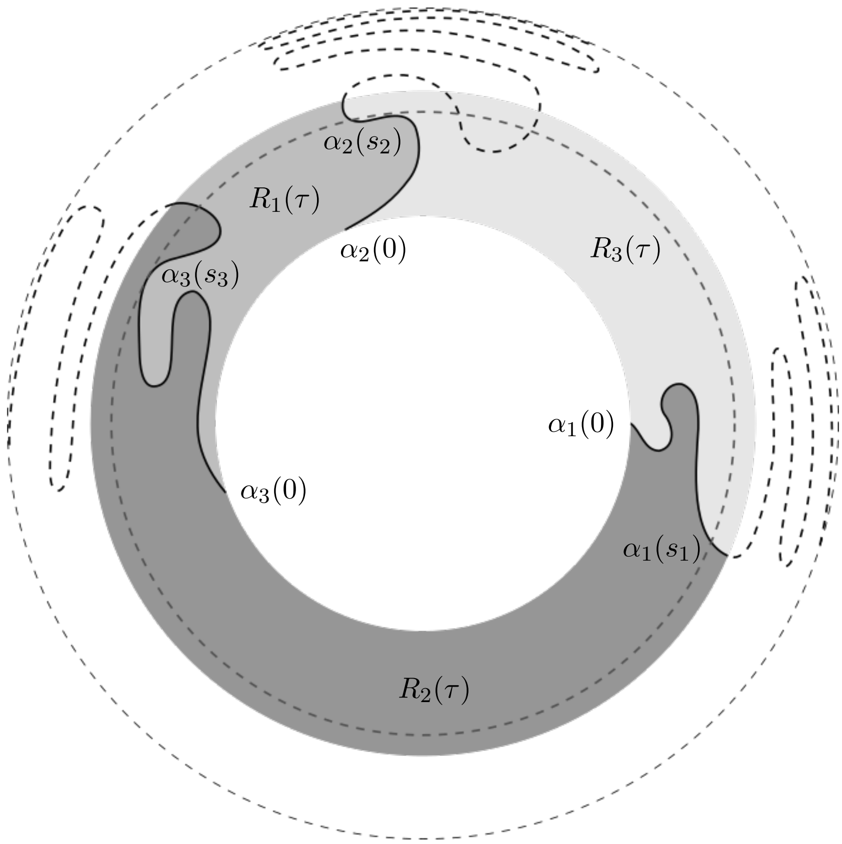
By Jordan's closed curve theorem for piecewise smooth curves, the points $\alpha_i(0)$ and $\alpha_i(r_i(\tau))$ are in the same order on the circles $\left\{|z| = \frac{1}{2}\right\}$ and $\{|z| = \tau\}$. The curves $\alpha_i([0, r_i(\tau)])$ and the circles $\left\{|z| = \frac{1}{2}\right\}$, $\{|z| = \tau\}$ bound three disjoint domains $R_1(\tau)$, $R_2(\tau)$, $R_3(\tau)$; we adopt the convention that $R_i(\tau)$ is the region whose closure is disjoint from $\alpha_i([0, r_i(\tau)])$. Let

$$\delta := \inf_{\tau \in \left(\frac{3}{4}, 1\right)} \min_i d_{\mathbb{R}^2}(\alpha_i(0), \overline{R_i(\tau)}) \tag{3.3.1}$$

and notice that, since

$$d_{\mathbb{R}^2}(\alpha_i(0), \overline{R_i(\tau)}) = d_{\mathbb{R}^2}(\alpha_i(0), \partial R_i(\tau)) = d_{\mathbb{R}^2}(\alpha_i(0), \partial R_i(\tau) \setminus \partial B_\tau^2(0)),$$

we have $\delta > 0$.



Assume now $T_j = (K', \gamma_1', \gamma_2', \gamma_3')$ satisfies $d(T_{j_0}, T_j) < \varepsilon$. If ε is small enough, we have:

- $\operatorname{spt}(T_j) \subseteq U$ (this is obtained arguing as in the first case), so that we can define $\alpha_i'(t) := \upsilon \circ \gamma_i'(t)$ for $t \in [0,1]$;
- $\upsilon(K') \subseteq \left\{\frac{3}{4} < |z| < 1\right\}$;
- $\max_{t\in[0,s_i]} |\alpha_i'(t) - \alpha_i(t)| < \delta'$, for some $\delta' \le \delta$ to be chosen later;
- $\alpha_i'([s_i,1]) \subseteq \left\{|z| > \frac{1}{2}\right\}$.

Let $t_i' := \max\left(\left\{t : |\alpha_i'(t)| = \frac{1}{2}\right\} \cup \{0\}\right) < s_i$. We claim that

$$\left|\alpha_i'(t_i') - \alpha_i(0)\right| < \delta. \tag{3.3.2}$$

If $t_i' = 0$ this is trivial, while otherwise $|\alpha_i'(t_i')| = \frac{1}{2}$ and

$$\operatorname{dist}(\alpha_i'(t_i'), \alpha_i([0,s_i])) \le \left|\alpha_i'(t_i') - \alpha_i(t_i')\right| < \delta',$$

which yields our claim once δ' is chosen so small that

$$\left\{z : |z| = \frac{1}{2}, \operatorname{dist}(z, \alpha_i([0,s_i])) < \delta'\right\} \subseteq B^2_\delta(\alpha_i(0))$$

(if such δ' did not exist, we could find points $|z_k| = \frac{1}{2}$ with $\operatorname{dist}(z_k, \alpha_i([0,s_i])) \to 0$ and $|z_k - \alpha_i(0)| \ge \delta$; up to subsequences we could assume $z_k \to z_\infty$, for some $z_\infty \in \alpha_i([0,s_i])$ with $|z_\infty| = \frac{1}{2}$, hence $z_\infty = \alpha_i(0)$, contradiction).

Fix now any τ such that $\max\{|z| : z \in \upsilon(\operatorname{spt}(T_j))\} < \tau < 1$. The connected set

$$\upsilon(K') \sqcup \alpha'_1((t'_1, 1)) \sqcup \alpha'_2((t'_2, 1)) \sqcup \alpha'_3((t'_3, 1))$$

is contained in $\left\{\frac{1}{2} < |z| < \tau\right\}$ and is disjoint from $\alpha_1([0,1)) \sqcup \alpha_2([0,1)) \sqcup \alpha_3([0,1))$, so it is contained in some region $R_{i_0}(\tau)$ and, in particular, $\alpha'_{i_0}([t'_{i_0}, 1]) \subseteq \overline{R_{i_0}(\tau)}$. But, using (3.3.1) and (3.3.2), we infer that $\alpha'_{i_0}(t'_{i_0}) \notin \overline{R_{i_0}(\tau)}$. This contradiction shows that such T_j with $d(T_{j_0}, T_j) < \varepsilon$ cannot exist, completing the treatment of the second case. □

Planar case

We now show the regularity in the special case where the parametrized varifold is contained in a plane.

Theorem 3.3.3. *Let (Ω, Φ, N) be a local parametrized stationary varifold in $\mathbb{R}^2 = \mathbb{C}$ defined on a bounded connected open set $\Omega \subset \mathbb{C}$. Assume that $\Phi^{-1}(p)$ is compact for all $p \in \mathbb{C}$. Then Φ is holomorphic or antiholomorphic.*

Proof. We recall that, under these hypotheses, $\Phi^{-1}(p)$ has always finitely many connected components and the upper semicontinuous function $\widetilde{N} \geq 1$ is everywhere finite (see Remark 3.2.11). It suffices to show that $\Delta\Phi = 0$: once this is done, since Φ is necessarily nonconstant we can pick any $z_0 \in \Omega$ such that $\nabla\Phi(z_0) \neq 0$ and, by weak conformality, there is an $r > 0$ such that $\partial_z \Phi\big|_{B^2_r(z_0)} = 0$ or $\partial_{\bar{z}} \Phi\big|_{B^2_r(z_0)} = 0$; the statement then follows by the analyticity of $\partial_z \Phi$ and $\partial_{\bar{z}} \Phi$.

We further make the following assumptions, which will be dropped in Step 4 below:

(i) Φ extends continuously to $\overline{\Omega}$ and $\Phi(\partial\Omega) \cap \Phi(\Omega) = \emptyset$;

(ii) $\Phi(\Omega) \subseteq \mathbb{C}$ is open and the varifold $\mathbf{v}_\Omega$ equals $\widetilde{N}(x_0)\mathbf{v}(\Phi(\Omega))$, for some x_0 in Ω, $\mathbf{v}(\Phi(\Omega))$ denoting the canonical varifold associated to $\Phi(\Omega)$.

We show that in this situation the theorem holds, by strong induction on $\widetilde{N}(x_0)$. Notice that $\widetilde{N}(x_0)$ is necessarily an integer, since $\mathbf{v}_\Omega$ has integer multiplicity.

Step 1. If $\widetilde{N}(x_0) = 1$ then $\widetilde{N} = 1$ everywhere: indeed, for every $z \in \Omega$ and every $K_z \subseteq \omega \subset\subset \Omega$ we have

$$1 \leq \widetilde{N}(z) \leq \lim_{s \to 0} \frac{\|\mathbf{v}_\omega\|(B^2_s(\Phi(z)))}{\pi s^2} \leq \lim_{s \to 0} \frac{\|\mathbf{v}_\Omega\|(B^2_s(\Phi(z)))}{\pi s^2} = 1.$$

By Proposition 3.2.8 we can replace N with $\widetilde{N}$ without affecting the stationarity of (Ω, Φ, N), hence by Theorem 3.2.13 we have $\Delta\Phi = 0$.

Assume now $\widetilde{N}(x_0) > 1$. Fix any $y \in \Omega$ and choose a point y_i in every connected component K_i of $\Phi^{-1}(\Phi(y))$. Choosing disjoint neighborhoods $K_i \subseteq \omega_i \subset\subset \Omega$ we have

$$\sum_i \widetilde{N}(y_i) = \lim_{s \to 0} \sum_i \frac{\|\mathbf{v}_{\omega_i}\|(B^2_s(\Phi(y)))}{\pi s^2} = \lim_{s \to 0} \frac{\|\mathbf{v}_\Omega\|(B^2_s(\Phi(y)))}{\pi s^2} = \widetilde{N}(x_0),$$

since $\Phi(y) \notin \overline{\Phi(\Omega \setminus \bigcup_i \omega_i)}$. We deduce that the following dichotomy is true: for any $y \in \Omega$, either $\widetilde{N}(y) = \widetilde{N}(x_0)$ and $\Phi^{-1}(\Phi(y))$ is connected, or $\widetilde{N}(y) \le \widetilde{N}(x_0) - 1$ and $\Phi^{-1}(\Phi(y))$ has at least two components. We can assume that $\widetilde{N}$ is not identically equal to $\widetilde{N}(x_0)$, since otherwise we are done as in the base case of the induction.

Step 2. We claim that, by inductive hypothesis, Φ is holomorphic or antiholomorphic on each connected component of the set

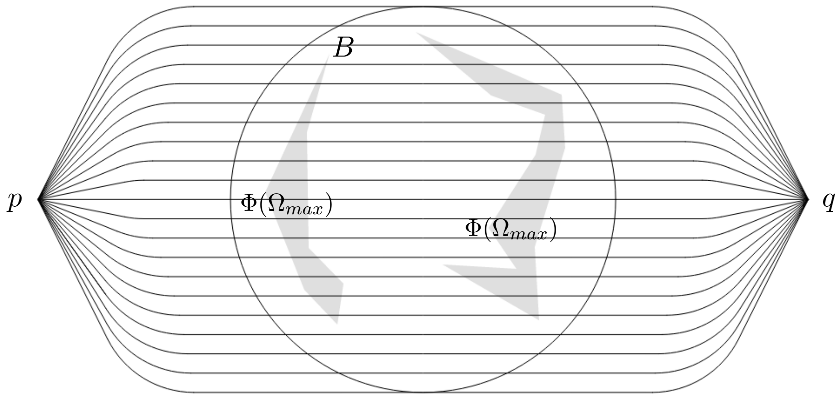
$$\Omega_0 := \{\widetilde{N} \le \widetilde{N}(x_0) - 1\} = \{\widetilde{N} < \widetilde{N}(x_0)\},$$

which is open by Proposition 3.2.8. For any $y_0 \in \Omega_0$ we can take an open set $K_{y_0} \subseteq \omega \subset\subset \Omega_0$ with $\overline{\omega}$ disjoint from $\Phi^{-1}(\Phi(y_0)) \setminus K_{y_0}$. Possibly replacing ω with the connected component of $\omega \setminus \Phi^{-1}(\Phi(\partial\omega))$ containing y_0, we observe that ω satisfies the same hypotheses as Ω, as well as (i)–(ii): the only nontrivial task is to check (ii), which we do below.

By the constancy theorem [98, Theorem 41.1] applied to $\mathbf{v}_\omega$, which is stationary in $\mathbb{C} \setminus \Phi(\partial\omega)$, the varifold $\mathbf{v}_\omega$ equals a nontrivial constant multiple of $\mathbf{v}(W)$, where W is the connected component of $\mathbb{C} \setminus \Phi(\partial\omega)$ containing the connected set $\Phi(\omega)$. Since $\Phi(\omega)$ is relatively closed in W, we deduce $W = \Phi(\omega)$. Finally, by definition of $\widetilde{N}(y_0)$ we must have $\mathbf{v}_\omega = \widetilde{N}(y_0)\mathbf{v}(W)$. Thus, the inductive hypothesis applies and we deduce $\Delta\Phi = 0$ on ω. Since y_0 was arbitrary, we get $\Delta\Phi = 0$ on Ω_0 and our claim is established.

Step 3. Notice that $\Phi(\Omega_0)$ is nonempty and open, being Φ nonconstant on every connected component of Ω_0. We call $D \subset \Omega_0$ the relatively closed, discrete set of points where $\nabla\Phi$ vanishes. The map $\Phi\big|_{\Omega_0} : \Omega_0 \to \Phi(\Omega_0)$ is proper, thanks to the fact that $\Phi(\Omega_0)$ and $\Phi(\overline{\Omega} \setminus \Omega_0)$ are disjoint, so $\Phi(D)$ is a relatively closed, discrete subset of $\Phi(\Omega_0)$. Hence, $D' := \Phi^{-1}(\Phi(D))$ is still relatively closed in Ω_0 and $\Phi\big|_{\Omega_0 \setminus D'}$, being a proper local diffeomorphism onto $\Phi(\Omega_0) \setminus \Phi(D)$, is a covering map.

Let $\Omega_{max} := \{\widetilde{N} = \widetilde{N}(x_0)\}$, which is closed in Ω. Due to Lemma 3.5.1, we can assume that $\Phi(\Omega_{max})$ is uncountable. Observe that $\Phi(\Omega_{max})$ is relatively closed in the open set $\Phi(\Omega)$, being Φ a proper map, and $\Phi(\Omega) = \Phi(\Omega_0) \sqcup \Phi(\Omega_{max})$ by the dichotomy of Step 1. Take two distinct points $p, q \in \Phi(\Omega_0)$ and choose any ball $p, q \notin \overline{B} \subseteq \Phi(\Omega)$ such that $\Phi(\Omega_{max}) \cap B$ is uncountable. We consider a foliation of curves on the connected set $\Phi(\Omega)$ as in the picture (which illustrates the position of p, q, B up to a diffeomorphism of $\Phi(\Omega)$).



We can assume that uncountably many of these curves intersect $\Phi(\Omega_{max})$: if this does not happen, it means that uncountably many points of $\Phi(\Omega_{max})$ lie on a single horizontal segment in B, so it suffices to apply a diffeomorphism which rotates B slightly. Uncountably many such curves do not intersect $\Phi(D) = \Phi(D')$, as well. For any such good curve $\gamma : [0,1] \to \Phi(\Omega)$ (with $\gamma(0) = p$, $\gamma(1) = q$) we let

$$a := \min\{t : \gamma(t) \in \Phi(\Omega_{max})\}, \quad 1 - b := \max\{t : \gamma(t) \in \Phi(\Omega_{max})\}.$$

We clearly have $0 < a \le 1 - b < 1$. We can lift $\gamma\big|_{[a/3,2a/3]}$ to two curves γ_1, γ_2 in $\Omega_0 \setminus D'$ and $\gamma\big|_{[1-2b/3,1-b/3]}$ to a curve γ_3 in $\Omega_0 \setminus D'$, thanks to the dichotomy observed in Step 1 and the fact that $\Phi\big|_{\Omega_0\setminus D'}$ is a covering map.

Finally, the compact set $K := \Phi^{-1}(\gamma([2a/3, 1 - 2b/3]))$ is connected: assume by contradiction that it splits into two disjoint compact sets $A \sqcup B$. For each point $z \in \gamma([2a/3, 1 - 2b/3])$ the fiber $\Phi^{-1}(z)$ lies either in A or in B: this is clear if $z \in \Phi(\Omega_{max})$, since then $\Phi^{-1}(z)$ is connected; if $z \notin \Phi(\Omega_{max})$ we can travel γ back or forward until we hit a point $w \in \Phi(\Omega_{max})$ and the corresponding lifted curves will necessarily accumulate against $\Phi^{-1}(w)$ (thanks to the properness of Φ), so they lie either all in A or all in B. We infer that $\gamma([2a/3, 1 - 2b/3]) = \Phi(A) \sqcup \Phi(B)$, which contradicts the connectedness of $[2a/3, 1 - 2b/3]$.

Thus any such good curve produces a generalized triod $(K, \gamma_1, \gamma_2, \gamma_3)$ and these triods are disjoint from each other. Since there are uncountably many such triods, this contradicts Lemma 3.3.2. The inductive proof is complete.

Step 4. We now drop the extra assumptions (i)–(ii). This is done with the same argument of Step 2: for any $y_0 \in \Omega$ we can find a neighborhood $\omega \subset\subset \Omega$ satisfying (i)–(ii), hence $\Delta\Phi = 0$ on ω. We deduce that $\Delta\Phi = 0$ on all of Ω. □

Corollary 3.3.4. *Under the same hypotheses, N is a.e. constant.*

Proof. Since $U := \{\nabla\Phi \neq 0\} \subseteq \Omega$ is connected, it suffices to show the claim locally in U. Fix $z_0 \in U$. We can find a connected open neighborhood $\omega \subset\subset U$ such that Φ is injective on $\overline{\omega}$. Arguing as in the proof of Theorem 3.3.3, $\Phi(\omega)$ is open and $\mathbf{v}_\omega = \theta\mathbf{v}(\Phi(\omega))$ for some θ. By definition of $\widetilde{N}$ and Proposition 3.2.8, $N = \widetilde{N} = \theta$ a.e. on ω. □

Conical case

We now gradually move to the case where the varifold is contained in a finite union of planes.

Lemma 3.3.5. *Let (Ω, Φ, N) be a local parametrized stationary varifold in $\mathbb{R}^Q$ defined on a bounded connected open set $\Omega \subset \mathbb{C}$. Assume that $\Phi^{-1}(p)$ is compact for all $p \in \mathbb{R}^Q$ and that Φ takes values in the union of two 2-dimensional closed half-planes H_a, H_b with common boundary. Then $\Delta\Phi = 0$.*

Proof. The idea is to straighten the two half-planes into a single plane and then apply Corollary 3.3.4. We can assume that $Q = 3$ and, by Theorem 3.3.3, that the two half-planes are not contained in a single plane. Up to translations and rotations, $\Phi(\Omega) \subseteq H_a \cup H_b$, where

$$H_a := \{\lambda v_1 + \mu v_2 \mid \lambda \in \mathbb{R}, \mu \in [0, \infty)\}, \ H_b := \{\lambda v_1 + \mu v_3 \mid \lambda \in \mathbb{R}, \mu \in [0, \infty)\},$$

$$v_1 := (1, 0, 0), \quad v_2 := (0, \cos\theta, \sin\theta), \quad v_3 := (0, -\cos\theta, \sin\theta),$$

for some $0 < \theta < \frac{\pi}{2}$. Let

$$S : \mathbb{R}^3 \to \mathbb{R}^2 = \mathbb{C}, \quad S(x, y, z) := \left(x, \frac{y}{\cos\theta}\right)$$

and $\Psi := S \circ \Phi$. This map is still weakly conformal: indeed, if x is a Lebesgue point for $\nabla\Phi$ and $\nabla\Phi(x) = 0$, then the same holds for Ψ; if instead $\nabla\Phi(x)$ has full rank, then $d\Phi(x)$ takes values in the linear span of v_1, v_2, or in the linear span of v_1, v_3 (since $\liminf_{r\to 0} r^{-1}\|\Phi(x + ry) - \Phi(x) - r\langle\nabla\Phi(x), y\rangle\|_{C^0(S^1)} = 0$, by Lemma A.4) and the claim follows from the chain rule.

We now show that (Ψ, N) is still a local parametrized stationary varifold, i.e. that

$$\int_\omega N\langle\nabla(X \circ \Psi); \nabla\Psi\rangle \, d\mathcal{L}^2 = 0$$

for any $\omega \subset\subset \Omega$ and any vector field $X \in C_c^\infty(\mathbb{R}^2 \setminus \Psi(\partial\omega), \mathbb{R}^2)$. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix}, \quad P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y := P^t A^{-1}(X \circ S).$$

Notice that, although Y does not have compact support, it vanishes in a neighborhood of $\Phi(\partial\omega)$. Since we know that $\Phi(\overline{\omega})$ is compact, we have

$$\int_\omega N\langle\nabla(Y \circ \Phi); \nabla\Phi\rangle \, d\mathcal{L}^2 = 0.$$

But, viewing S also as a matrix, $\nabla Y = P^t A^{-1}(\nabla X \circ S)S$ and $P = AS$, thus

$$\begin{aligned}
\langle\nabla(Y \circ \Phi); \nabla\Phi\rangle &= \left\langle P^t A^{-1}(\nabla X \circ S \circ \Phi)\nabla(S \circ \Phi); \nabla\Phi\right\rangle \\
&= \left\langle P^t A^{-1}\nabla(X \circ \Psi); \nabla\Phi\right\rangle \\
&= \left\langle S^t\nabla(X \circ \Psi); \nabla\Phi\right\rangle \\
&= \langle\nabla(X \circ \Psi); S\nabla\Phi\rangle \\
&= \langle\nabla(X \circ \Psi); \nabla\Psi\rangle.
\end{aligned}$$

This shows the stationarity of (Ψ, N). By Corollary 3.3.4, N is a.e. constant. By Theorem 3.2.13, this implies $\Delta\Phi = 0$. □

Lemma 3.3.6. *The same conclusion holds if Φ takes values in the union of finitely many (distinct) closed half-planes $\bigcup_{i=1}^k H_i \subseteq \mathbb{R}^Q$ with a common boundary $C = \partial H_i$.*

Proof. It suffices to show that $\Delta\Phi = 0$ near any point $x_0 \in \Omega$. By Theorem 3.3.3 we have $\Delta\Phi = 0$ on $\Phi^{-1}(H_i \setminus C)$, for all i, so we can assume $\Phi(x_0) \in C$. By Remark 3.2.12, shrinking Ω if necessary, we can further assume that $\Phi^{-1}(\Phi(y))$ is connected whenever $\widetilde{N}(y) \geq \widetilde{N}(x_0) - \frac{1}{2}$ and that Φ extends continuously to $\overline{\Omega}$ with $\Phi(\Omega) \cap \Phi(\partial\Omega) = \emptyset$.

By induction on $\lfloor 2\widetilde{N}(x_0) \rfloor$, $\lfloor\cdot\rfloor$ denoting the integer part, we can also assume that in this situation we have $\Delta\Phi = 0$ on the open set $\left\{\widetilde{N} < \widetilde{N}(x_0) - \frac{1}{2}\right\}$ (using e.g. $\lfloor 2\widetilde{N}(x_0) \rfloor = 1$ as the base case, which is vacuously true).

We pick any $0 < r < \operatorname{dist}(\Phi(x_0), \Phi(\partial\Omega))$ and let $\omega := \Phi^{-1}(B_r^Q(\Phi(x_0))) \subset\subset \Omega$, as well as $\omega_i := \omega \cap \Phi^{-1}(H_i \setminus C)$. If ω_i is nonempty for at most two values of i, then we are done by Lemma 3.3.5, applied to the connected components of ω. Thus, we suppose e.g. that $\omega_i \neq \emptyset$ for $i = 1, 2, 3$. Applying the constancy theorem to the varifold $\mathbf{v}_{\omega_i}$, which is a nontrivial stationary varifold in $B_r^Q(\Phi(x_0)) \cap (H_i \setminus C)$, and using the fact that $\Phi(\omega_i)$ is relatively closed in this set, we infer that

$$\Phi(\omega_i) = B_r^Q(\Phi(x_0)) \cap (H_i \setminus C).$$

for $i = 1, 2, 3$. Let $C' := C \cap \Phi\Big(\big\{x \in \omega : \widetilde{N}(x) \geq \widetilde{N}(x_0) - \frac{1}{2}\big\}\Big)$, which is closed in $B_r^Q(\Phi(x_0))$, being $\Phi\big|_\omega : \omega \to B_r^Q(\Phi(x_0))$ a proper map. If C' is countable, then we are done by applying Lemma 3.5.1 to ω.

Otherwise, we can pick for each $c \in C'$ a segment $\beta_i([0,1]) \subseteq B_r^Q(\Phi(x_0)) \cap H_i$ perpendicular to C with $\beta_i(1) = c$ (for $i = 1, 2, 3$). Apart from (at most) countably many exceptions, these segments do not intersect the singular values of $\Phi\big|_{\omega_i}$. So, arguing as in the proof of Theorem 3.3.3, the curves $\beta_i([0, 1/2])$ can be lifted to smooth regular curves γ_i in ω_i and, setting $K := \Phi^{-1}(c) \cup \bigcup_{i=1}^3 \beta_i([1/2, 1])$, $(K, \gamma_1, \gamma_2, \gamma_3)$ is a generalized triod. This gives an uncountable family of disjoint generalized triods, contradicting Lemma 3.3.2. □

Theorem 3.3.7. *Let (Ω, Φ, N) be a local parametrized stationary varifold in $\mathbb{R}^Q$, with Ω connected. Assume that Φ takes values in the union of finitely many (distinct) 2-dimensional planes $\Sigma_1, \ldots, \Sigma_k$ passing through the origin. Then Φ takes values in a single plane Σ_{i_0} and, once we identify it with $\mathbb{C}$, it is holomorphic or antiholomorphic (but possibly constant).*

Proof. Assume without loss of generality that Φ is nonconstant. It suffices to show that $\Delta\Phi = 0$: then, by weak conformality, $\Phi(\Sigma)$ cannot be contained in a finite union of lines, i.e. for some i_0 we have $\Phi^{-1}\Big(\Sigma_{i_0} \setminus \bigcup_{i \neq i_0} \Sigma_i\Big) \neq \emptyset$; thus on this open set we have $\pi_{\Sigma_{i_0}^\perp}\Phi = 0$ and we deduce that this holds on all of Ω by analyticity. The statement then follows by weak conformality, as in the proof of Theorem 3.3.3.

Fix now $x_0 \in \Omega \setminus \Phi^{-1}(0)$ and let $J := \{j : \Phi(x_0) \in \Sigma_j\}$. We pick any radius $r < \operatorname{dist}(x_0, \partial\Omega)$ such that $\Phi(\overline{B}^2_r(x_0))$ intersects only the planes Σ_j with $j \in J$. Notice that $\bigcup_{j\in J} \Sigma_j$ is a finite union of half-planes with common boundary $\{t\Phi(x_0) : t \in \mathbb{R}\}$. We assume that the weak gradient coincides with the classical one on the biggest open subset of Ω where Φ is smooth and we call $\mathcal{G}^f$ the set of Lebesgue points for $\nabla\Phi$ where $\nabla\Phi \neq 0$.

For $x \in B^2_r(x_0) \cap \mathcal{G}^f$ we have $\liminf_{s\to 0} s^{-1}\|\Phi(x + sy) - \Phi(x) - s\langle\nabla\Phi(x), y\rangle\|_{C^0(S^1)} = 0$, by Lemma A.4. Hence, we can find an open neighborhood $\omega \subset B^2_r(x_0)$ of x such that $\Phi(x) \notin \Phi(\partial\omega)$. Possibly replacing ω with the connected component of $\omega \setminus \Phi^{-1}(\Phi(\partial\omega))$ containing x, we can even assume $\Phi(\omega) \cap \Phi(\partial\omega) = \emptyset$. Hence we can apply Lemma 3.3.6 on ω, obtaining $\Delta\Phi = 0$ near x. In particular, this shows that $\mathcal{G}^f \setminus \Phi^{-1}(0)$ is open and $\Delta\Phi = 0$ on $\mathcal{G}^f \setminus \Phi^{-1}(0)$.

Using Fubini's theorem and [35, Theorem 4.21], we can pick an $r' < r$ such that the map $\Phi|_{\partial B^2_{r'}(x_0)}$ is absolutely continuous (with weak derivative given by the chain rule) and $\int_{\partial B^2_{r'}(x_0)\setminus\mathcal{G}^f} |\nabla\Phi|\, d\mathcal{H}^1 = 0$. For any relatively open subset $U \subseteq \partial B^2_{r'}(x_0)$ containing $\partial B^2_{r'}(x_0) \setminus \mathcal{G}^f$ we have

$$\mathcal{H}^1(\Phi(\partial B^2_{r'}(x_0) \cap U)) \le \int_U |\nabla\Phi|\, d\mathcal{H}^1,$$

by definition of $\mathcal{H}^1$. Since U is arbitrary, we deduce

$$\mathcal{H}^1(K) = 0, \quad K := \Phi(\partial B^2_{r'}(x_0) \setminus \mathcal{G}^f).$$

Fix now any $x \in B^2_{r'}(x_0) \setminus \Phi^{-1}(K)$. Assume $\partial B^2_s(x_0) \cap \Phi^{-1}(\Phi(x)) \neq \emptyset$ for all $|x - x_0| < s < r'$. Then we can find a sequence $s_\ell \uparrow r'$ and points $y_\ell \in \partial B^2_{s_k}(x_0)$ such that $\Phi(y_\ell) = \Phi(x)$ and $y_k \to y_\infty$, for some $y_\infty \in \partial B^2_{r'}(x_0)$. Necessarily we have $y_\infty \in \mathcal{G}^f$, as $\Phi(y_\infty) = \Phi(x) \notin K$, but this contradicts the fact that Φ is injective near y_∞.

Thus there exists a radius $s < r'$ such that $x \in B^2_s(x_0)$ and $\Phi(x) \notin \Phi(\partial B^2_s(x_0))$. Again we can let ω be the connected component of $B^2_s(x_0) \setminus \Phi^{-1}(\Phi(\partial B^2_s(x_0)))$ containing x and we can apply Lemma 3.3.6 on ω. This shows that $\Delta\Phi = 0$ on $B^2_{r'}(x_0) \setminus \Phi^{-1}(K)$.

Finally, by Lemma 3.5.1 and $\mathcal{H}^1(K) = 0$, we have $\Delta\Phi = 0$ on $B^2_{r'}(x_0)$. So $\Delta\Phi = 0$ on $\Omega \setminus \Phi^{-1}(0)$ and we deduce that $\Delta\Phi = 0$ on all of Ω, again by Lemma 3.5.1. □

3.4 Blow-up of a parametrized stationary varifold

Let (Ω, Φ, N) be a local parametrized stationary varifold. Let us fix a sequence of points $(x_k) \subseteq \Omega$ and a sequence of radii (r_k) such that $0 < r_k < \frac{1}{2}\operatorname{dist}(x_k, \partial\Omega)$. We let

$$\ell_k^2 := \int_{B^2_{r_k}(x_k)} |\nabla\Phi|^2\, d\mathcal{L}^2, \quad \Phi_k := \ell_k^{-1}(\Phi(x_k + r_k\cdot) - \Phi(x_k)), \quad N_k := N(x_k + r_k\cdot),$$

$$\nu_k := \frac{1}{2} N_k |\nabla\Phi_k|^2 \mathbf{1}_{B^2_2(0)}\mathcal{L}^2, \quad \mu_k := (\Phi_k)_*\nu_k.$$

Notice that the functions Φ_k, N_k are defined on $B^2_2(0)$, so the definition of the measures ν_k and μ_k, on $B^2_2(0)$ and $\mathbb{R}^Q$ respectively, makes sense. Throughout the section we will assume that there exist two constants $C', C'' \geq 1$ such that

- $0 < \int_{B^2_{2r_k}(x_k)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \leq C' \int_{B^2_{r_k}(x_k)} |\nabla\Phi|^2 \, d\mathcal{L}^2;$ (3.4.1)
- $\limsup_{k\to\infty} \mu_k(B^Q_s(p)) \leq C''\pi s^2$ for all $s > 0$ and all $p \in \mathbb{R}^Q$; (3.4.2)
- $\ell_k \to 0.$ (3.4.3)

We will show the following result.

Theorem 3.4.1. *Up to subsequences, there exist a map $\Phi_\infty \in W^{1,2}(B^2_2(0), \mathbb{R}^Q)$, a function $N_\infty \in L^\infty(B^2_2(0), \mathbb{N} \setminus \{0\})$ and a quasiconformal homeomorphism $\varphi_\infty \in W^{1,2}(B^2_2(0), \Omega_\infty)$ (for some bounded open set $\Omega_\infty \subseteq \mathbb{C}$), with $\varphi_\infty(0) = 0$, such that*

$$\Phi_k \to \Phi_\infty \quad \text{in } C^0_{loc}(B^2_2(0), \mathbb{R}^Q), \qquad \nabla\Phi_k \rightharpoonup \nabla\Phi_\infty \quad \text{in } L^2(B^2_2(0), \mathbb{R}^{Q\times 2}),$$

$$\frac{1}{2} N_k |\nabla\Phi_k|^2 \mathcal{L}^2 \rightharpoonup N_\infty |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \mathcal{L}^2 \quad \text{as Radon measures.}$$

Moreover, $\Phi_\infty \circ \varphi_\infty^{-1}$ is weakly conformal and $(\Omega_\infty, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ is a local parametrized stationary varifold in $\mathbb{R}^Q$.

We refer the reader to [56, Chapter 4] and [65] for the theory of quasiconformal homeomorphisms in the plane. Before proving this theorem we shall establish a number of intermediate results. Many arguments are similar to those used in the previous chapter.

First of all, since $\int_{B^2_2(0)} |\nabla\Phi_k|^2 \, d\mathcal{L}^2 \leq C'$ and $\nu_k(B^2_2(0)) \leq \frac{1}{2} C' \|N\|_{L^\infty}$, up to subsequences there exists $\Phi_\infty \in W^{1,2}(B^2_2(0), \mathbb{R}^Q)$ such that $\Phi_k \rightharpoonup \Phi_\infty$ in $W^{1,2}(B^2_2(0))$ and there exists a finite Radon measure ν_∞ on $B^2_2(0)$ such that $\nu_k \rightharpoonup \nu_\infty$ in $B^2_2(0)$. We can also assume that, for all $j \geq 0$,

$$\mu_{k,j} := (\Phi_k)_* \Big(\mathbf{1}_{B^2_{2-2^{-j}}(0)} \nu_k \Big) \rightharpoonup \mu_{\infty,j} \quad \text{as } k \to \infty$$

in $\mathbb{R}^Q$, for some finite measure $\mu_{\infty,j}$. Since $\mu_{\infty,j} \leq \mu_{\infty,j+1}$ and $\mu_{\infty,j}(\mathbb{R}^Q) \leq \frac{1}{2} C' \|N\|_{L^\infty}$, the measure $\mu_\infty := \lim_{j\to\infty} \mu_{\infty,j}$ is defined and is again finite.

Lemma 3.4.2. *The measure ν_∞ is absolutely continuous with respect to $\mathcal{L}^2$, i.e. $\nu_\infty = m\mathcal{L}^2$ for some nonnegative $m \in L^1(B^2_2(0))$. Moreover, Φ_∞ is continuous and $\Phi_k \to \Phi_\infty$ in $C^0_{loc}(B^2_2(0), \mathbb{R}^Q)$. Finally, for any open subset $\omega \subset\subset B^2_2(0)$,*

$$\Phi_\infty(\overline{\omega}) \subseteq \operatorname{conv}(\Phi_\infty(\partial\omega)),$$

where $\operatorname{conv}(\cdot)$ *denotes the convex hull.*

Proof. We introduce the *oscillation set*

$$\mathcal{O} := \left\{ x \in \operatorname{spt}(\nu_\infty) : \liminf_{r\to 0} \frac{\int_{B^2_{2r}(x)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2}{\nu_\infty(B^2_r(x))} = 0 \right\}.$$

Step 1. We show that ν_∞ is absolutely continuous with respect to $\mathcal{L}^2$ on the Borel set $B^2_2(0) \setminus \mathcal{O}$. Let $E \subseteq B^2_2(0) \setminus \mathcal{O}$ be a Borel set with $\mathcal{L}^2(E) = 0$. It suffices to show that $\nu_\infty(K) = 0$ for any compact subset $K \subseteq E \cap \operatorname{spt}(\nu_\infty)$ (since this implies that $\nu_\infty(E \cap \operatorname{spt}(\nu_\infty)) = 0$ and thus $\nu_\infty(E) = 0$, as required). We define the Borel sets

$$F_j := \left\{ x \in K : \inf_{0<r\le\bar{r}} \frac{\int_{B^2_{2r}(x)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2}{\nu_\infty(B^2_r(x))} \ge 2^{-j} \right\}, \quad \bar{r} := \frac{1}{2} \operatorname{dist}(K, \partial B^2_2(0))$$

and observe that $K = \bigcup_j F_j$. Fix j and an open set $E \subseteq V \subseteq B^2_2(0)$. Letting $r_V := \frac{1}{2}\operatorname{dist}(K, \mathbb{R}^2 \setminus V) \le \bar{r}$, we choose a maximal (finite) subset $\{x_i\}$ of F_j such that $|x_i - x_{i'}| \ge r_V$ for $i \ne i'$. We have

$$F_j \subseteq \bigcup_i B^2_{r_V}(x_i), \qquad \sum_i \mathbf{1}_{B^2_{2r_V}(x_i)} \le \mathfrak{N}$$

for some universal constant $\mathfrak{N}$. We deduce that

$$\nu_\infty(F_j) \le \sum_i \nu_\infty(B^2_{r_V}(x_i)) \le 2^j \sum_i \int_{B^2_{2r_V}(x_i)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2 \le 2^j \mathfrak{N} \int_V |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2.$$

Letting V range along a sequence of open sets $V_\ell \supseteq E$ with $\mathcal{L}^2(V_\ell) \to 0$, we deduce $\nu_\infty(F_j) = 0$. Hence,

$$\nu_\infty(K) \le \sum_j \nu_\infty(F_j) = 0.$$

Step 2. We show that $\mathcal{O} = \emptyset$. Fix any $x \in B^2_2(0)$ and any $r < \frac{1}{2}\operatorname{dist}(x, \partial B^2_2(0))$. Using Lemma A.5 and Lemma A.3 we select a radius $r' \in (r, 2r)$ such that $\Phi_{k_i}\big|_{\partial B^2_{r'}(x)} \to \Phi_\infty\big|_{\partial B^2_{r'}(x)}$ in L^∞, for some subsequence (Φ_{k_i}), and

$$\operatorname{diam} \Phi_\infty(\partial B^2_{r'}(x)) \le \sqrt{4\pi} \left(\int_{B^2_{2r}(x)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2 \right)^{1/2} \tag{3.4.4}$$

(we are implicitly referring to the continuous representative of $\Phi_\infty\big|_{\partial B^2_{r'}(x)}$). Since (Ω, Φ, N) is a local parametrized stationary varifold in $\mathcal{M}$, the varifolds $\mathbf{v}_{k_i}$ issued by (Φ_{k_i}, N_{k_i}) from the domain $B^2_{r'}(x)$ have generalized mean curvature bounded by $O(\ell_{k_i})$ (in L^∞) in $\mathbb{R}^Q \setminus \Phi_{k_i}(\partial B^2_{r'}(x))$. As a consequence of assumption (3.4.3), up to further subsequences they converge to a varifold $\mathbf{v}_\infty$ (in $\mathbb{R}^Q$) which is stationary in $\mathbb{R}^Q \setminus \Phi_\infty(\partial B^2_{r'}(0))$.

Moreover, by Proposition 3.2.4, $0 = \Phi_k(0) \in \operatorname{spt}(\|\mathbf{v}_k\|)$ unless $\operatorname{spt}(\|\mathbf{v}_k\|) = \emptyset$. Using estimate (3.2.5) we infer that the sets $\operatorname{spt}(\|\mathbf{v}_k\|)$ are all included in a unique compact set. It follows that $\mathbf{v}_\infty$ has compact support, hence by [98, Theorem 19.2] (which applies to general stationary varifolds) $\operatorname{spt}(\mathbf{v}_\infty) \subseteq K := \operatorname{conv}(\Phi_\infty(\partial B^2_{r'}(x)))$. It follows that

$$\sup_{z \in \overline{B}^2_{r'}(x)} \operatorname{dist}(\Phi_{k_i}(z), K) \to 0 : \tag{3.4.5}$$

if this were not the case, up to subsequences we could find $z_{k_i} \in B^2_{r'}(x)$ such that $\operatorname{dist}(\Phi_{k_i}(z_{k_i}), K) \geq \varepsilon$. Eventually $\Phi_{k_i}(x_{k_i}) \in \operatorname{spt}(\|\mathbf{v}_{k_i}\|)$ (by Proposition 3.2.4, since eventually Φ_{k_i} must be nonconstant on $B^2_{r'}(x)$), so we can assume that $\Phi_{k_i}(x_{k_i}) \to p_\infty$ and

$$\|\mathbf{v}_\infty\|(\overline{B}^Q_{\varepsilon/2}(p_\infty)) \geq \limsup_{i\to\infty} \|\mathbf{v}_{k_i}\|(\overline{B}^Q_{\varepsilon/2}(\Phi_{k_i}(x_{k_i}))) \geq \pi\frac{\varepsilon^2}{4},$$

thanks to the monotonicity formula and the fact that $\overline{B}^Q_{\varepsilon/2}(\Phi_{k_i}(x_{k_i})) \cap \Phi_{k_i}(\partial B^2_{r'}(x)) = \emptyset$ eventually. This, however, contradicts the fact that $\overline{B}^Q_{\varepsilon/2}(p_\infty) \cap K = \emptyset$. Using (3.4.5) and (3.4.2) we deduce that

$$\nu_\infty(B^2_{r'}(x)) \leq \liminf_{i\to\infty} \nu_{k_i}(B^2_{r'}(x)) \leq \liminf_{i\to\infty} \mu_{k_i}(\Phi_{k_i}(B^2_{r'}(x))) \leq C''\pi(\operatorname{diam} K)^2.$$

From (3.4.4) and the fact that the convex hull preserves the diameter, we have $(\operatorname{diam} K)^2 \leq 4\pi \int_{B^2_{2r}(x)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2$. Hence,

$$\liminf_{r\to 0} \frac{\int_{B^2_{2r}(x)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2}{\nu_\infty(B^2_r(x))} \geq \frac{1}{4\pi^2 C''}.$$

It follows that $\mathcal{O} = \emptyset$.

Step 3. We show that Φ_∞ has a continuous representative. Since $\Phi_{k_i} \to \Phi_\infty$ in $L^2(B^2_2(0), \mathbb{R}^Q)$, from (3.4.5) we infer that $\Phi_\infty(z) \in K$ for a.e. $z \in B^2_r(x)$. In particular, this must happen whenever z is a Lebesgue point. This, together with the estimate for $\operatorname{diam} K$, proves that Φ_∞ is locally uniformly continuous on the set of its Lebesgue points, hence it has a continuous representative.

Step 4. Assume now by contradiction that Φ_k does not converge locally uniformly to (the continuous representative of) Φ_∞. Then we can find a subsequence Φ_{k_i} and points x_{k_i}, lying in a compact subset of $B^2_2(0)$, such that

$$|\Phi_{k_i}(x_{k_i}) - \Phi_\infty(x_{k_i})| \geq \varepsilon.$$

We can further assume that $x_{k_i} \to x_\infty \in B^2_2(0)$. Let $r < \frac{1}{2}\operatorname{dist}(x_\infty, \partial B^2_2(0))$ be such that $4\pi \int_{B^2_{2r}(x_\infty)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2 \leq \frac{\varepsilon^2}{4}$. Up to subsequences, we can repeat the argument of the two previous steps and conclude that

$$\operatorname{dist}(\Phi_{k_i}(x_{k_i}), K) \to 0, \quad \Phi_\infty(x_\infty) \in K,$$

where again $K := \operatorname{conv}(\Phi_\infty(\partial B^2_{r'}(x_\infty)))$ for a suitable $r' \in (r, 2r)$. In particular we have $\limsup_{i\to\infty} |\Phi_{k_i}(x_{k_i}) - \Phi_\infty(x_\infty)| \leq \operatorname{diam} K \leq \frac{\varepsilon}{2}$, which is the desired contradiction.

Step 5. Finally, let us turn to the last part of the statement. We can assume that ω is connected. We already know that $\Phi_k \to \Phi_\infty$ uniformly on $\partial\omega$, so we can repeat the argument of Step 2, with $B^2_{r'}(x)$ replaced by ω, and conclude that $\operatorname{dist}(\Phi_k(z), \operatorname{conv}(\Phi_\infty(\partial\omega))) \to 0$ for all $z \in \omega$, from which it follows that $\Phi_\infty(z) \in \operatorname{conv}(\Phi_\infty(\partial\omega))$. □

Lemma 3.4.3. *We have $\mu_\infty = (\Phi_\infty)_*\nu_\infty$ and, for any $\omega \subset\subset B_2^2(0)$ with $\mathcal{L}^2(\partial\omega) = 0$,*

$$(\Phi_k)_*(\mathbf{1}_\omega \nu_k) \rightharpoonup (\Phi_\infty)_*(\mathbf{1}_\omega \nu_\infty)$$

as Radon measures in $\mathbb{R}^Q$.

Proof. We first show the second statement. Consider any nonnegative $\rho \in C_c^0(\mathbb{R}^Q)$. By Lemma 3.4.2 we have $\nu_\infty(\partial\omega) = 0$, so approximating $\mathbf{1}_\omega \rho(\Phi_\infty)$ from above and below by functions in $C_c^0(B_2^2(0))$ we get

$$\int_{B_2^2(0)} \rho(\Phi_\infty)\mathbf{1}_\omega \, d\nu_k \to \int_{B_2^2(0)} \rho(\Phi_\infty)\mathbf{1}_\omega \, d\nu_\infty.$$

Moreover, thanks to the local uniform convergence $\Phi_k \to \Phi_\infty$ and $\nu_k(B_2^2(0)) \le \frac{1}{2}C'\|N\|_{L^\infty}$,

$$\left| \int_{B_2^2(0)} \rho(\Phi_k)\mathbf{1}_\omega \, d\nu_k - \int_{B_2^2(0)} \rho(\Phi_\infty)\mathbf{1}_\omega \, d\nu_k \right| \le \frac{1}{2}C'\|N\|_{L^\infty}\|\rho(\Phi_k) - \rho(\Phi_\infty)\|_{L^\infty(\omega)} \to 0$$

as $k \to \infty$ and the claim follows, since

$$\int_{\mathbb{R}^Q} \rho \, d(\Phi_k)_*(\mathbf{1}_\omega \nu_k) = \int_{B_2^2(0)} \rho(\Phi_k)\mathbf{1}_\omega \, d\nu_k \to \int_{B_2^2(0)} \rho(\Phi_\infty)\mathbf{1}_\omega \, d\nu_\infty = \int_{\mathbb{R}^Q} \rho \, d(\Phi_\infty)_*(\mathbf{1}_\omega \nu_\infty).$$

Choosing $\omega = B^2_{2-2^{-j}}(0)$ we get

$$\int_{\mathbb{R}^Q} \rho \, d\mu_{\infty,j} = \lim_{k\to\infty} \int_{\mathbb{R}^Q} \rho \, d\mu_{k,j} = \int_{B^2_{2-2^{-j}}(0)} \rho(\Phi_\infty) \, d\nu_\infty.$$

The first statement now follows by letting $j \to \infty$. □

Lemma 3.4.4. *Let $\mathcal{G}' \subseteq B_2^2(0)$ denote the set of points z where $\nabla\Phi_\infty(z)$ has full rank and both $⨍_{B_r^2(z)} |m - m(z)| \, d\mathcal{L}^2$ and $⨍_{B_r^2(z)} |\nabla\Phi_\infty - \nabla\Phi_\infty(z)|^2 \, d\mathcal{L}^2$ are infinitesimal as $r \to 0$. Then $m = 0$ and $\nabla\Phi_\infty = 0$ a.e. on $B_2^2(0) \setminus \mathcal{G}'$.*

This statement was essentially already proved in Theorem 2.5.3. We present here a simpler argument.

Proof. Let $\mathcal{G}$ be the set of Lebesgue points for $\nabla\Phi_\infty$. It suffices to show that $m = 0$ a.e. on $\mathcal{G} \setminus \mathcal{G}'$ and $|\nabla\Phi_\infty|^2 \le 2m$ a.e. By Lemma A.2 and the area formula,

$$\mathcal{H}^2(\Phi_\infty(\mathcal{G} \setminus \mathcal{G}')) \le \int_{\mathcal{G}\setminus\mathcal{G}'} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \, d\mathcal{L}^2 = 0.$$

We can thus cover $\Phi_\infty(\mathcal{G} \setminus \mathcal{G}')$ with countably many balls $B^Q_{s_i}(p_i)$ such that $\sum_i s_i^2$ is arbitrarily small. By assumption (3.4.2), $\mu_\infty(B^Q_{s_i}(p_i)) \le \liminf_{k\to\infty} \mu_k(B^Q_{s_i}(p_i)) \le C''\pi s_i^2$. Hence, by Lemma 3.4.3,

$$\int_{\mathcal{G}\setminus\mathcal{G}'} m \, d\mathcal{L}^2 = \nu_\infty(\mathcal{G} \setminus \mathcal{G}') \le \nu_\infty\big(\Phi_\infty^{-1}\big(\textstyle\bigcup_i B^Q_{s_i}(p_i)\big)\big) = \mu_\infty\big(\textstyle\bigcup_i B^Q_{s_i}(p_i)\big) \le C''\pi \displaystyle\sum_i s_i^2$$

is arbitrarily small. We deduce that $\int_{\mathcal{G}\setminus\mathcal{G}'} m\, d\mathcal{L}^2 = 0$. Moreover, for any open sets $V \subset\subset W \subseteq B_2^2(0)$ we have

$$\int_V |\nabla\Phi_\infty|^2\, d\mathcal{L}^2 \leq \liminf_{k\to\infty} \int_V |\nabla\Phi_k|^2\, d\mathcal{L}^2 \leq 2\limsup_{k\to\infty} \nu_k(\overline{V}) \leq 2\nu_\infty(\overline{V}) \leq 2\int_W m\, d\mathcal{L}^2$$

and we infer that $\int_W |\nabla\Phi_\infty|^2\, d\mathcal{L}^2 \leq 2\int_W m\, d\mathcal{L}^2$. Since W is arbitrary, we deduce that $|\nabla\Phi_\infty|^2 \leq 2m$ a.e. and the claim follows. □

Lemma 3.4.5. *There exists $N_\infty \in L^\infty(B_2^2(0), \mathbb{N}\setminus\{0\})$ bounded above by C'', i.e. the constant in (3.4.2), and*

$$m = N_\infty|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \quad a.e.$$

Moreover, Φ_∞ satisfies $|\nabla\Phi_\infty|^2 \leq 2N_\infty|\partial_1\Phi_\infty\wedge\partial_2\Phi_\infty|$ a.e.

Proof. The proof is analogous (but simpler, since we have fewer error terms) to the one of Theorem 2.5.3. □

Proof of Theorem 3.4.1. We let $g_{ij} := \partial_i\Phi_\infty\cdot\partial_j\Phi_\infty$ and we define the Beltrami coefficient

$$\mu := \frac{g_{11}-g_{22}+2ig_{12}}{g_{11}+g_{22}+2\sqrt{g_{11}g_{22}-g_{12}^2}}\mathbf{1}_{B_2^2(0)\cap\mathcal{G}'} \quad \text{on } \mathbb{C}.$$

In particular, $\mu = 0$ on $\mathbb{C}\setminus B_2^2(0)$. Moreover, a.e. on the set $\mathcal{G}'$ we have

$$|\mu|^2 \leq \frac{(g_{11}-g_{22})^2+4g_{12}^2}{(g_{11}+g_{22})^2+4(g_{11}g_{22}-g_{12}^2)} = \frac{(g_{11}+g_{22})^2-4(g_{11}g_{22}-g_{12}^2)}{(g_{11}+g_{22})^2+4(g_{11}g_{22}-g_{12}^2)} \leq \frac{N_\infty^2-1}{N_\infty^2+1},$$

since $(g_{11}+g_{22})^2 = |\nabla\Phi_\infty|^4$, which by Lemma 3.4.5 is bounded by $4N_\infty^2|\partial_1\Phi_\infty\wedge\partial_2\Phi_\infty|^2 = 4N_\infty^2(g_{11}g_{22}-g_{12}^2)$. We know that $\|N_\infty\|_{L^\infty} \leq C''$, so by [56, Theorem 4.24] there exists a $(C'')^2$-quasiconformal homeomorphism $\varphi_\infty \in W^{1,2}_{loc}(\mathbb{C},\mathbb{C})$, with $\varphi_\infty(0) = 0$, satisfying a.e.

$$\partial_{\overline{z}}\varphi_\infty = \mu\,\partial_z\varphi_\infty. \tag{3.4.6}$$

We recall that the inverse map is also a $(C'')^2$-quasiconformal homeomorphism in $W^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ (see [56, Theorem 4.10 and Proposition 4.2]) and that both φ_∞ and φ_∞^{-1} map negligible sets to negligible sets (see [56, Lemma 4.12]). Moreover, using the chain rule (which holds by [65, Lemma III.6.4]), we get that φ_∞ has invertible differential a.e. and

$$0 = \partial_{\overline{w}}(\varphi_\infty\circ\varphi_\infty^{-1}(w)) = (\partial_z\varphi_\infty)\circ\varphi_\infty^{-1}\partial_{\overline{w}}(\varphi_\infty^{-1}) + (\partial_{\overline{z}}\varphi_\infty)\circ\varphi_\infty^{-1}\partial_{\overline{w}}\left(\overline{\varphi_\infty^{-1}}\right).$$

Being also $\partial_z\varphi_\infty \neq 0$ a.e. (by (3.4.6)), we deduce that

$$\partial_w\left(\overline{\varphi_\infty^{-1}}\right) = \overline{\partial_{\overline{w}}(\varphi_\infty^{-1})} = -(\overline{\mu}\circ\varphi_\infty^{-1})\partial_w(\varphi_\infty^{-1})$$

a.e. From now on, φ_∞ will denote the homeomorphism restricted to $B_2^2(0)$. Let $\Omega_\infty := \varphi_\infty(B_2^2(0))$. By the chain rule again, we have $\Phi_\infty\circ\varphi_\infty^{-1} \in W^{1,1}_{loc}(\Omega_\infty)$ and

$$\begin{aligned}&\partial_w(\Phi_\infty\circ\varphi_\infty^{-1})\cdot\partial_w(\Phi_\infty\circ\varphi_\infty^{-1})\\ &= ((\partial_z\Phi_\infty\cdot\partial_z\Phi_\infty - 2\overline{\mu}\,\partial_z\Phi_\infty\cdot\partial_{\overline{z}}\Phi_\infty + \overline{\mu}^2\partial_{\overline{z}}\Phi_\infty\cdot\partial_{\overline{z}}\Phi_\infty)\circ\varphi_\infty^{-1})(\partial_w(\varphi_\infty^{-1}))^2\end{aligned}$$

vanishes a.e., since $\overline{\mu} = \frac{g_{11}+g_{22}-\sqrt{(g_{11}+g_{22})^2-((g_{11}-g_{22})^2+4g_{12}^2)}}{g_{11}-g_{22}+2ig_{12}}$ on the subset of $\mathcal{G}'$ where $\nabla\Phi_\infty$ is not conformal, while on the complement of this set $\overline{\mu} = 0$. Hence, $\Phi_\infty \circ \varphi_\infty^{-1}$ is weakly conformal. By the area formula (see Lemma A.2),

$$\begin{aligned}
\int_{\Omega_\infty} \left|\nabla(\Phi_\infty \circ \varphi_\infty^{-1})\right|^2 d\mathcal{L}^2 &= 2\int_{\Omega_\infty} \left|\partial_1(\Phi_\infty \circ \varphi_\infty^{-1}) \wedge \partial_2(\Phi_\infty \circ \varphi_\infty^{-1})\right| d\mathcal{L}^2 \\
&= 2\int_{\Omega_\infty} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \circ \varphi_\infty^{-1} \left|\partial_1\varphi_\infty^{-1} \wedge \partial_2\varphi_\infty^{-1}\right| d\mathcal{L}^2 \\
&= 2\int_{B_2^2(0)} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \, d\mathcal{L}^2 \le \int_{B_2^2(0)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2,
\end{aligned}$$

which shows that $\Phi_\infty \circ \varphi_\infty^{-1} \in W^{1,2}(\Omega_\infty, \mathbb{R}^Q)$. The same computation shows that

$$\nu_\infty = N_\infty|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\mathcal{L}^2 = (\varphi_\infty^{-1})_*\left(\frac{1}{2}N_\infty \circ \varphi_\infty^{-1}\left|\nabla(\Phi_\infty \circ \varphi_\infty^{-1})\right|^2\mathcal{L}^2\right). \tag{3.4.7}$$

This, together with $(\Phi_\infty)_*\nu_\infty = \mu_\infty$ (by Lemma 3.4.3) and assumption (3.4.2), shows that $(\Omega_\infty, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ satisfies (3.2.6).

Finally, for any $\omega \subset\subset \Omega_\infty$ with smooth boundary, we show that the varifold $\mathbf{v}$ associated to $(\omega, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ is stationary in $\mathbb{R}^Q \setminus \Phi_\infty \circ \varphi_\infty^{-1}(\partial\omega)$. Setting $\omega' := \varphi_\infty^{-1}(\omega)$, from the C^0_{loc} convergence $\Phi_k \to \Phi_\infty$ we infer that the varifolds $\mathbf{v}_k := \mathbf{v}_{(\omega',\Phi_k,N_k)}$ converge (a priori only after extracting a subsequence) to a stationary varifold $\widetilde{\mathbf{v}}$ in $\mathbb{R}^Q \setminus \Phi_\infty(\partial\omega')$. This varifold is rectifiable (see [98, Theorem 42.4]). But, since $\mathcal{L}^2(\partial\omega') = 0$, Lemma 3.4.3 gives

$$\begin{aligned}
\|\mathbf{v}_k\| &= (\Phi_k)_*(\mathbf{1}_{\omega'}\nu_k) \rightharpoonup (\Phi_\infty)_*(\mathbf{1}_{\omega'}\nu_\infty) \\
&= (\Phi_\infty \circ \varphi_\infty^{-1})_*\left(\frac{1}{2}N_\infty \circ \varphi_\infty^{-1}\left|\nabla(\Phi_\infty \circ \varphi_\infty^{-1})\right|^2 \mathbf{1}_\omega\mathcal{L}^2\right) = \|\mathbf{v}\|
\end{aligned}$$

as Radon measures in $\mathbb{R}^Q$. Since $\|\mathbf{v}_k\| \rightharpoonup \|\widetilde{\mathbf{v}}\|$ in $\mathbb{R}^Q \setminus \Phi_\infty(\partial\omega') = \mathbb{R}^Q \setminus \Phi_\infty \circ \varphi_\infty^{-1}(\partial\omega)$ and a rectifiable varifold is uniquely determined by the associated mass measure, we deduce that on this open set $\widetilde{\mathbf{v}} = \mathbf{v}$. Since $\widetilde{\mathbf{v}}$ is stationary, the theorem follows. □

Theorem 3.4.1 admits an analogous statement in which $B_2^2(0)$ is replaced by $\mathbb{C}$. Let $(x_k) \subseteq \Omega$ be a sequence of points, together with a sequence of radii (r_k) such that $\lim_{k\to\infty} \frac{\operatorname{dist}(x_k,\partial\Omega)}{r_k} = \infty$. Assuming $\ell_k^2 := \int_{B^2_{r_k}(x_k)} |\nabla\Phi|^2 \, d\mathcal{L}^2 > 0$ eventually, we let

$$\Phi_k := \ell_k^{-1}(\Phi(x_k + r_k\cdot) - \Phi(x_k)), \quad N_k := N(x_k + r_k\cdot)$$

and notice that, for any $R > 0$, the functions Φ_k, N_k are eventually defined on $B_R^2(0)$. Assume moreover that

- $\limsup_{k\to\infty} \frac{\int_{B^2_{Rr_k}(x_k)} |\nabla\Phi|^2 \, d\mathcal{L}^2}{\int_{B^2_{r_k}(x_k)} |\nabla\Phi|^2 \, d\mathcal{L}^2} < \infty$ for all $R > 0$;
- $\limsup_{k\to\infty} (\Phi_k)_*\left(\frac{1}{2}N_k|\nabla\Phi_k|^2\mathbf{1}_{B_R^2(0)}\mathcal{L}^2\right)(B_s^Q(p)) \le C''\pi s^2$ for all $s > 0$, all $p \in \mathbb{R}^Q$ and all $R > 0$;

- $\ell_k \to 0$.

Theorem 3.4.6. *Up to subsequences, there exist $\Phi_\infty \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{R}^Q)$, $N_\infty \in L^\infty(\mathbb{C}, \mathbb{N} \setminus \{0\})$ and a quasiconformal homeomorphism $\varphi_\infty \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$, with $\varphi_\infty(0) = 0$, such that*

$$\Phi_k \to \Phi_\infty \quad \text{in } C^0_{loc}(\mathbb{C}, \mathbb{R}^Q), \qquad \nabla\Phi_k \rightharpoonup \nabla\Phi_\infty \quad \text{in } L^2_{loc}(\mathbb{C}, \mathbb{R}^{Q\times 2}),$$
$$\frac{1}{2}N_k|\nabla\Phi_k|^2\mathcal{L}^2 \rightharpoonup N_\infty|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\mathcal{L}^2 \quad \text{as Radon measures.}$$

Moreover, $\Phi_\infty \circ \varphi_\infty^{-1}$ is weakly conformal and $(\mathbb{C}, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ is a local parametrized stationary varifold in $\mathbb{R}^Q$.

Proof. Let $\Phi_\infty \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{R}^Q)$ be a local weak limit. Repeating the proof of Theorem 3.4.1 with $R = 2^j$ in place of 2 (for all $j \geq 1$) and using a diagonal argument, up to subsequences we get $\Phi_k \to \Phi_\infty$ in $C^0_{loc}(\mathbb{C}, \mathbb{R}^Q)$ and, assuming without loss of generality that

$$\nu_{\infty,j} := \lim_{k\to\infty} \frac{1}{2}N_k|\nabla\Phi_k|^2\mathbf{1}_{B^2_{2^j}(0)}\mathcal{L}^2$$

exists, we also get that

$$\nu_{\infty,j} = N_\infty|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\mathbf{1}_{B^2_{2^j}(0)}\mathcal{L}^2$$

for some $N_\infty \in L^\infty(\mathbb{C}, \mathbb{N} \setminus \{0\})$ with $\|N_\infty\|_{L^\infty} \leq C''$, as well as

$$|\nabla\Phi_\infty|^2 \leq 2N_\infty|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|$$

a.e. Assuming also that $\lim_{k\to\infty}(\Phi_k)_*\left(\frac{1}{2}N_k|\nabla\Phi_k|^2\mathbf{1}_{B^2_{2^j}(0)}\mathcal{L}^2\right)$ exists for all j, we can set

$$\nu_\infty := \lim_{j\to\infty} \nu_{\infty,j}, \quad \mu_\infty := \lim_{j\to\infty}\lim_{k\to\infty}(\Phi_k)_*\left(\frac{1}{2}N_k|\nabla\Phi_k|^2\mathbf{1}_{B^2_{2^j}(0)}\mathcal{L}^2\right)$$

and, with the same proof as Lemma 3.4.3, we have again $\mu_\infty = (\Phi_\infty)_*\nu_\infty$. The remainder of the proof is completely analogous to the one of Theorem 3.4.1, using [56, Theorem 4.30] in order to build the quasiconformal homeomorphism $\varphi_\infty : \mathbb{C} \to \mathbb{C}$. □

3.5 Regularity in the general case

This section is devoted to the proof of the main regularity result (see Theorem 3.5.7 and Corollary 3.5.8 below). We first show a removable singularity criterion. Its proof consists of a standard capacity argument in the target $\mathbb{R}^Q$ and could be well known to the expert community. We include it both for the reader's convenience and because it is the only place where the technical assumption (3.2.6) is used.

Lemma 3.5.1. *Let (Ω, Φ, N) be a local parametrized stationary varifold in $\mathcal{M}$ (possibly $\mathcal{M} = \mathbb{R}^Q$). Assume that Φ satisfies*

$$-\Delta\Phi = A(\Phi)(\nabla\Phi, \nabla\Phi)$$

in the distributional sense on $\Omega \setminus S$, for some relatively closed $S \subseteq \Omega$ with $\mathcal{H}^1(\Phi(S)) = 0$. Then the equation is satisfied on the whole Ω and, as a consequence, Φ is C^∞-smooth.

Proof. Let $v \in C_c^\infty(\Omega)$. For any $\varepsilon > 0$ we can cover the compact set $K := \Phi(S \cap \operatorname{spt}(v))$ by a finite union of balls $\bigcup_{i \in I} B^Q_{r_i}(p_i)$ with centers on this set and $\sum_i r_i < \varepsilon$. Let $\rho_i \in C^\infty(\mathbb{R}^Q)$ be a function which equals 0 on $B^Q_{r_i}(p_i)$, 1 on $\mathbb{R}^Q \setminus B^Q_{2r_i}(p_i)$ and has $\|\nabla \rho_i\|_{L^\infty} \leq 2r_i^{-1}$.

Since $v_\varepsilon := v \prod_i (\rho_i \circ \Phi)$ vanishes near $S \cap \operatorname{spt}(v)$, the function $v_\varepsilon \in W^{1,2} \cap L^\infty(\Omega)$ is supported in a compact subset of $\Omega \setminus S$. Thus, using the hypothesis and a standard approximation argument,

$$\int_\Omega \langle \nabla \Phi, \nabla v_\varepsilon \rangle \, d\mathcal{L}^2 = \int_\Omega A(\Phi)(\nabla \Phi, \nabla \Phi) v_\varepsilon \, d\mathcal{L}^2. \tag{3.5.1}$$

We claim that, as $\varepsilon \to 0$, the left-hand side converges to $\int_{\Omega \setminus \Phi^{-1}(K)} \langle \nabla \Phi, \nabla v \rangle \, d\mathcal{L}^2$. Indeed, let us write

$$\nabla v_\varepsilon = \Big(\prod_i \rho_i \circ \Phi \Big) \nabla v + v \sum_i \Big(\prod_{j \neq i} \rho_j \circ \Phi \Big) \nabla(\rho_i \circ \Phi).$$

The first term converges to $\mathbf{1}_{\Omega \setminus \Phi^{-1}(K)} \nabla v$ in $L^2(\Omega, \mathbb{R}^2)$. On the other hand, by (3.2.6),

$$\begin{aligned}
&\Big| \int_\Omega \Big\langle \nabla \Phi, v \sum_i \Big(\prod_{j \neq i} \rho_j \circ \Phi \Big) \nabla(\rho_i \circ \Phi) \Big\rangle \, d\mathcal{L}^2 \Big| \\
&\leq 2\|v\|_{L^\infty} \sum_i r_i^{-1} \int_{\Phi^{-1}(B^Q_{2r_i}(p_i))} |\nabla \Phi|^2 \, d\mathcal{L}^2 \\
&\leq 4\|v\|_{L^\infty} \sum_i r_i^{-1} \|\mathbf{v}_\Omega\|(B^Q_{2r_i}(p_i)) = 4\|v\|_{L^\infty} \sum_i O(r_i) \to 0
\end{aligned}$$

as $\varepsilon \to 0$. This establishes the claim. Moreover, the right-hand side of (3.5.1) converges to $\int_{\Omega \setminus \Phi^{-1}(K)} A(\Phi)(\nabla \Phi, \nabla \Phi) v \, d\mathcal{L}^2$. Finally, in order to establish (3.5.1) with v in place of v_ε, we observe that $\nabla \Phi = 0$ a.e. on $\Phi^{-1}(K)$: indeed, by the area formula (see Lemma A.2),

$$\int_{\Phi^{-1}(K)} |\nabla \Phi|^2 \, d\mathcal{L}^2 = 2 \int_K \Big(\sum_{y \in \Phi^{-1}(p) \cap \mathcal{G}} 1 \Big) \, d\mathcal{H}^2(p) = 0,$$

since $\mathcal{H}^2(K) = 0$. The smoothness of Φ follows from the continuity of Φ and [82, Section 3.4]. $\square$

Definition 3.5.2. Given a local parametrized stationary varifold (Ω, Φ, N), a point $x \in \Omega$ is said to be *admissible* if Φ is nonconstant in any neighborhood of x and

$$\liminf_{r \to 0} \frac{\int_{B^2_{2r}(x)} |\nabla \Phi|^2 \, d\mathcal{L}^2}{\int_{B^2_r(x)} |\nabla \Phi|^2 \, d\mathcal{L}^2} < \infty.$$

We call $\mathcal{A} \subseteq \Omega$ the Borel set of the admissible points.

We now show that the image of the set of non-admissible points is very small.

Lemma 3.5.3. *The set $\Phi(\Omega \setminus \mathcal{A})$ has Hausdorff dimension 0, i.e.*

$$\mathcal{H}^s(\Phi(\Omega \setminus \mathcal{A})) = 0$$

for any real $s > 0$.

Proof. We can assume that Ω is bounded. Fix $s > 0$ and two arbitrary parameters $\delta, \varepsilon > 0$ and choose a real number $\alpha > 4s^{-1}$. For any $x \in \Omega \setminus \mathcal{A}$ we have

$$\int_{B^2_{2^{-k-1}}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \le 2^{-\alpha} \int_{B^2_{2^{-k}}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2$$

for all $k \ge k_0$ (for some threshold $k_0 \ge 0$ depending on x), so

$$\int_{B^2_{2^{-k}}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \le (2^{-k})^\alpha \Big(2^{k_0\alpha} \int_{B^2_{2^{-k_0}}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2\Big) = o((2^{-k})^{4/s}).$$

Hence we can find, for all $x \in \Omega \setminus \mathcal{A}$, a radius $r_x < \frac{1}{2} \operatorname{dist}(x, \partial\Omega)$ such that

$$\int_{B^2_{2r_x}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \le \varepsilon r_x^{4/s}, \quad \operatorname{diam} \Phi(B^2_{r_x}(x)) < \delta$$

(by continuity of Φ). Finally, Besicovitch covering theorem gives us a countable subcollection of balls $(B^2_{r_i}(x_i))$ such that $\Omega \setminus \mathcal{A} \subseteq \bigcup_i B^2_{r_i}(x_i)$ and $\sum_i \mathbf{1}_{B^2_{r_i}(x_i)}$ is bounded everywhere by a universal constant. Thus, by inequality (3.2.3) (or (3.2.4) if $\mathcal{M} = \mathbb{R}^Q$),

$$\begin{aligned}\sum_i \big(\operatorname{diam} \Phi(B^2_{r_i}(x_i))\big)^s &\le C \sum_i \Big(\int_{B^2_{2r_i}(x_i)} |\nabla\Phi|^2 \, d\mathcal{L}^2\Big)^{s/2} \le C\varepsilon^{s/2} \sum_i r_i^2 \\ &\le C\varepsilon^{s/2} \sum_i \mathcal{L}^2(B^2_{r_i}(x_i)) \le C\varepsilon^{s/2} \mathcal{L}^2(\Omega).\end{aligned}$$

Since δ and ε were arbitrary, we deduce $\mathcal{H}^s(\Phi(\Omega \setminus \mathcal{A})) = 0$. □

Let (Ω, Φ, N) be a local parametrized stationary varifold. Assume moreover that Ω is bounded, Φ extends continuously to $\overline{\Omega}$ with $\Phi(\Omega) \cap \Phi(\partial\Omega) = \emptyset$ and $\mathbf{v}_\Omega$ is stationary in $\mathcal{M} \setminus \Phi(\partial\Omega)$. Recall that in this situation the upper semicontinuous function $\widetilde{N}$ is finite (see Remark 3.2.11). We also assume that $\sup_\Omega \widetilde{N}$ is finite. For any $x \in \Omega$ and any $r \le \operatorname{dist}(x, \partial\Omega)$ we define $\ell(x, r) := \Big(\int_{B^2_r(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2\Big)^{1/2}$.

We now use a compactness argument, together with Theorem 3.4.1 and Theorem 3.3.7, in order to prove that, under some technical assumptions, the Dirichlet energy does not decay too fast (in a uniform, quantitative way). The main underlying idea is that this happens for a holomorphic function at a zero whose order is controlled, but we have to take care of the possible distortion caused by the quasiconformal homeomorphism appearing in the blow-up.

Lemma 3.5.4. *For every $C' > 0$ there exists $\varepsilon = \varepsilon(\Omega, \Phi, N, C') < \frac{1}{2}$ with the following property: whenever*

- $x \in \omega \subset\subset \Omega$ *and* $0 < r < \frac{1}{2} \operatorname{dist}(x, \partial\omega)$,
- $\ell(x, r) < \varepsilon \operatorname{dist}(\Phi(x), \Phi(\partial\omega))$,
- $0 < \int_{B^2_{2r}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < C' \int_{B^2_r(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2$,
- $\frac{\|\mathbf{v}_\omega\|(B^Q_s(\Phi(x)))}{\pi s^2} \in (\widetilde{N}(x) - \varepsilon, \widetilde{N}(x) + \varepsilon)$ *for all* $0 < s < \varepsilon^{-1}\ell(x, r)$,

there exists $r' \in \left(\varepsilon r, \frac{r}{2}\right)$ *such that* $\int_{B^2_{2r'}(x)} |\nabla\Phi|^2\, d\mathcal{L}^2 < \overline{C} \int_{B^2_{r'}(x)} |\nabla\Phi|^2\, d\mathcal{L}^2$*, for some* $\overline{C}$ *depending only on* $\sup_\Omega \widetilde{N}$*.*

Proof. Assume by contradiction that the statement is false for all $\varepsilon = 2^{-k}$. We can then find a sequence of points $x_k \in \omega_k$ and radii $r_k < \frac{1}{2}\operatorname{dist}(x_k, \partial\omega_k)$ such that (3.4.1) and (3.4.3) are satisfied, as well as $\widetilde{N}(x_k) \to \overline{N} \in [1, \sup_\Omega \widetilde{N}]$ (up to subsequences),

$$\frac{\|\mathbf{v}_{\omega_k}\|(B^Q_s(\Phi(x_k)))}{\pi s^2} \in (\widetilde{N}(x_k) - 2^{-k}, \widetilde{N}(x_k) + 2^{-k}) \quad \text{for } 0 < s < 2^k \ell(x_k, r_k), \tag{3.5.2}$$

$$\int_{B^2_{2r'}(x_k)} |\nabla\Phi|^2\, d\mathcal{L}^2 \geq \overline{C} \int_{B^2_{r'}(x_k)} |\nabla\Phi|^2\, d\mathcal{L}^2 \quad \text{for } 2^{-k} r_k < r' < \frac{r_k}{2} \tag{3.5.3}$$

($\overline{C}$ will be chosen at the end of the proof). Moreover, using the notation introduced in Section 3.4, the varifolds $\mathbf{v}_k := (\ell_k^{-1}(\cdot - \Phi(x_k)))_* \mathbf{v}_{\omega_k}$ have generalized mean curvature bounded by $O(\ell_k)$ (in L^∞) in

$$\mathbb{R}^Q \setminus \ell_k^{-1}(\Phi(\partial\omega_k) - \Phi(x_k)) \supseteq B^Q_{2^k}(0).$$

Hence, up to subsequences, the varifolds $\mathbf{v}_k$ converge to a stationary varifold $\mathbf{v}_\infty$ in $\mathbb{R}^Q$. Moreover, by (3.5.2), we have

$$\frac{\|\mathbf{v}_\infty\|(B^Q_s(0))}{\pi s^2} = \frac{\|\mathbf{v}_\infty\|(\overline{B}^Q_s(0))}{\pi s^2} = \overline{N}.$$

The varifold $\mathbf{v}_\infty$ is rectifiable and conical, with density in $[1, \overline{N}]$ on $\operatorname{spt}(\mathbf{v}_\infty)$ (see e.g. the proofs of [98, Corollary 42.6 and Theorem 19.3]). In particular, since $\mu_k \leq \|\mathbf{v}_k\|$, (3.4.2) follows, with $C'' := \overline{N}$. So, up to subsequences, the conclusions of Theorem 3.4.1 hold. We remark that Φ_∞ satisfies

$$\int_{B^2_1(0)} N_\infty |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2 = \nu_\infty(B^2_1(0)) = \lim_{k\to\infty} \nu_k(B^2_1(0)) = \frac{1}{2}$$

and in particular it is nonconstant. The varifold $\mathbf{v}_\infty$ has also integer multiplicity by [3, Theorem 6.4], thus it can be expressed as a cone (with vertex 0) over some stationary integer 1-rectifiable varifold $\mathbf{w}$ in S^{Q-1} with density in $[1, \overline{N}]$ $\|\mathbf{w}\|$-a.e.

By the structure theorem in [5, Section 3], $\mathbf{w}$ is supported in a finite union of geodesic curves. Hence, $\mathbf{v}_\infty$ is supported in a finite union of planes through the origin. Letting $\Psi := \Phi_\infty \circ \varphi_\infty^{-1}$, by (3.4.7) and Lemma 3.4.3 we have

$$\left\|\mathbf{v}_{(\Omega_\infty, \Psi, N_\infty \circ \varphi_\infty^{-1})}\right\| = (\Phi_\infty)_* \nu_\infty = \mu_\infty \leq \|\mathbf{v}_\infty\|.$$

So, using Proposition 3.2.4, we deduce that $\Psi(\Omega_\infty) \subseteq \operatorname{spt}(\|\mathbf{v}_\infty\|)$. Hence, Theorem 3.3.7 applies: we obtain that Ψ takes values in a plane and is a holomorphic function (once this plane is suitably identified with $\mathbb{C}$). Furthermore, by Lemma 3.4.5, we can assume $\|N_\infty\|_\infty \leq C'' = \overline{N}$. Now, by Proposition 3.2.8 and Lemma 3.4.5,

$$\frac{1}{2}\widetilde{N}(x_k + r_k \cdot)|\nabla\Phi_k|^2 \mathbf{1}_{B^2_2(0)} \mathcal{L}^2 = \nu_k \rightharpoonup N_\infty |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty| \mathcal{L}^2,$$

so (3.5.3) gives

$$\int_{B^2_{2r}(0)} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2 \geq \frac{\overline{C}}{(\sup_\Omega \widetilde{N})^2} \int_{B^2_r(0)} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2 \quad \text{for } r < \frac{1}{2}.$$

Let $\overline{k} \geq 1$ be the biggest integer such that $|\Psi(w)| = O\Big(|w|^{\overline{k}}\Big)$. We claim that

$$\overline{k} \leq \overline{N}.$$

Indeed, since Ψ is nonconstant and holomorphic, we have $K_w = \{w\}$ for all $w \in \Omega_\infty$ and the function $\widetilde{N}$ for the local parametrized stationary varifold $(\Omega_\infty, \Psi, N_\infty \circ \varphi_\infty^{-1})$ is everywhere finite (see Remark 3.2.11). We call it N' in order not to confuse it with the same function for (Ω, Φ, N). Since the density of the varifold $\mathbf{v}_{(\Omega_\infty, \Psi, N_\infty \circ \varphi_\infty^{-1})}$ is bounded everywhere by $\overline{N}$ (being this true for $\mathbf{v}_\infty$), the same argument used to prove Proposition 3.2.6 gives

$$\overline{k} \leq \sum_{j=1}^{\overline{k}} N'(w_j) \leq \overline{N}.$$

whenever $w_1, \dots, w_{\overline{k}} \in \Omega_\infty$ are distinct points with the same image. Such points exist because Ψ is a $\overline{k}$-to-1 map near 0. This establishes our claim and we deduce that

$$\lim_{r\to 0} \frac{\int_{B^2_{2r}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2}{\int_{B^2_r(0)} |\nabla\Psi|^2\, d\mathcal{L}^2} = 2^{2\overline{k}} \leq 2^{2\overline{N}}.$$

As was shown in the proof of Theorem 3.4.1, φ_∞ is an $\overline{N}^2$-quasiconformal homeomorphism. Since $\overline{N} \leq \sup_\Omega \widetilde{N}$, we claim that there exists a constant $\mathcal{K} \in \mathbb{N}$ depending only on $\sup_\Omega \widetilde{N}$ such that, for $r > 0$ small enough, there exists $s = s(r) > 0$ with

$$B^2_s(0) \subseteq \varphi_\infty(B^2_r(0)) \subseteq \varphi_\infty(B^2_{2r}(0)) \subseteq B^2_{2^{\mathcal{K}}s}(0). \tag{3.5.4}$$

Let $s := \min_{z\in\partial B^2_r(0)} |\varphi_\infty(z)|$, $s' := \max_{z\in\partial B^2_{2r}(0)} |\varphi_\infty(z)|$ and call z_1 and z_2 two points where the minimum and the maximum are attained, respectively. If r is small enough we have $\overline{B}^2_{s'}(0) \subseteq \varphi_\infty(B^2_2(0))$. Letting $A := B^2_{s'}(0) \setminus \overline{B}^2_s(0)$ and denoting by $M(\cdot)$ the module of a ring domain (see [65, Section I.6.1] for the definition), by [65, Theorem I.7.1] we have

$$\log\left(\frac{s'}{s}\right) = M(A) \leq \overline{N}^2 M(\varphi_\infty^{-1}(A)). \tag{3.5.5}$$

The ring domain $\varphi_\infty^{-1}(A)$ separates 0 and z_1 from z_2 and ∞ (in $\widehat{\mathbb{C}}$), so Teichmüller's module theorem (see [65, Section II.1.3]), together with $|z_2| = 2|z_1|$, implies that $M(\varphi_\infty^{-1}(A))$ is bounded by a constant depending only on $\sup_\Omega \widetilde{N}$. Since $\varphi_\infty(B^2_r(0)) \supseteq B^2_s(0)$ and $\varphi_\infty(B^2_{2r}(0)) \subseteq B^2_{s'}(0)$, (3.5.4) follows from (3.5.5). Thus, by the area formula,

$$\frac{\overline{C}}{(\sup_\Omega \widetilde{N})^2} \leq \frac{\int_{B^2_{2r}(0)} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2}{\int_{B^2_r(0)} |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2} \leq \frac{\int_{B^2_{2^{\mathcal{K}}s(r)}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2}{\int_{B^2_{s(r)}(0)} |\nabla\Psi|^2\, d\mathcal{L}^2} \to (2^{2\overline{k}})^{\mathcal{K}} \leq 2^{2\mathcal{K}\overline{N}}$$

as $r \to 0$. We deduce $\overline{C} \leq (\sup_\Omega \widetilde{N})^2 2^{2\mathcal{K}\sup_\Omega \widetilde{N}}$ and this is a contradiction once we choose $\overline{C}$ so large that this inequality fails. $\square$

We need another technical result, which is again obtained by means of a compactness argument.

Lemma 3.5.5. *For every $\delta \in (0,1)$ there exists $\varepsilon' = \varepsilon'(\Omega, \Phi, N, \delta)$ with the following property: whenever*

- $x \in \omega \subset\subset \Omega$ *and* $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\omega)$,
- $\ell(x,r) < \varepsilon' \operatorname{dist}(\Phi(x), \Phi(\partial\omega))$,
- $0 < \int_{B^2_{2r}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C} \int_{B^2_r(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2$, *with* $\overline{C}$ *given by Lemma 3.5.4,*
- $\frac{\|\mathbf{v}_\omega\|(B^Q_s(\Phi(x)))}{\pi s^2} \in (\widetilde{N}(x) - \varepsilon', \widetilde{N}(x) + \varepsilon')$ *for all* $0 < s < (\varepsilon')^{-1}\ell(x,r)$,

we have $\int_{B^2_{\delta r}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \geq \varepsilon' \int_{B^2_r(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$.

Proof. Arguing by contradiction as in the proof of Lemma 3.5.4, we would get a local parametrized stationary varifold $(\Omega_\infty, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ with

$$\int_{B^2_\delta(0)} |\nabla\Phi_\infty|^2 \, d\mathcal{L}^2 = 0.$$

But then $\Psi := \Phi_\infty \circ \varphi_\infty^{-1}$ could be identified with a nonconstant holomorphic function, on the connected domain $\Omega_\infty = \varphi_\infty(B^2_2(0))$, with uncountably many zeroes. This is a contradiction. □

Corollary 3.5.6. *If x lies in the admissible set $\mathcal{A}$, then there exists an arbitrarily small $r > 0$ such that $\int_{B^2_{2r}(x)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C} \int_{B^2_r(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$. Moreover,*

$$\limsup_{r\to 0} \frac{\int_{B^2_{2r}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2}{\int_{B^2_r(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2} < \infty.$$

Proof. Since $x \in \mathcal{A}$, we have $\liminf_{r\to 0} \frac{\int_{B^2_{2r}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2}{\int_{B^2_r(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2} < C'$ for some finite C'. Let $K_x \subseteq \omega \subset\subset \Omega$ with $\overline{\omega}$ disjoint from $\Phi^{-1}(\Phi(x)) \setminus K_x$ and choose a radius r such that

$$\int_{B^2_{2r}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < C' \int_{B^2_r(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$$

and r so small that it satisfies the other hypotheses of Lemma 3.5.4 with both $\varepsilon = \varepsilon(\Omega, \Phi, N, C')$ and $\varepsilon = \varepsilon(\Omega, \Phi, N, \overline{C})$ (the density assumptions for $\|\mathbf{v}_\omega\|$ are eventually satisfied by definition of $\widetilde{N}$). Then, by Lemma 3.5.4, we can find

$$\varepsilon(\Omega, \Phi, N, C')r < r_1 < \frac{r}{2}$$

such that $\int_{B^2_{2r_1}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C} \int_{B^2_{r_1}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$. This new radius r_1 satisfies the hypotheses of Lemma 3.5.4 with $\varepsilon = \varepsilon(\Omega, \Phi, N, \overline{C})$, so there exists

$$\varepsilon(\Omega, \Phi, N, \overline{C})r_1 < r_2 < \frac{r_1}{2}$$

such that $\int_{B^2_{2r_2}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C} \int_{B^2_{r_2}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$. Again, r_2 satisfies the hypotheses of Lemma 3.5.4 with $\varepsilon = \varepsilon(\Omega, \Phi, N, \overline{C})$, so we can find $\varepsilon(\Omega, \Phi, N, \overline{C})r_2 < r_3 < \frac{r_2}{2}$ and so on. Eventually r_k satisfies the hypotheses of Lemma 3.5.5 with $\varepsilon' = \varepsilon'(\Omega, \Phi, N, \varepsilon(\Omega, \Phi, N, \overline{C}))$. Thus,

$$\int_{B^2_{r_{k+1}}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 \geq \varepsilon' \int_{B^2_{r_k}(0)} |\nabla\Phi|^2 \, d\mathcal{L}^2.$$

Any radius $s > 0$ small enough lies in some interval $[r_{k+1}, r_k]$ and $2s \leq r_{k-1}$. The result follows. □

We are finally ready to show the full regularity result for parametrized stationary varifolds.

Theorem 3.5.7. *Let (Ω, Φ, N) be a local parametrized stationary varifold in $\mathcal{M}$. Then Φ solves $-\Delta\Phi = A(\Phi)(\nabla\Phi, \nabla\Phi)$ and, on each connected component where Φ is nonconstant, Φ is a C^∞-smooth branched immersion and N is a.e. constant.*

For the definition of branched immersion, see e.g. [47, Definitions 1.2 and 1.6].

Proof. Assume that $\Omega' \subset\subset \Omega$ satisfies $\Phi(\Omega') \cap \Phi(\partial\Omega') = \emptyset$ and $\sup_{\Omega'} \widetilde{N} < \infty$. We show that the partial differential equation holds in Ω' (the full result will be obtained at the end of the proof). More precisely, letting

$$\gamma := \frac{1}{3}\varepsilon\big(\Omega', \Phi\big|_{\Omega'}, N\big|_{\Omega'}, \overline{C}\big) < 1$$

(where $\overline{C}$ is the constant given by Lemma 3.5.4, depending only on $\sup_{\Omega'} \widetilde{N}$), we will show that the equation holds on the open set $\Omega_k := \Omega' \cap \{\gamma^{-1}\widetilde{N} < k+1\}$, by induction on k. The base case $k = 0$ is trivial, since $\{\widetilde{N} < \gamma\} = \emptyset$. Assume that the equation holds for $k-1$.

We call $\mathcal{C}_k$ the set of accumulation points of $\Omega_k \setminus \Omega_{k-1}$ in Ω_k, i.e. its derived set in Ω_k. Notice that $\mathcal{C}_k \subseteq \Omega_k \setminus \Omega_{k-1}$ is closed in Ω_k. We also set $\mathcal{A}_k := \mathcal{C}_k \cap \mathcal{A}$ and $\mathcal{B}_k := \mathcal{C}_k \setminus \mathcal{A}_k$. Notice that, by Lemma 3.5.1, the equation holds in the open set $\Omega_k \setminus \mathcal{C}_k$, since here the points with $k\gamma \leq \widetilde{N}(\cdot) < (k+1)\gamma$ form a discrete set.

Step 1. We first show that $\mathcal{A}_k$ is relatively open in $\mathcal{C}_k$, so that $\mathcal{B}_k$ is closed in Ω_k. Let $x_0 \in \mathcal{A}_k$. First of all, by Remark 3.2.12, we can find $K_{x_0} \subseteq \omega \subset\subset \Omega'$ with $\overline{\omega}$ disjoint from $\Phi^{-1}(\Phi(x_0)) \setminus K_{x_0}$, $\Phi(\omega) \cap \Phi(\partial\omega) = \emptyset$ and $\omega \cap \Phi^{-1}(\Phi(y)) = K_y$ whenever $y \in \omega$ has $k\gamma \leq \widetilde{N}(y) < (k+1)\gamma$ (thanks to the fact that $\gamma < 1$).

Let $\varepsilon := \varepsilon\big(\Omega', \Phi\big|_{\Omega'}, N\big|_{\Omega'}, \overline{C}\big)$ and assume that $x_j \to x_0$, with $x_j \in \omega \cap \mathcal{C}_k$. In particular, by definition of $\widetilde{N}$, the density of $\mathbf{v}_\omega$ at $\Phi(x_j)$ coincides with $\widetilde{N}(x_j)$. Using Corollary 3.5.6, we choose a radius $0 < r < \frac{1}{2}\operatorname{dist}(x_0, \partial\omega)$ with

$$\int_{B^2_{2r}(x_0)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C} \int_{B^2_r(x_0)} |\nabla\Phi|^2 \, d\mathcal{L}^2, \quad \ell(x_0, r) < \varepsilon \operatorname{dist}(\Phi(x_0), \Phi(\partial\omega)),$$

$$\frac{\|\mathbf{v}_\omega\|(\overline{B}^Q_{\varepsilon^{-1}\ell(x_0,r)}(\Phi(x_0)))}{\pi(\varepsilon^{-1}\ell(x_0,r))^2} < \widetilde{N}(x_0) + \gamma.$$

Eventually all the assumptions of Lemma 3.5.4 are satisfied by $x_j \in \omega$ (with Ω' and $C' := \overline{C}$), provided $\ell(x_0, r)$ is small enough: eventually we have $B^Q_{\varepsilon^{-1}\ell(x_j,r)}(\Phi(x_j)) \cap \Phi(\partial\omega) = \emptyset$, so by the monotonicity formula we get, for $0 < s < \varepsilon^{-1}\ell(x_j, r)$,

$$\widetilde{N}(x_j) \le e^{(\sqrt{2}\|A\|_\infty)s} \frac{\|\mathbf{v}_\omega\|(B^Q_s(\Phi(x_j)))}{\pi s^2} \le e^{(\sqrt{2}\|A\|_\infty)\varepsilon^{-1}\ell(x_j,r)} \frac{\|\mathbf{v}_\omega\|(B^Q_{\varepsilon^{-1}\ell(x_j,r)}(\Phi(x_j)))}{\pi(\varepsilon^{-1}\ell(x_j,r))^2}$$
$$\le e^{(\sqrt{2}\|A\|_\infty)\varepsilon^{-1}\ell(x_j,r)}(\widetilde{N}(x_0)+\gamma) \le (e^{(\sqrt{2}\|A\|_\infty)\varepsilon^{-1}\ell(x_0,r)} + o_j(1))\frac{\widetilde{N}(x_0)+\gamma}{\widetilde{N}(x_0)+2\gamma}(\widetilde{N}(x_j)+\varepsilon)$$

eventually and, since $\frac{\widetilde{N}(x_j)}{\widetilde{N}(x_j)-\varepsilon} \ge \frac{\widetilde{N}(x_0)-\gamma}{\widetilde{N}(x_0)-2\gamma}$, it suffices to impose additionally that

$$e^{(\sqrt{2}\|A\|_\infty)\varepsilon^{-1}\ell(x_0,r)} < \min\left\{\frac{\widetilde{N}(x_0)-\gamma}{\widetilde{N}(x_0)-2\gamma}, \frac{\widetilde{N}(x_0)+2\gamma}{\widetilde{N}(x_0)+\gamma}\right\}.$$

By Lemma 3.5.4 applied to the parametrized varifold $\left(\Omega', \Phi\big|_{\Omega'}, N\big|_{\Omega'}\right)$ and the point $x_j \in \omega$, for j big enough there exists $r' < \frac{r}{2}$ (depending on j) such that $\int_{B^2_{2r'}(x_j)} |\nabla\Phi|^2 \, d\mathcal{L}^2 < \overline{C}\int_{B^2_{r'}(x_j)} |\nabla\Phi|^2 \, d\mathcal{L}^2$. Since r' satisfies again all the hypotheses of Lemma 3.5.4, we can iterate and deduce that $x_j \in \mathcal{A}$. Hence, $x_j \in \mathcal{A}_k$ eventually.

Step 2. We now claim that $\widetilde{N}(x)$ is an integer for any admissible point $x \in \Omega' \cap \mathcal{A}$. Indeed, as in the proof of Lemma 3.5.4, we can apply Theorem 3.4.1 with $x_k := x$ and a suitable sequence of radii $r_k \to 0$: whenever $K_x \subseteq \widetilde{\omega} \subset\subset \Omega'$ has its closure disjoint from $\Phi^{-1}(\Phi(x)) \setminus K_x$, the varifolds $(\ell_k^{-1}(\cdot - \Phi(x_k)))_*\mathbf{v}_{\widetilde{\omega}}$ converge to a stationary cone $\mathbf{v}_\infty$ having density at most $\widetilde{N}(x)$ at 0, so we have $\|\mathbf{v}_\infty\|(B^Q_s(p)) \le \widetilde{N}(x)\pi s^2$ (see the proof of [98, Theorem 42.4]) and thus (3.4.2) holds with $C'' := \widetilde{N}(x)$.

We obtain a local parametrized stationary varifold $(\Omega_\infty, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ with $\Psi := \Phi_\infty \circ \varphi_\infty^{-1}$ nonconstant and holomorphic (again by Theorem 3.3.7, since the mass measure of this parametrized varifold is bounded by the mass measure of $\mathbf{v}_\infty$, which is an integer rectifiable stationary cone).

Let $\omega' \subset\subset B^2_2(0)$ be a smooth neighborhood of 0 with $0 \notin \Phi_\infty(\partial\omega')$ and $\Phi_\infty^{-1}(0) \cap \omega' = \{0\}$. Using the notation of Section 3.4, from the locally uniform convergence $\Phi_k \to \Phi_\infty$ we infer that eventually $\Phi(x) \notin \Phi(x + r_k\partial\omega')$ and, for all $0 < s < \operatorname{dist}(0, \Phi_\infty(\partial\omega'))$,

$$\widetilde{N}(x) \le \lim_{k\to\infty} \frac{\|\mathbf{v}_{x+r_k\omega'}\|(B^Q_{\ell_k s}(\Phi(x)))}{\pi(\ell_k s)^2} = \lim_{k\to\infty} \frac{(\Phi_k)_*(\mathbf{1}_{\omega'}\nu_k)(B^Q_s(0))}{\pi s^2}$$
$$= \frac{(\Phi_\infty)_*(\mathbf{1}_{\omega'}\nu_\infty)(B^Q_s(0))}{\pi s^2},$$

by the definition of $\widetilde{N}$, the monotonicity formula and Lemma 3.4.3. We deduce that, with the same notation as in the proof of Lemma 3.5.4, $N'(0) \ge \widetilde{N}(x)$. We also have the converse inequality $N'(0) \le \widetilde{N}(x)$, since the density of μ_∞ is everywhere at most $C'' = \widetilde{N}(x)$. This argument also shows that $\Psi^{-1}(0) = 0$ and $\sum_{i=1}^r N'(z_i) \le \widetilde{N}(x_0)$ whenever $z_i \in \Omega_\infty$ are distinct points in a fiber $\Psi^{-1}(p)$ (since z_i contributes by $N'(z_i)$ to the density of μ_∞ at p).

Finally, arguing as in the proof of Theorem 3.3.3, we conclude that $N'(0)$ is integer since it equals the constant density of a suitable localization of $(\Omega_\infty, \Psi, N_\infty \circ \varphi_\infty^{-1})$ (in an open subset of $\Psi(\Omega_\infty)$). This establishes our claim.

Step 3. We show by contradiction that there cannot be any $x_0 \in \partial\mathcal{A}_k \cap \mathcal{A}_k \cap \Omega_k$. Indeed, if this happens, then we can find $x_1 \in \Omega_k \setminus \mathcal{A}_k$ with

$$|x_1 - x_0| < \frac{1}{2} \min\{\mathrm{dist}(x_0, \mathcal{B}_k), \mathrm{dist}(x_0, \partial\Omega_k)\},$$

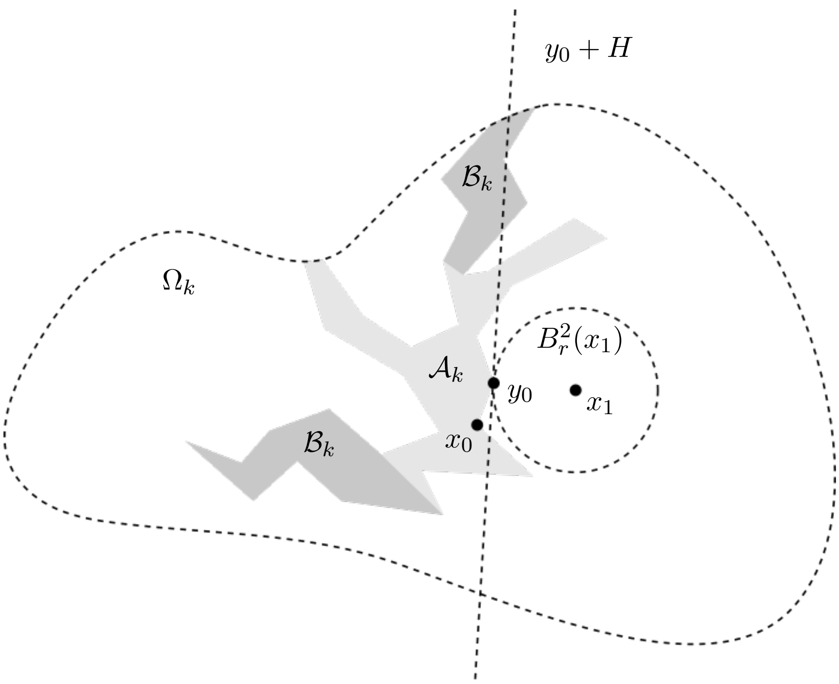
thanks to the fact that the latter is positive, as $\mathcal{B}_k$ is closed in Ω_k. We infer that

$$r := \mathrm{dist}(x_1, \mathcal{C}_k) < \frac{1}{2} \min\{\mathrm{dist}(x_0, \mathcal{B}_k), \mathrm{dist}(x_0, \partial\Omega_k)\},$$

there exists $y_0 \in \mathcal{C}_k$ with $|y_0 - x_1| = r$ and necessarily we have

$$y_0 \in \mathcal{A}_k, \quad B_r^2(x_1) \subseteq \Omega_k \setminus \mathcal{C}_k.$$

Let $H \subset \mathbb{C}$ be the unique open half-plane with $0 \in \partial H$ and $B_r^2(x_1) \subseteq y_0 + H$. By definition of $\mathcal{C}_k$, we can find a sequence $y_j \to y_0$ with $y_j \in \Omega_k \setminus \Omega_{k-1}$ and $y_j \neq y_0$. We can assume that $\frac{y_j - y_0}{|y_j - y_0|} \to \overline{y}$. Let $r_j := |y_j - y_0|$ and set $\ell_j^2 := \int_{B^2_{r_j}(y_0)} |\nabla\Phi|^2 \, d\mathcal{L}^2$, $\Phi_j := \ell_j^{-1}(\Phi(y_0 + r_j \cdot) - \Phi(y_0))$, $N_j := N(y_0 + r_j \cdot)$. Thanks to Corollary 3.5.6, we can apply Theorem 3.4.6 and obtain, up to subsequences, a limiting local parametrized stationary varifold $(\mathbb{C}, \Phi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$.



With the same notation and the same argument used in Step 2, we get $N'(0) = \widetilde{N}(y_0)$. Actually, we also have $N'(\varphi_\infty(\overline{y})) \geq k\gamma$: letting ω'' be a smooth neighborhood of $\overline{y}$ with $\Phi_\infty(\overline{y}) \notin \Phi_\infty(\partial\omega'')$ and $\Phi_\infty^{-1}(\Phi_\infty(\overline{y})) \cap \omega'' = \{\overline{y}\}$, as soon as $\Phi_j(r_j^{-1}(y_j - y_0)) \notin \Phi_j(\partial\omega'')$ and $r_j^{-1}(y_j - y_0) \in \omega''$ the varifold

$$\mathbf{v}_{(\omega'', \Phi_j, N_j)} = (\ell_j^{-1}(\cdot - \Phi(y_0)))_* \mathbf{v}_{(y_0 + r_j\omega'', \Phi, N)}$$

has density at least $\widetilde{N}(y_j) \geq k\gamma$ at $\ell_j^{-1}(\Phi(y_j) - \Phi(y_0)) = \Phi_j(r_j^{-1}(y_j - y_0)) \to \Phi_\infty(\overline{y})$, has infinitesimal mean curvature in $\mathbb{R}^Q \setminus \Phi_j(\partial\omega'')$ and its mass measure converges to $(\Phi_\infty)_*(\mathbf{1}_{\omega''}\nu_\infty)$, by Lemma 3.4.3. Thus, by the monotonicity formula, $N'(\varphi_\infty(\overline{y}))$ (i.e. the density of this last measure at $\Phi_\infty(\overline{y})$) is at least $k\gamma$.

Since $k\gamma + 1 > \widetilde{N}(y_0)$, as observed in Step 2 we must have $\Phi_\infty^{-1}(0) = \{0\}$ and $\Phi_\infty^{-1}(\Phi_\infty(\overline{y})) = \{\overline{y}\}$. Recall that, as in Step 2, the map $\Psi := \Phi_\infty \circ \varphi_\infty^{-1}$ takes values in a plane and is an entire holomorphic function, up to suitable identification of this plane with $\mathbb{C}$. Since it has two values having only one preimage, by Picard's great theorem it does not have an essential singularity at ∞ and is thus a polynomial. Actually, Picard's great theorem can be easily avoided: by Corollary 3.5.6 the Dirichlet energy $\int_{B_R^2(0)} |\nabla\Phi_\infty|^2\, d\mathcal{L}^2$ grows at most polynomially in R and, by inspecting the proof of [56, Theorem 4.30] (as well as inequalities (4.21) and (4.24) in [56]), we see that $\sup_{z\in B_R^2(0)} \left|\varphi_\infty^{-1}\right|(z)$ also grows at most polynomially, hence the same is true for $\int_{B_R^2(0)} |\nabla\Psi|^2\, d\mathcal{L}^2$ and thus (by the mean value property for harmonic functions) for $\sup_{z\in B_R^2(0)} |\nabla\Psi|(z)$, i.e. Ψ is a polynomial. Since $\Psi^{-1}(0) = \{0\}$, it must have the form

$$\Psi(z) = cz^{\overline{k}}$$

for some $\overline{k}$ and finally the fact that $\Psi^{-1}(\Phi_\infty(\overline{y}))$ is a singleton gives $\overline{k} = 1$. We deduce that $\Psi'(0) \neq 0$. Arguing as in the proof of Corollary 3.3.4, we get $N_\infty \circ \varphi_\infty^{-1} = N'(0) \geq k\gamma$ a.e. near 0.

We finally show that $\Phi_j \to \Phi_\infty$ in $W^{1,2}_{loc}(H, \mathbb{R}^Q)$. In particular, for any small ball $B \subset\subset H$ close enough to 0, this will contradict the estimate

$$\begin{aligned}
\kappa\gamma \int_B |\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|\, d\mathcal{L}^2 &\leq \nu_\infty(B) = \lim_{j\to\infty} \nu_j(B) \\
&= \frac{1}{2}\lim_{j\to\infty} \int_B N_j |\nabla\Phi_j|^2\, d\mathcal{L}^2 \leq \alpha \lim_{j\to\infty} \int_B |\partial_1\Phi_j \wedge \partial_2\Phi_j|\, d\mathcal{L}^2,
\end{aligned}$$

where α is the biggest integer smaller than $\kappa\gamma$ (the last inequality comes from the fact that $N_j \in \mathbb{N}$ and $N_j = \widetilde{N}(y_0 + r_j\cdot)$ a.e. on $\{\nabla\Phi_j \neq 0\}$, together with the fact that eventually $y_0 + r_j B \subseteq B_r^2(x_1) \subseteq \Omega_k \setminus \mathcal{C}_k$).

Fix any $U \subset\subset H$. Since eventually $-\Delta\Phi_j = \ell_j A(\Phi(y_0 + r_j\cdot))(\nabla\Phi_j, \nabla\Phi_j)$ on U and the right-hand side converges to 0 in $L^1(U, \mathbb{R}^Q)$, we get

$$-\Delta\Phi_\infty = 0$$

on U and hence (since U was arbitrary) on H. Fix any nonnegative $\rho \in C_c^\infty(H)$ with $\rho = 1$ on U. Setting $\Psi_j := \Phi_j - \Phi_\infty$, we have $-\Delta\Psi_j = \ell_j A(\Phi(y_0 + r_j\cdot))(\nabla\Phi_j, \nabla\Phi_j)$ and thus

$$\begin{aligned}
\int_H \rho|\nabla\Psi_j|^2\, d\mathcal{L}^2 + \int_H \Psi_j \cdot \langle\nabla\rho, \nabla\Psi_j\rangle\, d\mathcal{L}^2 &= \int_H \langle\nabla(\rho\Psi_j); \nabla\Psi_j\rangle\, d\mathcal{L}^2 \\
= \ell_j \int_H \rho\Psi_j \cdot A(\Phi(y_0 + r_j\cdot))(\nabla\Phi_j, \nabla\Phi_j)\, d\mathcal{L}^2.
\end{aligned}$$

But both the right-hand side and the second term in the left-hand side converge to 0. Hence,

$$\int_U |\nabla\Psi_j|^2\, d\mathcal{L}^2 \leq \int_H \rho|\nabla\Psi_j|^2\, d\mathcal{L}^2 \to 0.$$

Step 4. From the previous step we have $\partial\mathcal{A}_k \cap \Omega_k \subseteq \mathcal{B}_k$, hence $\mathcal{A}_k$ is open. Since $\gamma < 1$ and $\widetilde{N}$ is integer-valued on $\mathcal{A}_k$ (by Step 2), $\widetilde{N}$ takes exactly a single value here. We can then apply Theorem 3.2.13 (as replacing N with $\widetilde{N}$ does not affect the stationarity) and obtain that the partial differential equation holds on $\mathcal{A}_k$.

Step 5. From the two previous steps, it follows that $-\Delta\Phi = A(\Phi)(\nabla\Phi, \nabla\Phi)$ on the open set $\Omega_k \setminus \mathcal{B}_k$. Using Lemma 3.5.3 and Lemma 3.5.1, we deduce that the partial differential equation holds on the whole Ω_k (and Φ is C^∞-smooth on Ω_k). This completes the induction.

Step 6. The extra assumptions made at the beginning of the proof can be dropped by arguing as in the proof of Theorem 3.3.7 and using the upper semicontinuity of $\widetilde{N}$. For the fact that Φ is a branched immersion on a connected component Ω'' where it is nonconstant, we refer the reader to the proofs of [50, Theorems 1 and 2] and [47, Lemmas 2.1 and 2.2]. Finally, let $D \subset \Omega''$ denote the discrete subset where $\nabla\Phi = 0$. Whenever a connected $\omega \subset\subset \Omega'' \setminus D$ is such that $\Phi(\omega) \cap \Phi(\partial\omega) = \emptyset$ and $\Phi\big|_\omega$ is an embedding, the constancy theorem (see [98, Theorem 41.1]) implies that $\widetilde{N}$ is constant on ω. Since $\Omega'' \setminus D$ is connected, Proposition 3.2.8 gives that N is a.e. constant on $\Omega'' \setminus D$, hence on Ω''. □

Corollary 3.5.8. *If (Σ, Φ, N) is a parametrized stationary varifold, with Σ connected and Φ nonconstant, then Φ solves $-\Delta\Phi = A(\Phi)(\nabla\Phi, \nabla\Phi)$ in local conformal coordinates, Φ is a C^∞-smooth branched immersion and N is a.e. constant.*

Remark 3.5.9. We notice that the converse statement holds as well: namely, if $\Phi : \Sigma \to \mathcal{M}$ is a nonconstant weakly conformal, weakly harmonic map and N is a positive integer, then (Σ, Φ, N) is a parametrized stationary varifold. Indeed, for almost every $\omega \subseteq \Sigma$, the continuous representative of $\Phi\big|_{\partial\omega}$ coincides with the trace (by [35, Theorem 5.7] this holds whenever $\mathcal{H}^1$-a.e. point of $\partial\omega$ is a Lebesgue point for Φ). Thus, for any smooth $F \in C_c^\infty(\mathbb{R}^Q \setminus \Phi(\partial\omega), \mathbb{R}^Q)$, $F(\Phi)\big|_\omega$ has zero trace on $\partial\omega$ and (3.2.1) follows. Notice that we did not need Hélein's regularity result to show this assertion: on the contrary, we can immediately deduce the continuity of Φ (and hence the smoothness) from Proposition 3.2.4.

3.6 An application to the conductivity equation

In this section we illustrate an application of Theorem 3.5.7 to the regularity theory for the conductivity equation

$$-\operatorname{div}(N\nabla\Phi) = 0 \quad \text{on } B_1^2(0).$$

This partial differential equation was already investigated by many authors: see e.g. [11, 36, 66]. We show below that, assuming $\Phi \in W^{1,2}(B_1^2(0), \mathbb{R}^Q)$ weakly conformal and $N \in L^\infty(\mathbb{N} \setminus \{0\})$, Φ is necessarily harmonic and N is a.e. constant, unless Φ is itself constant.

This statement initially originated as a possible intermediate step in order to achieve Theorem 3.5.7, but as a matter of fact we are able to prove the former only as a consequence of the latter. It would be interesting to find an independent, purely PDE-theoretic proof.

We can do this in the case $Q = 2$, where the following slightly stronger result holds.

Theorem 3.6.1. *Assume $\Phi \in W^{1,2}(B_1^2(0), \mathbb{R}^2)$ is weakly conformal, $N \in L^\infty(B_1^2(0))$ is bounded below by a positive constant and*

$$-\operatorname{div}(N\nabla\Phi) = 0 \quad \text{in } \mathcal{D}'(B_1^2(0), \mathbb{R}^2). \tag{3.6.1}$$

Then $\Phi_1 + i\,\Phi_2$ is holomorphic or antiholomorphic and, if Φ is nonconstant, N is a.e. constant.

Proof. First of all, Φ is continuous (see e.g. [44, Section 4.4]). We can assume that Φ is nonconstant, so that the set $\mathcal{G}^f \neq \emptyset$ of Lebesgue points z for $\nabla\Phi$ with $\nabla\Phi(z) \neq 0$ is nonempty. Notice that Φ_1 and Φ_2 are both nonconstant: otherwise e.g. $\nabla\Phi_1$ would be a.e. 0 and in particular we would have $\nabla\Phi_1 = 0$ on $\mathcal{G}^f$, contradicting the weak conformality. From (3.6.1) and standard Hodge theory, we can find real functions $\Psi_k \in W^{1,2}(B_1^2(0))$ with

$$N\nabla\Phi_k = -\nabla^\perp\Psi_k, \quad \nabla^\perp := (-\partial_2, \partial_1),$$

for $k = 1, 2$. This equation can be equivalently rewritten as

$$N\partial_z\Phi_k = i\partial_z\Psi_k \quad \text{or} \quad N\partial_{\overline{z}}\Phi_k = -i\partial_{\overline{z}}\Psi_k. \tag{3.6.2}$$

Let $f_k := \Phi_k + i\,\Psi_k$. We have

$$\partial_{\overline{z}} f_k = (1 - N)\partial_{\overline{z}}\Phi_k, \quad \partial_z f_k = (1 + N)\partial_z\Phi_k.$$

We define the Beltrami coefficient μ_k on $\mathbb{C}$ as

$$\mu_k := \frac{(1 - N)\partial_{\overline{z}}\Phi_k}{(1 + N)\partial_z\Phi_k}\mathbf{1}_{B_1^2(0)\cap\mathcal{G}^f}.$$

Notice that, by weak conformality, $\partial_z\Phi_k \neq 0$ on $\mathcal{G}^f$. Our hypotheses on N clearly imply

$$\|\mu_k\|_{L^\infty} < 1$$

and f_k satisfies the Beltrami equation

$$\partial_{\overline{z}} f_k = \mu_k\partial_z f_k$$

on $B_1^2(0)$. Let φ_k be the normal solution of

$$\partial_{\overline{z}}\varphi_k = \mu_k\partial_z\varphi_k$$

(see [56, Theorem 4.24]). As already pointed out in the proof of Theorem 3.4.1, $\varphi_k, \varphi_k^{-1} \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ are homeomorphisms of $\mathbb{C}$ mapping negligible sets to negligible sets and

$$\partial_{\overline{w}}\varphi_k^{-1} = -(\mu_k \circ \varphi_k^{-1})\overline{\partial_w\varphi_k^{-1}}.$$

By the chain rule (see [65, Lemma III.6.4]), $h_k := f_k \circ \varphi_k^{-1}$ (defined on $\varphi_k(B_1^2(0))$) satisfies

$$\partial_{\overline{w}} h_k = (\partial_z f_k \circ \varphi_k^{-1}) \partial_{\overline{w}} \varphi_k^{-1} + (\partial_{\overline{z}} f_k \circ \varphi_k^{-1}) \overline{\partial_w \varphi_k^{-1}} = 0$$

and is thus a nonconstant holomorphic function. Pick now any $z_0 \in B_1^2(0)$ such that the points $\varphi_1(z_0)$ and $w_0 := \varphi_2(z_0)$ satisfy $h_1'(\varphi_1(z_0)) \neq 0$ and $h_2'(w_0) \neq 0$: by holomorphicity of h_1 and h_2, this holds true for all z_0 outside a discrete, relatively closed subset $D \subset B_1^2(0)$.

By the Cauchy–Riemann equations, the harmonic map $\Phi_2 \circ \varphi_2^{-1} = \Re h_2$ has nonzero differential at w_0. By the inverse function theorem, there exists a local chart ψ centered at w_0, with some ball $B_\delta^2(0)$ as its image, such that

$$\Phi_2 \circ \varphi_2^{-1} \circ \psi^{-1}(y) - \Phi_2(z_0) = y_2 \quad \text{on } B_\delta^2(0). \tag{3.6.3}$$

Since Φ is weakly conformal, we have $(\partial_z \Phi_1)^2 + (\partial_z \Phi_2)^2 = 0$ a.e., hence there exists a measurable function $\varepsilon \in L^\infty(B_1^2(0), \{-1, 1\})$ such that

$$\partial_z \Phi_1(z) = i\varepsilon(z) \partial_z \Phi_2(z) \quad \text{a.e. on } B_1^2(0). \tag{3.6.4}$$

Combining (3.6.2) and (3.6.4) we obtain

$$\nabla \Psi_1 = \varepsilon N \nabla \Phi_2$$

a.e. Using (3.6.3) and the chain rule again, we get

$$\partial_1(\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1}) = 0, \quad \partial_2(\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1}) = (\varepsilon N) \circ \varphi_2^{-1} \circ \psi^{-1}$$

a.e. and, since $\partial_{12}^2(\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1}) = 0$ distributionally, we deduce that

$$(\varepsilon N) \circ \varphi_2^{-1} \circ \psi^{-1}(y) = \partial_2(\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1})(y) = g(y_2)$$

a.e. on $B_\delta^2(0)$, for a suitable $g \in L^\infty((-\delta, \delta))$, as is immediately verified e.g. by mollification.

Let G be a Lipschitz primitive of g on $(-\delta, \delta)$. We have

$$\nabla(\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1}) = \nabla(G(y_2)) \quad \text{on } B_\delta^2(0),$$

so up to subtracting a constant from G we obtain

$$\Psi_1 \circ \varphi_2^{-1} \circ \psi^{-1}(y) = G(y_2)$$

and finally, using (3.6.3),

$$\Psi_1 = G \circ (\Phi_2 - \Phi_2(z_0)) \quad \text{on } \varphi_2^{-1} \circ \psi^{-1}(B_\delta^2(0)) \ni z_0. \tag{3.6.5}$$

Since $f_1 = \Phi_1 + i\Psi_1$ is injective in a neighborhood of z_0 (being $h_1'(\varphi_1(z_0)) \neq 0$), from (3.6.5) we deduce that Φ is injective on some neighborhood $B_\eta^2(z_0)$.

We claim that, as a consequence, $\det(\nabla \Phi)$ has a constant sign on $B_\eta^2(z_0) \cap \mathcal{G}^f$: indeed, the induced map

$$\Phi_* : H_1(\partial B_r^2(z)) \to H_1(\mathbb{C} \setminus \{\Phi(z)\})$$

is clearly independent of $z \in B^2_\eta(z_0)$ and $0 < r < \operatorname{dist}(z, \partial B^2_\eta(z_0))$, once the two groups are canonically identified with $\mathbb{Z}$. But, for any $z \in B^2_\eta(z_0) \cap \mathcal{G}^f$, applying Lemma A.4 to $\Phi - \Phi(z) - \langle \nabla\Phi(z), \cdot - z\rangle$ we can find a small radius r such that $\Phi|_{\partial B^2_r(z)}$ is homotopic to $\Phi(z) + \langle \nabla\Phi(z), \cdot - z\rangle$ (as a map $\partial B^2_r(z) \to \mathbb{C} \setminus \{\Phi(z)\}$). Thus the above map Φ_* coincides with the multiplication by $\operatorname{sgn}\det(\nabla\Phi(z))$ and our claim follows.

From this fact and the weak conformality assumption, on $B^2_\eta(z_0)$ either (Φ_1, Φ_2) or $(\Phi_1, -\Phi_2)$ satisfy the Cauchy–Riemann equations. In particular, Φ is real analytic on the connected set $B^2_1(0) \setminus D$. Since locally we have either $\partial_{\bar z}(\Phi_1 + i\Phi_2) = 0$ or $\partial_z(\Phi_1 + i\Phi_2) = 0$, by analyticity $\Phi_1 + i\Phi_2$ is globally holomorphic or antiholomorphic on $B^2_1(0) \setminus D$, hence also on $B^2_1(0)$. Finally, (3.6.1) gives

$$\partial_1\Phi_k\partial_1 N + \partial_2\Phi_k\partial_2 N = 0 \quad \text{in } \mathcal{D}'(B^2_1(0))$$

for $k = 1, 2$ (as $\Delta\Phi_k = 0$). Since $\nabla\Phi_1$ and $\nabla\Phi_2$ are smooth and linearly independent outside a closed discrete set, we infer $\nabla N = 0$ here and thus, by connectedness, N is a.e. constant. □

We now prove the result for arbitrary Q.

Theorem 3.6.2. *Assume $\Phi \in W^{1,2}(B^2_1(0), \mathbb{R}^Q)$ is weakly conformal, $N \in L^\infty(B^2_1(0), \mathbb{N}\setminus\{0\})$ and*

$$-\operatorname{div}(N\nabla\Phi) = 0 \quad \text{in } \mathcal{D}'(B^2_1(0), \mathbb{R}^Q).$$

Then $\Delta\Phi = 0$ and, if Φ is nonconstant, N is a.e. constant.

Proof. As in the previous proof, we notice that Φ is continuous. We can assume that Φ is not constant. The triple $(B^2_1(0), \Phi, N)$ satisfies Definition 3.2.9 (with $\mathcal{M} = \mathbb{R}^Q$), except possibly for the technical condition (3.2.6): indeed, for any $\omega \subset\subset \mathcal{B}^2_1(0)$ and any $F \in C^\infty_c(\mathbb{R}^Q \setminus \Phi(\partial\omega), \mathbb{R}^Q)$, $F(\Phi)\mathbf{1}_\omega$ lies in $W^{1,2}_0(B^2_1(0), \mathbb{R}^Q)$ and thus $\int_\omega N\langle\nabla(F(\Phi)); \nabla\Phi\rangle\, d\mathcal{L}^2 = 0$.

Assume, without loss of generality, that Φ_1 is not constant and let Ψ_1, f_1, φ_1 and h_1 be the functions constructed as in the preceding proof. Let $D \subset \varphi_1(B^2_1(0))$ be the discrete set of points where $h_1' = 0$, or equivalently (by the Cauchy–Riemann equations) where $\nabla(\Phi_1 \circ \varphi_1^{-1}) = 0$. From the chain rule and the fact that φ_1 and φ_1^{-1} map negligible sets to negligible sets (see [65, Lemma III.6.4] and [56, Lemma 4.12]) we deduce that, for a.e. $x \in B^2_1(0)$, $\nabla\varphi_1(x)$ is invertible, $\nabla(\Phi_1 \circ \varphi_1^{-1})(\varphi_1(x)) \neq 0$ and $\nabla\Phi_1(x)$ is given by the composition of these differentials. Hence, $\nabla\Phi \neq 0$ a.e. and thus has full rank a.e. (by weak conformality).

Let $S \subseteq B^2_1(0)$ denote the complement of the biggest open subset where $\Delta\Phi = 0$. We remark that, given $x \in B^2_1(0)$, if there exists a neighborhood $\omega \subset\subset B^2_1(0)$ with $\Phi(x) \cap \Phi(\partial\omega) = \emptyset$ then $x \notin S$: indeed, $\mathbf{v}_\omega$ is stationary in $\mathbb{R}^Q \setminus \Phi(\partial\omega)$, so by the monotonicity formula $\mathbf{v}_\omega$ satisfies (3.2.6) for $p \in B^Q_\rho(\Phi(x))$ and $s < \rho$, where $\rho := \frac{1}{2}\operatorname{dist}(\Phi(x), \Phi(\partial\omega))$. Thus, replacing ω with $\omega \cap \Phi^{-1}(B^2_{\rho/2}(\Phi(x)))$, the triple (ω, Φ, N) is a local parametrized stationary varifold and Theorem 3.5.7 gives $\Delta\Phi = 0$ near x. In particular, arguing as in the proof of Theorem 3.3.7, we infer that $x \notin S$ for any Lebesgue point x for $\nabla\Phi$ with $\nabla\Phi(x) \neq 0$. Being $\nabla\Phi \neq 0$ a.e., we get that $\mathcal{L}^2(S) = 0$ and that Φ is nonconstant on any ball outside S.

Moreover, N is a.e. constant on every connected component U of $B^2_1(0) \setminus S$, since on U we have $\partial_1\Phi_k\partial_1 N + \partial_2\Phi_k\partial_2 N = 0$ for all $k = 1, \dots, Q$ and the (classical) differential $\nabla\Phi$ has full rank except for a discrete set (being Φ a nonconstant harmonic function on U), which does not disconnect U.

It suffices to show that $S \subseteq \varphi_1^{-1}(D)$, since then any point of S is a removable singularity and thus $S = \emptyset$. Assume by contradiction that there exists a point $x_0 \in S \setminus \varphi_1^{-1}(D)$. Let $\psi : V \to (-1,1)^2$ be a local chart centered at $\varphi_1(x_0)$ and such that

$$\Phi_1 \circ \varphi_1^{-1} \circ \psi^{-1}(y) = \Phi_1(x_0) + y_2 \quad \text{on } (-1,1)^2. \tag{3.6.6}$$

Let $\Xi := \Phi \circ \varphi_1^{-1} \circ \psi^{-1}$. We claim that the negligible set $S' := \psi(V \cap \varphi_1(S))$, relatively closed in $(-1,1)^2$, has the following property: if $y = (y_1, y_2) \in S'$, then $\Xi^{-1}(\Xi(y))$ contains either $(-1, y_1] \times \{y_2\}$ or $[y_1, 1) \times \{y_2\}$. If this were not true, we could find $a \in (-1, y_1)$ and $b \in (y_1, 1)$ with $\Xi(a, y_2), \Xi(b, y_2) \neq \Xi(y)$. Given any $\varepsilon < 1 - |y_2|$, by (3.6.6) we would have $\Xi(y) \notin \Xi(\partial([a,b] \times [y_2 - \varepsilon, y_2 + \varepsilon]))$. But, as remarked earlier, this would imply $y \notin S'$.

This horizontal segment, i.e. either $(-1, y_1] \times \{y_2\}$ or $[y_1, 1) \times \{y_2\}$, has to be contained in S' (being Ξ locally injective at points outside S', with at most countably many exceptions). As a consequence we have $\mathcal{L}^1(\{t : (t, y_2) \in S'\}) > 0$ and thus, by Fubini's theorem and $\mathcal{L}^2(S') = 0$, we infer

$$((-1,1) \times \{t\}) \cap S' = \emptyset \quad \text{for a.e. } t \in (-1,1). \tag{3.6.7}$$

Pick now any $\mu > 0$ with $((-1,1) \times \{\mu\}) \cap S' = \emptyset$. Recall that $(0,0) \in S'$ and let $\lambda := \max\{t < \mu : (0,t) \in S'\} \geq 0$. From (3.6.7) it follows that the open set $((-1,1) \times (\lambda, \mu)) \setminus S'$ is connected. We infer that N is a.e. constant on $\varphi_1^{-1} \circ \psi^{-1}((-1,1) \times (\lambda,\mu))$, hence $\Delta\Phi = 0$ on this open set. However, this contradicts the weak conformality assumption: let

$$\omega := \varphi_1^{-1} \circ \psi^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(\lambda, \frac{\lambda + \mu}{2}\right)\right),$$

on which $\Delta\Phi = 0$, and take any homeomorphism $\upsilon : \overline{\omega} \to \overline{B}^2_1(0)$ biholomorphic on ω (using Riemann's mapping theorem and [40, Theorem I.3.1]). We have $\Phi_1 \circ \upsilon^{-1} > \lambda$ on $B^2_1(0)$, as well as $\Phi_1 \circ \upsilon^{-1} = \lambda$ and $\Phi \circ \upsilon^{-1}$ constant on some open arc A in $\partial B^2_1(0)$. Since $\Delta(\Phi \circ \upsilon^{-1}) = 0$ on $B^2_1(0)$, by standard regularity theory $\Phi \circ \upsilon^{-1}$ is smooth in a neighborhood of A in $\overline{B}^2_1(0)$. But, by Hopf's maximum principle, the radial derivative of $\Phi_1 \circ \upsilon^{-1}$ is nonzero on A. So, by weak conformality of $\Phi \circ \upsilon^{-1}$, the tangential derivative of $\Phi \circ \upsilon^{-1}$ does not vanish on A. This contradicts the fact that $\Phi \circ \upsilon^{-1}$ is constant on A. $\square$

4 Multiplicity one for parametrized stationary varifolds arising variationally

4.1 Introduction

Recall the main result of the second chapter, namely that a certain sequence of maps Φ_k which are (almost) critical for E_{σ_k} converge, in the varifold sense, to a *parametrized stationary varifold* (Σ', Φ, N).

A consequence of the theory contained in the previous chapter is that the multiplicity N is locally constant. This result, which is optimal for the class of parametrized stationary varifolds, leaves nonetheless open the question whether one can have $N > 1$ on some connected component of Σ'.

This question is similar to the *multiplicity one conjecture* by Marques and Neves. In [75], the following upper bound for the Morse index of a minimal hypersurface with locally constant multiplicity is established: if

$$\Sigma = \sum_{j=1}^{\ell} n_j \Sigma_j$$

is a minimal hypersurface with locally constant multiplicity, given by a min-max with k parameters in the context of Almgren–Pitts theory, then

$$\operatorname{index}(\operatorname{spt}(\Sigma)) \leq k, \qquad \operatorname{spt}(\Sigma) := \bigsqcup_{j=1}^{\ell} \Sigma_j.$$

In other words, this is a bound for the Morse index of the hypersurface obtained by replacing all the multiplicities n_j with 1. In order for this estimate to give more information about Σ, or at least its unstable part, the authors make the following conjecture.

Conjecture 4.1.1 (Multiplicity one conjecture). *For generic metrics on M^m, with $3 \leq m \leq 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one.*

The importance of this conjecture has already been explained in Section 1.2, where we mentioned several results in the literature for other variational frameworks. In this chapter we establish the natural counterpart of this conjecture in our setting, namely for minimal surfaces produced by the viscous relaxation method.

Theorem 4.1.2. *We have $N \equiv 1$.*

Although we present only the closed case in this chapter, since the proof is local it applies also to the free boundary case.

We stress that this result holds in *arbitrary codimension* and *without any genericity assumption*. This should be seen as a multiplicity one statement from the perspective of the parametrization *domain*, in that localization in the domain (away from branch points) gives a genuine embedded minimal surface, but a priori it does not exclude multiple covers of the image surface globally. It seems to be optimal for a min-max approach involving parametrizations, rather than, for example, approaches involving level sets of functions, and it is sufficient to obtain an upper bound on the Morse index. This bound, detailed in [93], relies on having a branched immersion at our disposal, for which a good definition of Morse index is available.

We remark that, in view of earlier work from [92], namely the regularity theory when N is constant, Theorem 4.1.2 would imply by itself the regularity result of the previous chapter, at least for parametrized stationary varifolds arising from the min-max framework. However, the proof of Theorem 4.1.2 relies substantially on the regularity result itself, needed in several compactness arguments.

Most of the work is contained in Section 4.5. A detailed discussion of the strategy, together with an informal explanation of the technical statements contained in Section 4.5, is deferred to the beginning of that section.

Corollary 4.1.3. *If there is no bubbling or degeneration of the underlying conformal structure, we have strong $W^{1,2}$-convergence $\Phi_k \to \Phi_\infty = \Phi$. In general we have a bubble tree convergence.*

Theorem 4.1.2 and Corollary 4.1.3 allow to obtain meaningful Morse index bounds. Indeed, although Theorem 4.1.2 does not rule out the possibility of having a surface covered multiple times by Φ, a crucial advantage of having a parametrization at our disposal is that we have a reasonable definition of Morse index and nullity: they are defined with respect to the area functional and variations in $C_c^\infty(\Sigma' \setminus \{z_1, \dots, z_s\})$, the points $z_1, \dots, z_s$ being the branch points of the immersion Φ.[1]

[1] Although we are dealing with a weakly conformal map Φ, for which area and energy are the same, it is important to remark that the Morse indices for area and energy, denoted index_A and index_E respectively, should not be expected to agree. The relationship between the two is a subtle problem: in this direction, we mention the inequality $\mathrm{index}_E(\Psi) \leq \mathrm{index}_A(\Psi) \leq \mathrm{index}_E(\Psi) + r$ established in [34], for a branched minimal conformal immersion Ψ, where $r = r(g, b)$ depends on the genus g and the number b of branch points of Ψ.

For suitable min-maxes with k parameters, the natural expected inequalities would be

$$\operatorname{index}(\Phi) \le k \le \operatorname{index}(\Phi) + \operatorname{nullity}(\Phi).$$

An abstract framework to show upper bounds for the Morse index, dealing with general relaxed functionals on Banach manifolds, is developed in [77]. Combining Corollary 4.1.3 with the general result obtained in [77] and with [93], we reach the following conclusion (we refer the reader to [77] for the notion of *admissible family*).

Corollary 4.1.4. *Given an admissible family $\mathcal{F}$ of compact subsets of the set of immersions $\Sigma \to \mathcal{M}$, of dimension k, and calling*

$$\beta := \inf_{A \in \mathcal{F}} \max_{\Phi \in A} \operatorname{area}(\Phi)$$

the width *of $\mathcal{F}$, there exists a (possibly disconnected, branched) minimal immersion Φ of a closed surface S into $\mathcal{M}$ such that*

(i) $\operatorname{genus}(S) \le \operatorname{genus}(\Sigma)$,

(ii) $\beta = \operatorname{area}(\Phi)$,

(iii) $\operatorname{index}(\Phi) \le k$.

4.2 Notation

- We will assume, without loss of generality, that $\mathcal{M}$ is isometrically embedded in some Euclidean space $\mathbb{R}^Q$. Given $p \in \mathcal{M}$ and $\ell > 0$, we set $\mathcal{M}_{p,\ell} := \ell^{-1}(\mathcal{M} - p)$.

- In what follows, Π will always denote a 2-plane through the origin, which we identify with the corresponding orthogonal projection $\Pi : \mathbb{R}^Q \to \Pi$. We call $\Pi^\perp$ the orthogonal $(Q-2)$-subspace, identified with the corresponding orthogonal projection. Given 2-planes Π, Π', we denote by $\operatorname{dist}(\Pi, \Pi')$ an arbitrary distance on the Grassmannian $\operatorname{Gr}_2(\mathbb{R}^Q)$, e.g. the one induced by Plücker's embedding of $\operatorname{Gr}_2(\mathbb{R}^Q)$ into the projectivization of $\Lambda_2\mathbb{R}^Q$.

  The adjoint maps, which are just the inclusions $\Pi \hookrightarrow \mathbb{R}^Q$ and $\Pi^\perp \hookrightarrow \mathbb{R}^Q$, are denoted Π^* and $(\Pi^\perp)^*$, so that

  $$\mathrm{id}_{\mathbb{R}^Q} = \Pi^*\Pi + (\Pi^\perp)^*\Pi^\perp.$$

  Also, Π_0 is the canonical 2-plane, so that $\Pi_0 : \mathbb{R}^Q \to \mathbb{R}^2$ is the projection onto the first two coordinates, while $\Pi_0^\perp : \mathbb{R}^Q \to \mathbb{R}^{Q-2}$ is the projection onto the remaining $Q-2$.

- We call $B_r^2(x)$ the open ball of center x and radius r in the plane $\mathbb{C} = \mathbb{R}^2$, while $B_s^Q(p)$ will denote the open ball of center p and radius s in $\mathbb{R}^Q$. Given $p \in \Pi$, we call $B_s^\Pi(p)$ the two-dimensional ball with center p and radius s in Π, i.e. $B_s^\Pi(p) := B_s^Q(p) \cap \Pi$. When the center is not specified, it is always meant to be the origin.

- Given a function $\Psi \in W^{1,2}(B^2_r(x))$ and $0 < s \leq r$, the notation $\Psi\big|_{\partial B^2_s(x)}$ always refers to the trace of Ψ on the circle $\partial B^2_s(x)$.

- Given $K \geq 1$, we define the following set of Beltrami coefficients:
$$\mathcal{E}_K := \Big\{\mu \in L^\infty(\mathbb{C},\mathbb{C}) : \|\mu\|_{L^\infty} \leq \frac{K-1}{K+1}\Big\}.$$
We let $\mathcal{D}_K$ denote the set of K-quasiconformal homeomorphisms $\varphi : \mathbb{C} \to \mathbb{C}$ such that
$$\varphi(0) = 0, \qquad \min_{x \in \partial B^2_1} |\varphi(x)| = 1.$$
If $\varphi \in \mathcal{D}_K$, we have $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ and $\partial_{\bar{z}}\varphi = \mu \partial_z \varphi$ for some $\mu \in \mathcal{E}_K$, in the weak sense; we refer the reader to [56, Chapter 4] for the basic theory of K-quasiconformal homeomorphisms in the plane. Moreover, it is immediate to check that a linear map φ is in $\mathcal{D}_K$ if and only if $\varphi(e_1) = e'_1$ and $\varphi(e_2) = \lambda e'_2$, for suitable orthonormal bases (e_1, e_2), (e'_1, e'_2) inducing the canonical orientation and a suitable $1 \leq \lambda \leq K$.

- We define
$$D(K) := \sup\Big\{|\varphi(x)|; x \in \overline{B}^2_1, \varphi \in \mathcal{D}_K\Big\}, \quad s(K) := \inf\Big\{\big|\varphi^{-1}(y)\big|; |y| \geq \frac{1}{2}, \varphi \in \mathcal{D}_K\Big\},$$
so that $\varphi(\overline{B}^2_1) \subseteq \overline{B}^2_{D(K)}$ and $\varphi(\overline{B}^2_{s(K)}) \subseteq \overline{B}^2_{1/2}$ for all $\varphi \in \mathcal{D}_K$. The fact that $D(K) < \infty$ and $s(K) > 0$ is guaranteed by Corollary A.4. We also set
$$\eta(K) := \frac{1}{4} \inf\Big\{|\varphi(x)|; x \in \partial B^2_{s(K)^2}, \varphi \in \mathcal{D}_K\Big\} > 0.$$

- We let $\mathcal{D}^\Pi_K$ denote the set of maps having the form $\Pi^* \circ R \circ \varphi$, where $\varphi \in \mathcal{D}_K$ and $R : \mathbb{R}^2 \to \Pi$ is a linear isometry. Given $0 < \delta < 1$, we call $\mathcal{R}^\Pi_{K,\delta}$ the set of maps in $W^{1,2}(B^2_1, \mathbb{R}^Q)$ which are close to some $\psi \in \mathcal{D}^\Pi_K$ on the circles of radii $1, s(K), s(K)^2$, namely we set
$$\mathcal{R}^\Pi_{K,\delta} := \Big\{\Psi \in W^{1,2}(B^2_1, \mathbb{R}^Q) : \min_{\psi \in \mathcal{D}^\Pi_K} \max_{r \in \{1, s(K), s(K)^2\}} \Big\|\Psi\big|_{\partial B^2_r}(r\cdot) - \psi(r\cdot)\Big\|_{L^\infty(\partial B^2_1)} \leq \delta\Big\}.$$

- Given $\Psi \in C^1(\Omega, \mathbb{R}^Q)$, a ball $B^2_r(z) \subset\subset \Omega$ and a 2-plane Π, we define the *projected multiplicity* function
$$N(\Psi, B^2_r(z), \Pi) : \Pi \to \mathbb{N} \cup \{\infty\}, \qquad N(\Psi, B^2_r(z), \Pi)(p) := \#((\Pi \circ \Psi)^{-1}(p) \cap B^2_r(z))$$
and, given $p \in \Pi$ and $t > 0$, we also define the *macroscopic multiplicity*
$$n(\Psi, B^2_r(z), B^\Pi_t(p)) := \Big\lfloor \fint_{B^\Pi_t(p)} N(\Psi, B^2_r(z), \Pi) + \frac{1}{2} \Big\rfloor \in \mathbb{N}. \tag{4.2.1}$$
The mean appearing in (4.2.1) is finite by the area formula and $\lfloor\cdot\rfloor$ denotes the integer part. Note that, if the mean is close to an integer k, then the macroscopic multiplicity is precisely k. Note also that for any $p \in \mathbb{R}^Q$ we have
$$n(\Psi, B^2_r(z), B^\Pi_t(\Pi(p))) = n\Big(\frac{\Psi(z + r\cdot) - p}{t}, B^2_1, B^\Pi_1\Big).$$

4.3 Background on parametrized stationary varifolds

Let $\mathcal{M} \subset \mathbb{R}^Q$ be a (smooth, closed) embedded submanifold. Assume we have a smooth conformal immersion $\Phi : B_1^2 \to \mathcal{M}$, σ^5-critical for the functional

$$E_\sigma(\Phi) = \int_{B_1^2} \mathrm{vol}_\Phi + \sigma^4 \int_{B_1^2} |\mathbb{I}^\Phi|_{g_\Phi}^4 \, \mathrm{vol}_\Phi \,. \tag{4.3.1}$$

Here $g_\Phi := \Phi^* g_{\mathbb{R}^Q}$ and $\mathbb{I}^\Phi$ is the second fundamental form of Φ. The σ^5-criticality means that, for any infinitesimal variation w supported in B_1^2, we have

$$|dE_\sigma(\Phi)[w]| \le \sigma^5 \|w\|_\Phi,$$

with the last norm defined as in Section 2.2. In the sequel, for simplicity, we will just say that Φ is *almost critical*.

Assume that the following *entropy condition*

$$\sigma^4 \log(\sigma^{-1}) \int_{B_1^2} |\mathbb{I}^\Phi|^4 \, \mathrm{vol}_\Phi \le \varepsilon \int_{B_1^2} \mathrm{vol}_\Phi \tag{4.3.2}$$

holds for some $\varepsilon > 0$. Note that

$$g_\Phi = \frac{1}{2}|\nabla\Phi|^2 \delta, \qquad \int_{B_1^2} \mathrm{vol}_\Phi = \frac{1}{2}\int_{B_1^2} |\nabla\Phi|^2$$

by conformality of Φ.

Given any $0 < \ell < 1$ and $p \in \mathcal{M}$, recall that $\mathcal{M}_{p,\ell} = \ell^{-1}(\mathcal{M} - p)$. The rescaled map

$$\Psi : B_1^2 \to \mathcal{M}_{p,\ell}, \qquad \Psi := \ell^{-1}(\Phi - p)$$

is almost critical for the functional

$$\int_{B_1^2} \mathrm{vol}_\Psi + \tau^4 \int_{B_1^2} |\mathbb{I}^\Psi|^4 \, \mathrm{vol}_\Psi, \qquad \tau := \sigma/\ell, \tag{4.3.3}$$

as we saw in the proof of Theorem 2.5.3; since $\tau^4 \log(\tau^{-1}) \le \ell^{-4}\sigma^4 \log(\sigma^{-1})$, it satisfies

$$\tau^4 \log(\tau^{-1}) \int_{B_1^2} |\mathbb{I}^\Psi|^4 \, \mathrm{vol}_\Psi \le \varepsilon \int_{B_1^2} \mathrm{vol}_\Psi, \tag{4.3.4}$$

where now $\mathbb{I}^\Psi$ denotes the second fundamental form of Ψ in $\mathcal{M}_{p,\ell}$ and its norm is meant with respect to the induced metric g_Ψ.

In the sequel, we will establish many intermediate results on maps Ψ arising in this way, by means of compactness arguments. The starting point in these arguments is that, heuristically, if we have sequences Ψ_k, p_k, $\ell_k \to 0$, $\tau_k \to 0$ and $\varepsilon_k \to 0$, then by (4.3.3) and (4.3.4) Ψ_k should have a subsequential limit which is a parametrized stationary varifold in the tangent space $T_{p_\infty}\mathcal{M}$ (where p_∞ is a subsequential limit of the sequence p_k).

Since the theory from the previous chapters is crucially used in many intermediate steps towards the proof of Theorem 4.1.2, we give a precise statement that summarizes all the information we need to extract from those.

Theorem 4.3.1. *Assume that $\Psi_k \in C^2(\overline{B}^2_R, \mathcal{M}_{p_k,\ell_k})$ is a sequence of conformal immersions such that Ψ_k is almost critical for the functional (4.3.3) on the interior B^2_R (with τ_k, ℓ_k in place of τ, ℓ) and*

- $\Psi_k(\partial B^2_R) \to \Gamma_\infty$ *in the Hausdorff topology, for some $\Gamma_\infty \subseteq \mathbb{R}^Q$ compact,*
- $\frac{1}{2}\int_{B^2_R} |\nabla \Psi_k|^2 \le E,$
- $\tau_k^4 \log(\tau_k^{-1}) \int_{B^2_R} \left|\mathrm{I\!I}^{\Psi_k}\right|^4 \mathrm{vol}_{\Psi_k} \to 0,$
- $\ell_k, \tau_k \to 0.$

Then, up to subsequences, $\Psi_k \rightharpoonup \Psi_\infty$ in $W^{1,2}(B^2_R, \mathbb{R}^Q)$, for some Ψ_∞ which is continuous (in the interior) and satisfies the convex hull property, namely

$$\Psi_\infty(\overline{\omega}) \subseteq \mathrm{co}(\Psi_\infty(\partial\omega)) \qquad \text{for all } \omega \subset\subset B^2_R.$$

The image measures $(\Psi_k)_*\big(\frac{1}{2}|\nabla\Psi_k|^2\mathcal{L}^2\big)$ in $\mathbb{R}^Q$ form a tight sequence.

Given $\omega \subset\subset B^2_R$ with $\Psi_\infty(\overline{\omega}) \subseteq \mathbb{R}^Q \setminus \Gamma_\infty$, there exist a quasiconformal homeomorphism φ_∞ of $\mathbb{R}^2$ and a multiplicity $N_\infty \in L^\infty(\omega, \mathbb{Z}^+)$ such that the 2-varifolds induced by $\Psi_k\big|_\omega$ subsequentially converge on $\mathbb{R}^Q \setminus \Psi_\infty(\partial\omega)$ to a (local) parametrized stationary varifold

$$(\varphi_\infty(\omega), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$$

in the varifold sense, namely in duality with $C^0_c((\mathbb{R}^Q \setminus \Phi_\infty(\partial\omega)) \times \mathrm{Gr}_2(\mathbb{R}^Q))$. Also, on ω we have the convergence of Radon measures

$$\frac{1}{2}|\nabla\Psi_k|^2\mathcal{L}^2 \rightharpoonup N_\infty|\partial_1\Psi_\infty \wedge \partial_2\Psi_\infty|\mathcal{L}^2. \tag{4.3.5}$$

We have $N_\infty \le \frac{E}{\pi\,\mathrm{dist}(\Psi_\infty(\overline{\omega}),\Gamma_\infty)^2}$ a.e. and the distortion constant of φ_∞ is bounded by $\left(\frac{E}{\pi\,\mathrm{dist}(\Psi_\infty(\overline{\omega}),\Gamma_\infty)^2}\right)^2$.

Proof. The proof is essentially already contained in the previous chapters, so we just present the required adaptations.

Up to subsequences, we can assume that Ψ_k has a weak limit Ψ_∞ in $W^{1,2}(B^2_R, \mathbb{R}^Q)$ and that the varifolds $\mathbf{v}_k$ induced by Ψ_k converge to a varifold $\mathbf{v}_\infty$ in $\mathbb{R}^Q$.

The arguments used in Section 2.5 and in Section 3.2 show that Ψ_∞ has a continuous representative (on the interior B^2_R), satisfying the convex hull property. Also, from Theorem 2.3.4 and Proposition 2.5.6 we have that

$$\mathbf{v}_\infty \text{ is stationary in } U := \mathbb{R}^Q \setminus \Gamma_\infty \tag{4.3.6}$$

and is an integer rectifiable varifold. We claim that the measures $\|\mathbf{v}_k\| = (\Psi_k)_*\big(\frac{1}{2}|\nabla\Psi_k|^2\big)$ on $\mathbb{R}^Q$ form a tight sequence. If this were not true, up to subsequences we could find points

$q_k \in \mathbb{R}^Q$ with $|q_k| \to \infty$ and such that the argument of Proposition 2.4.2 applies (with q_k in place of q), on the region $\mathbb{R}^Q \setminus \Psi_k(\partial B^2_R)$. Hence,

$$\liminf_{k\to\infty} \|\mathbf{v}_k\|(B^Q_1(q_k)) > 0.$$

So the varifolds $\mathbf{v}_k - q_k$ converge subsequentially to a nontrivial varifold $\mathbf{v}'_\infty$ in $\mathbb{R}^Q$, stationary on $U' := \mathbb{R}^Q$: indeed, the proof of (4.3.6) can be repeated with $\Psi_k - q_k$ in place of Ψ_k (and U' in place of U), using the fact that, for all $s > 0$, the image of $\Psi_k|_{\partial B^2_R} - q_k$ is eventually disjoint from B^Q_s. Its total mass $\|\mathbf{v}'_\infty\|(\mathbb{R}^Q)$ must be bounded by $\liminf_{k\to\infty} \|\mathbf{v}_k\|(\mathbb{R}^Q) \le E$; however, the monotonicity formula implies that $\|\mathbf{v}'_\infty\|(\mathbb{R}^Q) = \infty$, a contradiction.

We also claim that the concentration set is empty in our setting. Indeed, by the tightness of the measures $\|\mathbf{v}_k\|$, a concentration point would produce, in the limit, a nontrivial stationary varifold in $\mathbb{R}^Q$. Again, its mass would be bounded by E, contradicting the monotonicity formula.

Now Theorem 2.5.2 and Theorem 2.5.3 give, up to further subsequences, the limit

$$\nu_k := \frac{1}{2}|\nabla \Psi_k|^2 \rightharpoonup \nu_\infty, \quad \text{with } \nu_\infty = m\,\mathcal{L}^2$$

in the sense of Radon measures (i.e. in duality with $C^0_c(B^2_R)$). The function $m(z) \ge 0$ equals $N_\infty(z)|\partial_1\Phi_\infty \wedge \partial_2\Phi_\infty|(z)$ a.e., for a positive integer $N_\infty(z)$ which is bounded by the density of $\mathbf{v}_\infty$ at $\Psi_\infty(z)$, whenever $\Psi_\infty(z) \in U$.

Let $\omega \subset\subset B^2_R$ be such that $\Psi_\infty(\overline{\omega}) \subseteq \mathbb{R}^Q \setminus \Gamma_\infty$. Defining $s := \operatorname{dist}(\Psi_\infty(\overline{\omega}), \Gamma_\infty) > 0$, note that

$$B^Q_s(q) \subseteq U \qquad \text{for all } q \in \Psi_\infty(\overline{\omega}).$$

Hence, by the monotonicity formula, the density of $\mathbf{v}_\infty$ at such points q is bounded by $\frac{E}{\pi s^2}$. This gives the upper bound for N_∞. As explained in detail in Section 3.4, there exists a quasiconformal homeomorphism φ_∞ of the plane, with distortion constant bounded by the square of the (essential) supremum of $N_\infty|_\omega$, such that $\Psi_\infty \circ \varphi_\infty^{-1}$ is weakly conformal on $\varphi_\infty(\omega)$. Finally, it is the main outcome of the second chapter that the 2-varifolds induced by $\Phi_k|_\omega$ converge to the (local) parametrized stationary varifold $(\varphi_\infty(\omega), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ (whose mass measure is bounded above by $\|\mathbf{v}_\infty\|$), in the complement of $\Psi_\infty(\partial\omega)$.[2] □

Theorem 4.3.2. *In the situation of Theorem 4.3.1, $\Psi_\infty \circ \varphi_\infty^{-1} : \varphi_\infty(\omega) \to \mathbb{R}^Q$ is harmonic. Also, if ω is connected and $\Psi_\infty|_\omega$ is not constant, N_∞ equals a constant integer (a.e.) on ω and $\Psi_\infty \circ \varphi_\infty^{-1}$ is a minimal branched immersion.*

Proof. This is a special case of Theorem 3.5.7. □

[2] The convergence actually holds on all of $\mathbb{R}^Q$ (or, more precisely, on $\mathbb{R}^Q \times \mathrm{Gr}_2(\mathbb{R}^Q)$) if $\nu_\infty(\partial\omega) = 0$.

4.4 Two lemmas on harmonic maps

Lemma 4.4.1. *Let $\gamma_k \in C^0(\partial B_1^2, \mathbb{R}^2)$ be a sequence of Jordan curves converging (in C^0) to a Jordan curve γ_∞ and let $f_k \in C^0(\partial B_1^2)$ be a sequence converging uniformly to a function f_∞. Let D_k be the domain bounded by γ_k, let $u_k \in C^0(\overline{D}_k)$ be the harmonic extension of $f_k \circ \gamma_k^{-1}$, and similarly define D_∞ and u_∞. Then $u_k \to u_\infty$ in $C^0_{loc}(D_\infty)$. Moreover, if $y_k \to y_\infty$ with $y_k \in \overline{D}_k$ and $y_\infty \in \overline{D}_\infty$, then $u_k(y_k) \to u_\infty(y_\infty)$.*

Note that such harmonic extensions exist and are unique, since by Carathéodory's theorem there exist homeomorphisms $\overline{B}_1^2 \to \overline{D}_k$ restricting to biholomorphisms $B_1^2 \to D_k$ (and similarly for D_∞), allowing one to reduce matters to the well-known existence and uniqueness of the harmonic extension on the unit disk.

Proof. Since the functions f_k are equibounded, from the maximum principle and interior estimates it follows that the functions u_k are equibounded in $C^2(\overline{\omega})$, for any $\omega \subset\subset D_\infty$, and hence by Ascoli–Arzelà theorem the convergence $u_k \to u_\infty$ in $C^0_{loc}(D_\infty)$ follows from the second claim.

It suffices to show that the second claim holds for a subsequence: once this is done, it can be obtained for the full sequence by a standard contradiction argument (given a sequence $y_k \to y_\infty$, if $u_k(y_k)$ does not converge to $u_\infty(y_\infty)$, we can find a subsequence such that it converges to a different value; then we reach a contradiction along a further subsequence where the second claim holds).

Up to removing a finite set of indices, we can suppose that there is a point p such that $p \in D_k$ for all $k \in \mathbb{N} \cup \{\infty\}$. By Carathéodory's theorem, we can find homeomorphisms $\upsilon_k : \overline{B}_1^2 \to \overline{D}_k$ restricting to biholomorphisms from B_1^2 to D_k, so that $\upsilon_k\big|_{\partial B_1^2} = \gamma_k \circ \beta_k$, for suitable homeomorphisms $\beta_k : \partial B_1^2 \to \partial B_1^2$ (for all $k \in \mathbb{N}$), and $\upsilon_k(0) = p$.

Since the maps υ_k and υ_k^{-1} are equibounded and harmonic, we can assume that

$$\upsilon_k \to \upsilon_\infty, \qquad \zeta_k := \upsilon_k^{-1} \to \zeta_\infty \tag{4.4.1}$$

in $C^\infty_{loc}(B_1^2)$ and $C^\infty_{loc}(D_\infty)$, respectively. Note that υ_∞ is a holomorphic map taking values into $\overline{D}_\infty$, whereas ζ_∞ is holomorphic and takes values into B_1^2 (by the maximum principle, since $\zeta_\infty(p) = 0$ and $|\zeta_\infty| \le 1$). So for any $w \in D_\infty$ the set $\{\zeta_k(w) \mid k \in \mathbb{N}\} \cup \{\zeta_\infty(w)\} \subset B_1^2$ is compact and we infer

$$\upsilon_\infty \circ \zeta_\infty(w) = \lim_{k\to\infty} \upsilon_k \circ \zeta_k(w) = w. \tag{4.4.2}$$

Hence υ_∞ is surjective and thus an open map. So $\upsilon_\infty(B_1^2) = D_\infty$ and, by [94, Theorem 10.43] (applied with $f := \upsilon_\infty - w$, $g := \upsilon_k - w$, for a fixed $w \in D_\infty$ and an arbitrary circle $\partial B_r^2 \subseteq B_1^2$ avoiding $\upsilon_\infty^{-1}(w)$, with k large enough), it is also injective. By Carathéodory's theorem, it extends continuously to a homeomorphism (still denoted υ_∞) from $\overline{B}_1^2$ to $\overline{D}_\infty$ and we have $\upsilon_\infty\big|_{\partial B_1^2} = \gamma_\infty \circ \beta_\infty$ for a suitable homeomorphism $\beta_\infty : \partial B_1^2 \to \partial B_1^2$.

Up to subsequences, applying Helly's selection principle to suitable lifts $\overline{\beta}_k : \mathbb{R} \to \mathbb{R}$, we can assume that $\beta_k \to \widetilde{\beta}_\infty$ everywhere, for some order-preserving $\widetilde{\beta}_\infty$.[3] On the other hand, since $\sup_k \int_{B_1^2} |v_k'|^2 = \sup_k \mathcal{L}^2(D_k)$ is finite, we have weak convergence $v_k \rightharpoonup v_\infty$ in $W^{1,2}(B_1^2)$ and thus weak convergence $\gamma_k \circ \beta_k \rightharpoonup \gamma_\infty \circ \beta_\infty$ in $L^2(\partial B_1^2)$. The everywhere convergence $\gamma_k \circ \beta_k \to \gamma_\infty \circ \widetilde{\beta}_\infty$ implies $\gamma_\infty \circ \beta_\infty = \gamma_\infty \circ \widetilde{\beta}_\infty$ a.e. and thus $\beta_\infty = \widetilde{\beta}_\infty$ a.e. In particular, β_∞ is also order-preserving. Since β_∞ is continuous and both maps are order-preserving, we conclude that $\beta_\infty = \widetilde{\beta}_\infty$ everywhere. Using again the continuity of β_∞, as well as the everywhere convergence of the order-preserving maps $\beta_k \to \beta_\infty$, we also get that $\beta_k \to \beta_\infty$ uniformly.

With v_k being the harmonic extension of $\gamma_k \circ \beta_k$ (for $k \in \mathbb{N} \cup \{\infty\}$), we conclude that $v_k \to v_\infty$ in $C^0(\overline{B}_1^2)$. Let $U_k \in C^0(\overline{B}_1^2)$ be the harmonic extension of $f_k \circ \beta_k$ and note that $U_k \to U_\infty$ in $C^0(\overline{B}_1^2)$. By conformal invariance, $u_k := U_k \circ v_k^{-1}$ is the harmonic extension of $f_k \circ \gamma_k^{-1}$ on D_k (for $k \in \mathbb{N} \cup \{\infty\}$).

Finally, we claim that in the situation of the second claim we have $v_k^{-1}(y_k) \to v_\infty^{-1}(y_\infty)$. This easily follows from the injectivity of v_∞: if we had $\left|v_k^{-1}(y_k) - v_\infty^{-1}(y_\infty)\right| \geq \varepsilon$ along some subsequence (for some $\varepsilon > 0$), we would have a subsequential limit point $x_\infty \in \overline{B}_1^2$ with $\left|x_\infty - v_\infty^{-1}(y_\infty)\right| \geq \varepsilon$ and $v_\infty(x_\infty) = \lim_{k\to\infty} y_k = y_\infty$, which is a contradiction. Hence,

$$u_k(y_k) = U_k(v_k^{-1}(y_k)) \to U_\infty(v_\infty^{-1}(y_\infty)) = u_\infty(y_\infty), \tag{4.4.3}$$

as desired. □

Remark 4.4.2. In the situation of Lemma 4.4.1, if $D_k \supseteq D$ for all $k \in \mathbb{N} \cup \{\infty\}$ then $u_k \to u_\infty$ uniformly on $\overline{D}$. Indeed, if this were not true, then we could find points $y_k \in \overline{D} \subseteq \overline{D}_k$ such that $|u_k(y_k) - u_\infty(y_k)| \geq \varepsilon$ (along a subsequence) for some $\varepsilon > 0$. Assuming without loss of generality $y_k \to y_\infty$, we would get

$$\liminf_{k\to\infty} |u_k(y_k) - u_\infty(y_\infty)| \geq \varepsilon$$

by continuity of u_∞ on $\overline{D}$. This would however contradict the last part of Lemma 4.4.1.

Lemma 4.4.3. *Given $K \geq 1$ and $s, \varepsilon > 0$, there exists a constant $0 < \delta_0 < \varepsilon$, depending only on Q, K, s, ε, with the following property: whenever*

- $\Psi \in W^{1,2} \cap C^0(\overline{B}_1^2, \mathbb{R}^Q)$ *has* $\left\|\Psi|_{\partial B_1^2} - \psi(s\cdot)|_{\partial B_1^2}\right\|_{C^0(\partial B_1^2)} \leq \delta_0$ *for some* $\psi \in \mathcal{D}_K^\Pi$,
- $\Psi \circ \varphi^{-1}$ *is harmonic and weakly conformal on* $\varphi(B_1^2)$, *where* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *is a* K*-quasiconformal homeomorphism,*[4]

then $\Pi \circ \Psi \circ \varphi^{-1}$ *is a diffeomorphism from* $\varphi(\overline{B}_{1/2}^2)$ *onto its image and*

$$\operatorname{dist}(\Pi, \Pi(x)) < \varepsilon, \qquad \Pi(x) := \text{2-plane spanned by } \nabla(\Psi \circ \varphi^{-1})(x), \tag{4.4.4}$$

for all $x \in \varphi(\overline{B}_{1/2}^2)$. *In particular,* $\Pi \circ \Psi$ *is injective on* $\overline{B}_{1/2}^2$.

[3] The map $\widetilde{\beta}_\infty$ could also be order-reversing: this happens precisely if β_k reverses the orientation along the subsequence. For simplicity, we assume β_k, $\widetilde{\beta}_\infty$ to be order-preserving (the other case being analogous).

[4] The maps ψ and φ are not necessarily related to each other.

Proof. Assume by contradiction that, for a sequence $\delta_k \downarrow 0$, there exist maps $\Psi_k : B_1^2 \to \mathbb{R}^Q$, planes Π_k and homeomorphisms $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^2$ such that the claim fails with $\delta_0 = \delta_k$. By Corollary A.4, up to subsequences we have $\Pi_k \to \Pi_\infty$ and $\Psi_k\big|_{\partial B_1^2} \to \Gamma$ (uniformly), where $\Gamma : \partial B_1^2 \to \mathbb{R}^Q$ is a Jordan arc in Π_∞.

We can assume that $\varphi_k \in \mathcal{D}_K$ (replacing φ_k with $\frac{\varphi_k - \varphi_k(0)}{\min_{\partial B_1^2} |\varphi_k - \varphi_k(0)|}$). By Corollary A.4, we can assume that $\varphi_k \to \varphi_\infty$ and $\varphi_k^{-1} \to \varphi_\infty^{-1}$ in $C^0_{loc}(\mathbb{R}^2)$, for some homeomorphism $\varphi_\infty : \mathbb{R}^2 \to \mathbb{R}^2$.

By harmonicity, up to subsequences we get $\Theta_k := \Psi_k \circ \varphi_k^{-1} \to \Theta_\infty$ in $C^2_{loc}(\varphi_\infty(B_1^2))$, for some $\Theta_\infty : \varphi_\infty(B_1^2) \to \mathbb{R}^Q$, so that Θ_∞ is weakly conformal and harmonic.

On the other hand, by Lemma 4.4.1 applied to the sequence of harmonic maps Θ_k on the Jordan domains $\varphi_k(B_1^2)$, Θ_∞ is the harmonic extension of $\Gamma \circ \varphi_\infty^{-1}$ and $\Psi_k \to \Theta_\infty \circ \varphi_\infty =: \Psi_\infty$ in $C^0(\overline{B}_1^2)$. By the maximum principle we have $\Pi_\infty^\perp \circ \Theta_\infty = 0$ and thus $\Pi_\infty \circ \Theta_\infty$ is either holomorphic or antiholomorphic on $\varphi_\infty(B_1^2)$ (once Π_∞ is identified with $\mathbb{C}$).

Now, given two Jordan domains $U, V \subset \mathbb{C}$, if a holomorphic map $h : U \to \mathbb{C}$ extends to a continuous map $h : \overline{U} \to \mathbb{C}$ mapping ∂U onto ∂V homeomorphically, then h maps U diffeomorphically onto V.[5] With $\Pi_\infty \circ \Theta_\infty\big|_{\partial\varphi_\infty(B_1^2)} = \Pi_\infty \circ \Gamma \circ \varphi_\infty^{-1}$ being a Jordan curve, we deduce that $\Pi_\infty \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(B_1^2)$ onto its image.

Fix now a compact neighborhood F of $\varphi_\infty(\overline{B}_{1/2}^2)$ in $\varphi_\infty(B_1^2)$, with smooth boundary. Since $\Theta_k \to \Theta_\infty$ in $C^1_{loc}(\varphi_\infty(B_1^2))$, we obtain that eventually $\Pi_k \circ \Theta_k$ is a diffeomorphism of F onto its image, with

$$\operatorname{dist}(\Pi_k, \Pi_k(x)) < \varepsilon, \qquad x \in F.$$

The fact that eventually $\varphi_k(\overline{B}_{1/2}^2) \subseteq F$ yields the desired contradiction. $\square$

4.5 Technical iteration lemmas

Informal discussion of the results

Since the intermediate results contained in this section have rather involved statements, with several different constants and thresholds appearing along the way, we find it helpful to provide an informal explanation of the meaning of these statements and constants, as well as a rough sketch of the underlying ideas in the proofs.

This section contains four important intermediate results, namely Lemmas 4.5.2, 4.5.3, 4.5.5 and 4.5.6, which all invoke Theorems 4.3.1 and 4.3.2 (except for Lemma 4.5.6) by means of a compactness-and-contradiction argument. All statements are about a conformal immersion $\Psi : \overline{B}_r^2(z) \to \mathcal{M}_{p,\ell}$, almost critical for the functional (4.3.3) (on the interior). For

[5] Indeed, $h\big|_U$ must be an open map, hence $h(\overline{U}) \setminus \partial V$ is closed and open in $\mathbb{C} \setminus \partial V$ and it follows that $h(U) = V$. We can find biholomorphisms $u : B_1^2 \to U$ and $v : B_1^2 \to V$ extending to homeomorphisms of the closures. The map $g := v^{-1} \circ h \circ u$ satisfies $g(B_1^2) \subseteq B_1^2$ and maps ∂B_1^2 to itself homeomorphically. Given $w \in B_1^2$, for $r < 1$ close enough to 1 the loop $g(re^{i\theta}) - w$ is homotopic to $g(e^{i\theta})$ in $\mathbb{C} \setminus \{0\}$, so the classical argument principle gives $\#g^{-1}(w) = 1$.

simplicity, in this discussion we assume $z = 0$ and $r = 1$. The first three statements require the following:

(i) a control of the shape of the images of three circles, selected by a distortion constant K; namely we require that

$$\Psi \in \mathcal{R}^{\Pi}_{K,\delta_0}$$

for some 2-plane Π and some small δ_0; recall from Section 4.2 that this means (assuming $\Pi = \mathbb{R}^2 \subseteq \mathbb{R}^Q$, up to rotations of $\mathbb{R}^Q$) that Ψ is C^0-close to a K-quasiconformal homeomorphism $\varphi \in \mathcal{D}_K$, $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \subseteq \mathbb{R}^Q$ on the three circles ∂B^2_1, $\partial B^2_{s(K)}$, $\partial B^2_{s(K)^2}$ (it would be far too restrictive to ask for C^0-closeness on all of B^2_1);

(ii) an upper bound E on the Dirichlet energy $\frac{1}{2}\int_{B^2_1} |\nabla\Psi|^2$;

(iii) an upper bound V on the area (divided by π) of the immersed surface $\Psi(B^2_1) \cap B^Q_1$, taking into account multiplicity; namely,

$$\int_{\Psi^{-1}(B^Q_1)} \mathrm{vol}_\Psi = \frac{1}{2}\int_{\Psi^{-1}(B^Q_1)} |\nabla\Psi|^2 \le V\pi,$$

where g_Ψ is the pullback of the Euclidean metric, which equals $\frac{1}{2}|\nabla\Psi|^2\delta$ by conformality; this upper bound will give a crucial improvement on the last conclusions of Theorem 4.3.1, as discussed below.

Also, in the same spirit as Theorem 4.3.1, these lemmas assume $\tau, \ell \ll 1$ and

$$\tau^4 \log(\tau^{-1}) \int_{B^2_1} |\mathbb{I}^\Psi|^4 \,\mathrm{vol}_\Psi \ll \int_{B^2_1} \mathrm{vol}_\Psi\,.$$

In Lemmas 4.5.3 and 4.5.5, the closeness in (i) is measured by a threshold δ_0 (which will be specified according to Lemma 4.4.3), whereas other closeness or smallness constraints will be measured by thresholds $\varepsilon_0, \varepsilon_0', \varepsilon_0''$ in Lemmas 4.5.2, 4.5.3, 4.5.5, respectively.

Observe that the hypotheses guarantee that $\Pi \circ \Psi$ maps $\partial B^2_{s(K)}$ to a subset of $B^{\Pi}_{1/2}$ and ∂B^2_1 to a subset of $\Pi \setminus B^{\Pi}_1$ (approximately); hence, $\Psi(\partial B^2_{s(K)})$ is far away from $\Psi(\partial B^2_1)$. Hence, when arguing by contradiction, we can apply the last part of Theorem 4.3.1 and obtain in the limit a parametrized stationary varifold close to $\Psi|_\omega$ (we will choose either $\omega := B^2_{s(K)}$ or the smaller domain $\omega := B^2_{s(K)^2}$). The reason to impose the geometric control on three circles, rather than two, is merely technical and is convenient for the proofs.

Lemma 4.5.2 says that the *projected multiplicity* $N(\Psi, B^2_{s(K)^2}, \Pi)$ (introduced in Section 4.2) issued by Ψ from the ball $B^2_{s(K)^2}$ has an average close to a positive integer k, on the ball $B^{\Pi}_{\eta(K)}$. It also asserts that this holds for 2-planes Π' close enough to Π. As a consequence, the corresponding *macroscopic multiplicity* will be precisely k.

Observe that the hypotheses guarantee that $\Pi \circ \Psi$ maps $B^2_{s(K)^2}$ approximately to a superset of $B^{\Pi}_{\eta(K)}$ (see Section 4.2 for the definition of these geometrical constants). Hence, arguing by contradiction, we obtain in the limit a (parametrized) stationary varifold which is close, in the varifold sense, to $\Psi(B^2_{s(K)^2})$. The constraints on Ψ force this limiting varifold to lie on a 2-plane, so by the constancy theorem it has constant integer multiplicity on $B^\Pi_{\eta(K)}$, giving a contradiction. Note that the volume constraint V is not used here.

As already mentioned in the introduction, we would now like to find a decreasing sequence of radii $r_0 := 1, \dots, r_k \approx \tau$, with r_j comparable to r_{j+1}, such that the maps $\Psi(r_j \cdot)$ satisfy the same assumptions (with different scales $\ell_0 := \ell, \dots, \ell_k$ in the target). The strategy to get Theorem 4.1.2 is then to show that the corresponding macroscopic multiplicities n_j *do not change* from one scale to the next one: $n_0 = n_1 = \cdots = n_k$. At the smallest scale, we will be able to say that the immersed surface $\ell_k^{-1}\Psi(B^2_{r_k})$ has small second fundamental form in L^4, implying a strong graphical control that allows one to conclude $n_k = 1$ and, thus, $n_0 = 1$. In the situation where we will apply this strategy (namely, in Section 4.6), upon careful selection of the center z, it will be easy to impose the "maximal" bounds

$$(\ell')^{-2}\int_{\Psi^{-1}(B^Q_{\ell'})} \mathrm{vol}_\Psi \le V\pi, \qquad \tau^4\log(\tau^{-1})\int_{B^2_{r'}} |\mathrm{I\!I}^\Psi|^4 \,\mathrm{vol}_\Psi \ll \int_{B^2_{r'}} \mathrm{vol}_\Psi,$$

for *all* $0 < \ell' < 1$ and $0 < r' < 1$, by means of covering arguments. However, we *cannot* a priori impose similar bounds on the Dirichlet energy and on the shape of the images of small circles (items (ii) and (i)). Note that if $(\ell')^{-1}\Psi(r'\cdot)$ satisfies (i), then we can bound the Dirichlet energy of this rescaled map on the ball $B^2_{s(K)}$, in terms of V, as Ψ maps $B^2_{s(K)r'}$ into $B^Q_{\ell'}$ (approximately). So (i) would give (ii) for free (on a smaller domain ball), with a uniform bound (depending on K, V) in place of E.

Lemma 4.5.3 is the main technical workhorse, and essentially says that we can circumvent this difficulty: namely, the hypotheses (i)–(iii) are still satisfied for a smaller radius $r' \le \frac{r}{2}$ in the domain, with a smaller scale $\ell'\ell \le \frac{\ell}{2}$ in the codomain. Note that the reference point p also changes; this will in principle destroy the maximal volume bound, but we can still recover (iii) in the new situation, exploiting the fact that the multiplicity is quantized in the limit (see the proof of Lemma 4.5.3 and Definition 4.5.4 for the details).

The idea of the proof of Lemma 4.5.3 is that, up to a quasiconformal homeomorphism φ, Ψ is close (in the weak $W^{1,2}$-topology) to a conformal harmonic map with small oscillation with respect to Π. Hence, by Lemma 4.4.3, it will be arbitrarily close to an affine injective conformal map L on smaller and smaller balls $B^2_{r'}$. If φ were the identity, given a (finite) collection of circles centered at 0 we would get the C^0-closeness of $\frac{\Psi(r'\cdot)}{r'}$ to L on all these circles, for some r' small, and we would be done.

The important observation now is that the distortion constant of φ can be bounded solely in terms of V: indeed, as in the proof of Theorem 4.3.1, $\Psi(B^2_1) \cap B^Q_1$ is close to a stationary varifold $\mathbf{v}$ (in B^Q_1), whose density on $B^Q_{1/2}$ is bounded in terms of V. Since $\Psi(B^2_{s(K)}) \subseteq B^Q_{1/2}$ (approximately), the upper bound on the distortion constant given by Theorem 4.3.1 can be improved to a constant $K'(V)$ depending *only* on V. Hence, we get (i) also for a smaller radius r' (with $K'(V)$ replacing K) and, as already said, this gives also (ii) with a bound $E'(V)$ in place of E. Our sequence of radii is now obtained by iterated application of Lemma 4.5.3 with parameters $K'(V)$, $E'(V)$, V.

Given constants K'', E'' and V, which will be chosen when applying these results in Section 4.6, we then fix $K_0 := \max\{K'(V), K''\}$ and $E_0 := \max\{E'(V), E''\}$, so that all the statements apply for all radii $r_0, r_1, \dots, r_k$.

Lemma 4.5.5 says that the macroscopic multiplicity does not change after applying Lemma 4.5.3, namely when replacing the domain and codomain scales r, ℓ with r', $\ell'\ell$ (and p, Π with p', Π'). Its proof uses Lemma 4.4.3 to claim that Ψ is approximately a graph over Π, and then applies the constancy theorem (in the limiting situation).

Finally, as it will be clear along the proof of Theorem 4.6.1, Lemma 4.5.6 concerns the behavior of a conformal immersion $\Phi : B_1^2 \to \mathcal{M}$ at a scale (comparable to) $\ell := \sigma$ in the codomain, when Φ satisfies (4.3.2). Assume that $\Phi(B_r^2)$ has diameter approximately ℓ^2, assume the smallness

$$\sigma^4 \int_{B_r^2} |\mathrm{I\!I}^\Phi|^4 \,\mathrm{vol}_\Phi \ll \int_{B_r^2} \mathrm{vol}_\Phi \tag{4.5.1}$$

and the bound $\int_{B_r^2} \mathrm{vol}_\Phi \le C\ell^2$. When dilating the codomain by a factor ℓ^{-1}, (4.5.1) becomes $\int_{B_r^2} |\mathrm{I\!I}^\Psi|^4 \,\mathrm{vol}_\Psi \ll 1$, for $\Psi := \ell^{-1}(\Phi - \Phi(0))$. As Ψ is conformal, we have $\Delta\Psi = 2H_\Psi e^{2\lambda}$ (where e^λ is the conformal factor). Thus, we get that $\Delta\Psi$ is small in L^4 provided we can obtain an upper bound on λ; once this is done, by Sobolev's embedding we obtain a C^1-control on Ψ, which implies that the macroscopic multiplicity is 1 at this scale.

In order to bound λ, we use a result by Hélein (belonging to a broad class of phenomena of integrability by compensation, whose study dates back to the discovery of Wente's inequality), guaranteeing the existence of an orthonormal frame $\{e_1(z), e_2(z)\}$ for the tangent space of the immersed surface Ψ, with a bound on $\|\nabla e_i\|_{L^2}$ depending only on the L^2-norm of the second fundamental form. Then we show that

$$-\Delta\lambda = \partial_1 e_1 \cdot \partial_2 e_2 - \partial_2 e_1 \cdot \partial_1 e_2$$

and we compare λ with the solution μ to the same equation, with zero boundary conditions on a ball. A pointwise bound for μ now follows from Wente's inequality, from which one easily deduces the desired upper bound for λ. Although not necessary, we will also show how to obtain a pointwise lower bound on λ in this situation.

While reading Sections 4.5 and 4.6, it can be useful to refer to the following diagrams, illustrating how the constants depend on each other:

$$
\begin{tikzcd}
K \arrow[dd] \arrow[dr] \arrow[drrr] & & V \arrow[dl] \arrow[dr] & E \arrow[dll] \arrow[d] \\
& \varepsilon_0 \arrow[dl] & & \varepsilon_0' \\
\delta_0 \arrow[urrr] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & V \arrow[dl] \arrow[dr] \arrow[ddd] & & \\
K'' \arrow[dr] & K'(V) \arrow[d] & & E'(V) \arrow[d] & E'' \arrow[dl] \\
& K_0 \arrow[dr] & & E_0 \arrow[dl] & \\
& \varepsilon_0' \arrow[r] & \varepsilon_0'' & &
\end{tikzcd}
$$

(an arrow $A \to B$ means that B depends on A).

Rigorous statements and proofs

We now make the above discussion rigorous. In a first reading, it can be helpful to pretend that all quasiconformal homeomorphisms appearing in the proofs coincide with the identity.

Definition 4.5.1. Given $V > 0$ with $V = \lfloor V \rfloor + \frac{1}{2}$, we define the constants

$$K'(V) := (16V)^2, \qquad E'(V) := 2\pi K'(V) D(K'(V))^2.$$

In the sequel, it will be convenient to assume always that $V \in \mathbb{N} + \frac{1}{2}$, so that $V = \lfloor V \rfloor + \frac{1}{2}$.

Lemma 4.5.2. *There exists $0 < \varepsilon_0 < \eta(K)$, depending on $E, V > 0$, $K \geq 1$ and $\mathcal{M}$, such that whenever $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell})$ is a conformal immersion, almost critical for the functional (4.3.3) on $B_r^2(z)$, and Π, Π' are 2-planes satisfying*

- $\Psi(z + r\cdot) \in \mathcal{R}^{\Pi}_{K,\varepsilon_0}$,
- $\displaystyle \frac{1}{2}\int_{B_r^2(z)} |\nabla \Psi|^2 \leq E$,
- $\displaystyle \int_{\Psi^{-1}(B_1^Q)} \mathrm{vol}_\Psi = \frac{1}{2}\int_{\Psi^{-1}(B_1^Q)} |\nabla \Psi|^2 \leq V\pi$,
- $\displaystyle \tau^4 \log(\tau^{-1}) \int_{B_r^2(z)} \left|\mathbb{I}^\Psi\right|^4 \mathrm{vol}_\Psi \leq \varepsilon_0$ *for some* $0 < \tau \leq \varepsilon_0$,
- $\mathrm{dist}(\Pi, \Pi') \leq \varepsilon_0$ *and* $0 < \ell \leq \varepsilon_0$,

then the projected multiplicity $N(\Psi, B^2_{s(K)^2 r}(z), \Pi)$ satisfies

$$\mathrm{dist}\Big(\fint_{B^{\Pi}_{\eta(K)}} N(\Psi, B^2_{s(K)^2 r}(z), \Pi),\, \mathbb{Z}^+ \Big) < \frac{1}{8}, \tag{4.5.2}$$

$$\left| \fint_{B^{\Pi}_{\eta(K)}} N(\Psi, B^2_{s(K)^2 r}(z), \Pi) - \fint_{B^{\Pi'}_{\eta(K)}} N(\Psi, B^2_{s(K)^2 r}(z), \Pi') \right| < \frac{1}{8}, \tag{4.5.3}$$

where $\mathbb{Z}^+$ is the set of positive integers.

Proof. We can assume $z = 0$ and $r = 1$. Suppose by contradiction that there exist sequences $\varepsilon_k \downarrow 0$, τ_k, ℓ_k, points p_k, maps Ψ_k and planes Π_k, Π'_k making the claim false for $\varepsilon_0 = \varepsilon_k$. Up to subsequences, we can assume that $\Pi_k, \Pi'_k \to \Pi_\infty$, that Ψ_k has a weak limit Ψ_∞ in $W^{1,2}(B_1^2, \mathbb{R}^Q)$, with traces $\Psi_\infty\big|_{\partial B_s^2}(s\cdot) = \psi(s\cdot)$ for some $\psi \in \mathcal{D}_K^{\Pi_\infty}$ and all $s \in \{1, s(K), s(K)^2\}$ (thanks to Corollary A.4), and that the varifolds $\mathbf{v}_k$ induced by Ψ_k converge to a varifold $\mathbf{v}_\infty$ in $\mathbb{R}^Q$.

We now invoke Theorem 4.3.1. Recalling the definition of $\mathcal{D}_K^{\Pi_\infty}$ and $s(K)$ from Section 4.2, the convex hull property satisfied by Ψ_∞ gives

$$\Psi_\infty(\overline{B}^2_{s(K)}) \subseteq \operatorname{co}\left(\Psi_\infty(\partial B^2_{s(K)})\right) = \operatorname{co}\left(\psi(\partial B^2_{s(K)})\right) \subseteq \overline{B}^Q_{1/2}, \tag{4.5.4}$$

so that, with $\Gamma_\infty = \Psi_\infty(\partial B_1^2) = \psi(\partial B_1^2)$ being disjoint from B_1^Q (as $|\psi(x)| \geq 1$ for $x \in \partial B_1^2$, by definition of $\mathcal{D}_K$),

$$\operatorname{dist}(\Psi_\infty(x), \Gamma_\infty) \geq \frac{1}{4} \quad \text{for } x \in \overline{B}^2_{s(K)}. \tag{4.5.5}$$

Theorem 4.3.1 gives the varifold convergence $\mathbf{v}'_k \rightharpoonup \mathbf{v}'_\infty$ and $\mathbf{v}''_k \rightharpoonup \mathbf{v}''_\infty$ as $k \to \infty$, as well as the tightness of the sequences of mass measures $\|\mathbf{v}'_k\|$ and $\|\mathbf{v}''_k\|$, where $\mathbf{v}'_k$ and $\mathbf{v}''_k$ are the varifolds issued by $\Psi_k\big|_{B^2_{s(K)}}$ and $\Psi_k\big|_{B^2_{s(K)^2}}$ respectively, while $\mathbf{v}'_\infty$ and $\mathbf{v}''_\infty$ are the ones issued by $(\varphi_\infty(B^2_{s(K)}), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$ and $(\varphi_\infty(B^2_{s(K)^2}), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$.[6]

Although not needed in the present proof, let us remark the following improvement on the last statement in Theorem 4.3.1, which will be used in the proof of Lemma 4.5.3: we have

$$N_\infty \leq \frac{V\pi}{\pi\left(\frac{1}{4}\right)^2} = 16V$$

and the distortion constant of φ_∞ is bounded by $K'(V) = (16V)^2$. Indeed, since $\mathbf{v}_\infty$ is stationary in B_1^Q and $\|\mathbf{v}_\infty\|(B_1^Q) \leq V\pi$, by the monotonicity formula its density is bounded by $\frac{V\pi}{\pi(1-|p|)^2}$ at any $p \in B_1^Q$. In particular, (4.5.4) gives an upper bound $4V$ at points in $\Psi_\infty(B^2_{s(K)})$, which implies our claim.

The support of $\mathbf{v}''_\infty$ is contained in the plane Π_∞, by the convex hull property enjoyed by Ψ_∞ and the fact that Ψ_∞ maps $\partial B^2_{s(K)^2}$ to Π_∞. Since $\Psi_\infty(\partial B^2_{s(K)^2})$ does not intersect $B^{\Pi_\infty}_{\eta(K)}$, the varifold $\mathbf{v}''_\infty$ is stationary here and thus, by the constancy theorem [98, Theorem 41.1], it has a constant density $\nu \in \mathbb{N}$. We must have $\nu > 0$, since $\Psi_\infty(B^2_{s(K)^2})$ is a superset of $B^{\Pi_\infty}_{\eta(K)}$ by Lemma A.1 (applied to $\eta(K)^{-1}\Psi_\infty(s(K)^2\cdot)$). The area formula and the tightness of $\|\mathbf{v}''_k\|$ then give

$$\fint_{B^{\Pi_k}_{\eta(K)}} N(\Psi_k, B^2_{s(K)^2}, \Pi_k) = \frac{\|(\Pi_k)_*\mathbf{v}''_k\|(B^{\Pi_k}_{\eta(K)})}{\pi\eta(K)^2} \to \frac{\|(\Pi_\infty)_*\mathbf{v}''_\infty\|(B^{\Pi_\infty}_{\eta(K)})}{\pi\eta(K)^2} = \nu.$$

Similarly, $\fint_{B^{\Pi'_k}_{\eta(K)}} N(\Psi_k, B^2_{s(K)^2}, \Pi'_k) \to \nu$ as $k \to \infty$. Hence the claim is eventually true, yielding the desired contradiction. □

[6] The fact that one can choose the same multiplicity N_∞ and the same quasiconformal homeomorphism $\varphi_\infty : \mathbb{R}^2 \to \mathbb{R}^2$ for both domains is evident from the proof of Theorem 4.3.1.

Assume $K \geq K'(V)$. We now specify $\delta_0 < \frac{1}{8}$ so that Lemma 4.4.3 applies, with $\varepsilon := \varepsilon_0$ and $s := s(K)$. Note that $\delta_0 < \varepsilon_0 < \eta(K)$ and that ε_0 and δ_0 still depend on V, K and E.

Lemma 4.5.3. *Given $E > 0$ and $K \geq 1$ there exists a constant $0 < \varepsilon_0' < \varepsilon_0$ (depending on $E, V, K, \mathcal{M}$) with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell})$ is almost critical for the functional (4.3.3) (on the interior) and satisfies*

- $\Psi(z + r\cdot) \in \mathcal{R}^{\Pi}_{K,\delta_0}$,
- $\displaystyle\frac{1}{2}\int_{B_r^2(z)} |\nabla\Psi|^2 \leq E$,
- $\displaystyle\frac{1}{\pi}\int_{\Psi^{-1}(B_1^Q)} \mathrm{vol}_\Psi,\ \frac{1}{\pi\eta(K)^2}\int_{\Psi^{-1}(B^Q_{\eta(K)})} \mathrm{vol}_\Psi \leq V$,
- $\displaystyle\tau^4\log(\tau^{-1})\int_{B_r^2(z)} \left|\mathrm{I\!I}^\Psi\right|^4 \mathrm{vol}_\Psi \leq \varepsilon_0'$ *for some* $0 < \tau \leq \varepsilon_0'$,
- $0 < \ell \leq \varepsilon_0'$,

then there exist a new point $p' \in \mathcal{M}_{p,\ell}$, new scales r', ℓ' and a new 2-plane Π' with

- $\varepsilon_0' r < r' < s(K)r$,
- $\varepsilon_0' < \ell' < \dfrac{1}{2}$,
- $\mathrm{dist}(\Pi, \Pi') < \varepsilon_0$,
- $(\ell')^{-1}(\Psi(z + r'\cdot) - p') \in \mathcal{R}^{\Pi'}_{K'(V),\delta_0}$,
- $\displaystyle\frac{1}{2}\int_{B^2_{r'}(z)} \left|\nabla\Psi'\right|^2 < E'(V)$, *for $\Psi' := (\ell')^{-1}(\Psi - p')$ (defined on $B^2_{r'}(z)$),*
- $\displaystyle\frac{1}{\pi}\int_{(\Psi')^{-1}(B_1^Q)} \mathrm{vol}_{\Psi'},\ \frac{1}{\pi\eta(K)^2}\int_{(\Psi')^{-1}(B^Q_{\eta(K)})} \mathrm{vol}_{\Psi'} < \left\lfloor\left(\frac{\eta(K)}{\eta(K) - \varepsilon_0}\right)^2 V\right\rfloor + \frac{1}{2}$.

Proof. We can assume $z = 0$ and $r = 1$. By contradiction, suppose that there is a sequence $\varepsilon_k \downarrow 0$ such that the claim fails (with $\varepsilon_0' = \varepsilon_k$) for all radii $\varepsilon_k < r' < s(K)$, for some Ψ_k and Π_k satisfying all the hypotheses.

Assuming also that $\Pi_k \to \Pi_\infty$ and $p_k \to p_\infty$, by Corollary A.4 we still have $\Psi_\infty \in \mathcal{R}^{\Pi_\infty}_{K,\delta_0}$: indeed, if $\psi_k \in \mathcal{D}^{\Pi_k}_K$ are such that $\left\|\Psi_k\big|_{\partial B_s^2}(s\cdot) - \psi_k(s\cdot)\right\|_{L^\infty(\partial B_1^2)} \leq \delta_0$ for $s = 1, s(K), s(K)^2$, then there exists $\psi_\infty \in \mathcal{D}^{\Pi_\infty}_K$ such that $\psi_k \to \psi_\infty$ (up to subsequences), uniformly on the three circles; for any bounded measurable function $\chi : \partial B_1^2 \to \mathbb{R}^Q$ with $\|\chi\|_{L^1} \leq 1$, weak convergence of the traces $\Psi_k\big|_{\partial B_s^2} \rightharpoonup \Psi_\infty\big|_{\partial B_s^2}$ in L^2 gives

$$\int_{\partial B_1^2} \chi \cdot (\Psi_\infty\big|_{\partial B_s^2}(s\cdot) - \psi_\infty(s\cdot)) = \lim_{k\to\infty}\int_{\partial B_1^2} \chi \cdot (\Psi_k\big|_{\partial B_s^2}(s\cdot) - \psi_k(s\cdot)) \leq \delta_0$$

for $s = 1, s(K), s(K)^2$; thus, with χ being arbitrary, the desired inequality holds also for $k = \infty$. Also, $\Psi_k(\partial B_1^2)$ converges to a compact set Γ_∞ included in the closed δ_0-neighborhood of $\psi_\infty(\partial B_1^2)$ (up to subsequences).

As observed in the proof of Lemma 4.5.2, up to subsequences we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in $\mathbb{R}^Q$, where $\Theta_\infty = \Psi_\infty \circ \varphi_\infty^{-1}$ and $\Omega_\infty = \varphi_\infty(B_{s(K)}^2)$ for a suitable $K'(V)$-quasiconformal homeomorphism φ_∞ of the plane (since $\operatorname{dist}(\Psi_\infty(B_{s(K)}^2), \Gamma_\infty) \geq \frac{1}{4}$). By Theorem 4.3.2, Θ_∞ is harmonic. Also, it takes values in the tangent space T at p_∞ (translated to the origin).

We can assume that $\varphi_\infty(0) = 0$. By definition of δ_0 and Lemma 4.4.3, applied to $\Psi_\infty(s(K)\cdot)$ and $\varphi_\infty(s(K)\cdot)$, Θ_∞ is a diffeomorphism from $\varphi_\infty(\overline{B}_{s(K)/2}^2)$ onto its image and the differential $\nabla\Theta_\infty(0)$ is a conformal linear map of full rank, spanning a plane Π' with $\operatorname{dist}(\Pi_\infty, \Pi') < \varepsilon_0$.

The varifolds $\mathbf{v}_k$ induced by $\Psi_k\big|_{B_{s(K)^2}^2}$ converge to $\mathbf{v}_\infty$, induced by $(\varphi_\infty(B_{s(K)^2}^2), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$. Using Lemma A.1, applied to $\eta(K)^{-1}\Pi_\infty \circ \Psi_\infty(s(K)^2\cdot)$, and the fact that $\delta_0 < \eta(K)$, we deduce the existence of a point $y \in B_{s(K)^2}^2$ such that $\Pi_\infty \circ \Psi_\infty(y) = 0$. By the convex hull property enjoyed by Ψ_∞, it follows that

$$|\Psi_\infty(y)| = \left|\Pi_\infty^\perp \circ \Psi_\infty(y)\right| \leq \delta_0,$$

as $\Psi_\infty(\partial B_{s(K)^2}^2) \subseteq \left\{p : \left|\Pi_\infty^\perp(p)\right| \leq \delta_0\right\}$. Since $\|\mathbf{v}_\infty\|(B_{\eta(K)}^Q) \leq V\pi\eta(K)^2$, the stationarity of $\mathbf{v}_\infty$ on $B_{\eta(K)}^Q$ implies that its density at $\Psi_\infty(y)$ is at most

$$\frac{V\pi\eta(K)^2}{\pi(\eta(K) - \delta_0)^2} \leq \left(\frac{\eta(K)}{\eta(K) - \varepsilon_0}\right)^2 V. \tag{4.5.6}$$

With $\mathbf{v}_\infty$ being stationary in the embedded surface $\Theta_\infty(\varphi_\infty(B_{s(K)^2}^2))$, the constancy theorem gives that its density θ is a constant integer here. This also follows from the fact that N_∞ is constant, by Theorem 4.3.2. Thus we have

$$\|\mathbf{v}_\infty\|(\overline{B}_t^Q(p'_\infty)) < \left(\left\lfloor\left(\frac{\eta(K)}{\eta(K) - \varepsilon_0}\right)^2 V\right\rfloor + \frac{1}{2}\right)\pi t^2, \qquad p'_\infty := \Theta_\infty(0) \in T, \tag{4.5.7}$$

for all $t > 0$ small enough. Fix now any $r' < s(K)$ such that we have the strong convergence $\Psi_k(r'\cdot) \to \Psi_\infty(r'\cdot)$ in $C^0(\partial B_1^2 \cup \partial B_{s(K)}^2 \cup \partial B_{s(K)^2}^2)$ along a subsequence.[7] Note that $\lambda^{-1}\varphi_\infty(r'\cdot) \in \mathcal{D}_{K'(V)}$, where $\lambda := \min_{|x|=r'} |\varphi_\infty(x)|$. Also, the fact that $\Psi_\infty = \Theta_\infty \circ \varphi_\infty$ and the smoothness of Θ_∞ give

$$\left|\Psi_\infty(r'x) - \Psi_\infty(0) - \left\langle\nabla\Theta_\infty(0), \varphi_\infty(r'x)\right\rangle\right| < \frac{\delta_0|\nabla\Theta_\infty(0)|}{2\sqrt{2}D(K'(V))}\left|\varphi_\infty(r'x)\right| \leq \frac{\delta_0\ell'}{2} \tag{4.5.8}$$

[7] This can be obtained by applying e.g. Lemma A.5 to the weakly converging $\mathbb{R}^{3Q}$-valued maps

$$(\Psi_k, \Psi_k(s(K)\cdot), \Psi_k(s(K)^2\cdot)) \rightharpoonup (\Psi_\infty, \Psi_\infty(s(K)\cdot), \Psi_\infty(s(K)^2\cdot)).$$

if r' is chosen small enough, where $\ell' := \frac{|\nabla\Theta_\infty(0)|}{\sqrt{2}}\lambda$ and $x \in \overline{B}^2_1$. This implies

$$(\ell')^{-1}(\Psi_\infty(r'\cdot) - p'_\infty) \in \mathcal{R}^{\Pi'}_{K'(V),\delta_0/2}$$

by conformality of $\nabla\Theta_\infty(0)$. Shrinking r', we can also ensure that $\ell' < \frac{1}{2}$, as well as

$$\begin{aligned}\int_{B^2_{r'}} N_\infty|\partial_1\Psi_\infty \wedge \partial_2\Psi_\infty| &\le \frac{K'(V)}{2}\int_{B^2_{D(K'(V))\lambda}} |\nabla\Theta_\infty|^2 \\ &< K'(V)(D(K'(V))\lambda)^2\pi|\nabla\Theta_\infty(0)|^2.\end{aligned} \tag{4.5.9}$$

Calling $\mathbf{v}'_\infty$ the varifold induced by $(\varphi_\infty(B^2_{r'}), (\ell')^{-1}(\Theta_\infty - p'_\infty), N_\infty \circ \varphi_\infty^{-1})$, in view of (4.5.7) we can even guarantee that

$$\frac{\|\mathbf{v}'_\infty\|(\overline{B}^Q_1)}{\pi}, \frac{\|\mathbf{v}'_\infty\|(\overline{B}^Q_{\eta(K)})}{\pi\eta(K)^2} < \left\lfloor\left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V\right\rfloor + \frac{1}{2}.$$

Calling p'_k the closest point to p'_∞ in $\mathcal{M}_{p_k,\ell_k}$ (eventually defined and converging to p'_∞, since $\mathcal{M}_{p_k,\ell_k} \to T$), thanks to (4.5.8) and $\lambda^{-1}\varphi_\infty(r'\cdot) \in \mathcal{D}_{K'(V)}$, eventually we have

$$(\ell')^{-1}(\Psi_k(r'\cdot) - p'_k) \in \mathcal{R}^{\Pi'}_{K'(V),\delta_0}.$$

Moreover, (4.5.9) and (4.3.5) give

$$\frac{1}{2}\int_{B^2_{r'}(z)} |\nabla\Psi_k|^2 \to \int_{B^2_{r'}(z)} N_\infty|\partial_1\Psi_\infty \wedge \partial_2\Psi_\infty| < (\ell')^2E'(V).$$

From the convergence of the varifolds $\mathbf{v}'_k$ induced by $(\ell')^{-1}(\Psi_k - p'_k)\big|_{B^2_{r'}}$ to $\mathbf{v}'_\infty$ we get

$$\limsup_{k\to\infty}\frac{\|\mathbf{v}'_k\|(B^Q_1)}{\pi}, \limsup_{k\to\infty}\frac{\|\mathbf{v}'_k\|(B^Q_{\eta(K)})}{\pi\eta(K)^2} < \left\lfloor\left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V\right\rfloor + \frac{1}{2}.$$

So eventually $(\ell')^{-1}(\Psi_k(r'\cdot) - p'_k)$ satisfies all the conclusions. This yields the desired contradiction. □

Definition 4.5.4. Given constants $K'' \ge 1$ and $E'' \ge 1$, we define $K_0 := \max\{K'(V), K''\}$ and $E_0 := \max\{E'(V), E''\}$. We also let $s_0 := s(K_0)$ and $\eta_0 := \eta(K_0)$.

We fix ε_0 (and thus δ_0) and ε'_0 so that Lemmas 4.5.2 and 4.5.3 apply with $K := K_0$, $E := E_0$. Since ε_0 depends on V, we can assume that it is chosen so small that

$$\left\lfloor\left(\frac{\eta_0}{\eta_0-\varepsilon_0}\right)^2 V\right\rfloor + \frac{1}{2} = \lfloor V\rfloor + \frac{1}{2} = V. \tag{4.5.10}$$

This makes the last conclusion of Lemma 4.5.3 match one of the hypotheses, making it possible to iterate that result. On the other hand, the constants V, K'', E'' (upon which all the aforementioned constants depend) will be fixed only in Section 4.6.

Lemma 4.5.5. *There exists a constant $0 < \varepsilon_0'' < \varepsilon_0'$ with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell})$ satisfies the hypotheses of the previous lemma (with $\varepsilon_0'', E_0, K_0$ in place of ε_0', E, K), then the new point p', the new radius r' and the new scale ℓ' provided by Lemma 4.5.3 satisfy*

$$n(\Psi, B^2_{s_0^2 r}(z), B^{\Pi}_{\eta_0}) = n(\Psi - p', B^2_{s_0^2 r'}(z), B^{\Pi}_{\eta_0 \ell'}) = n(\Psi - p', B^2_{s_0^2 r'}(z), B^{\Pi'}_{\eta_0 \ell'}). \tag{4.5.11}$$

Proof. The second equality in (4.5.11) follows immediately from Lemma 4.5.2 (applied with $(\ell')^{-1}(\Psi - p')$ on $B^2_{r'}$), which gives

$$n(\Psi - p', B^2_{s_0^2 r'}(z), B^{\Pi}_{\eta_0 \ell'}) = n(\Psi - p', B^2_{s_0^2 r'}(z), B^{\Pi'}_{\eta_0 \ell'})$$

since $\operatorname{dist}(\Pi', \Pi) < \varepsilon_0$.

Assume again $z = 0$, $r = 1$ and, by contradiction, that the first equality in (4.5.11) fails, so that we have again two sequences $\varepsilon_k \downarrow 0$ and Ψ_k. We can assume that $\Pi_k \to \Pi_\infty$, $p_k' \to p_\infty'$, $\ell_k' \to \ell_\infty'$ and $r_k' \to r_\infty'$, with $p_\infty' \in \mathcal{M}$, $\varepsilon_0' \le \ell_\infty' \le \frac{1}{2}$ and $\varepsilon_0' \le r_\infty' \le s_0$. Moreover, as in the proof of Lemma 4.5.3, up to further subsequences we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in $\mathbb{R}^Q$, with $\Omega_\infty = \varphi_\infty(B^2_{s_0})$. From Theorem 4.3.2 we know that Θ_∞ is harmonic and $N_\infty = \nu$ is constant, so Lemma 4.4.3 gives that $\Pi_\infty \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(\overline{B}^2_{s_0/2})$ onto its image.

Calling $\mathbf{v}_k$ the varifold issued by $\Psi_k\big|_{B^2_{s_0^2}}$ and $\mathbf{v}_\infty$ the one issued by $(\varphi_\infty(B^2_{s_0^2}), \Theta_\infty, \nu)$, we have the varifold convergence $\mathbf{v}_k \rightharpoonup \mathbf{v}_\infty$ as $k \to \infty$. The area formula gives

$$\fint_{B^{\Pi_k}_{\eta_0}} N(\Psi_k, B^2_{s_0^2}, \Pi_k) = \frac{\|(\Pi_k)_* \mathbf{v}_k\|(B^{\Pi_k}_{\eta_0})}{\pi \eta_0^2} \to \frac{\|(\Pi_\infty)_* \mathbf{v}_\infty\|(B^{\Pi_\infty}_{\eta_0})}{\pi \eta_0^2} = \nu,$$

since $(\Pi_\infty)_* \mathbf{v}_\infty$ equals an open superset of $B^{\Pi_\infty}_{\eta_0}$ in Π_∞ (by Lemma A.1), equipped with the constant integer multiplicity ν. Hence, $n(\Psi_k, B^2_{s_0^2}, B^{\Pi_k}_{\eta_0}) = \nu$ eventually.

Similarly, calling $\mathbf{v}_k'$ the varifold induced by $\Psi_k\big|_{B^2_{s_0^2 r_k'}}$ and $\mathbf{v}_\infty'$ the varifold induced by $(\varphi_\infty(B^2_{s_0^2 r_\infty'}), \Theta_\infty, \nu)$, we have $\mathbf{v}_k' \rightharpoonup \mathbf{v}_\infty'$ as $k \to \infty$, as is readily seen by approximating with domains which do not vary along the sequence. Since $(\ell_\infty')^{-1}(\Psi_\infty(r_\infty' \cdot) - p_\infty') \in \mathcal{R}^{\Pi_\infty}_{K_0, \delta_0}$, again $(\Pi_\infty)_* \mathbf{v}_\infty'$ equals a superset of $B^{\Pi_\infty}_{\eta_0 \ell_\infty'}$ in Π_∞, with constant density ν. This gives again

$$\fint_{B^{\Pi_k}_{\eta_0 \ell_k'}(q_k)} N(\Psi_k, B^2_{s_0^2 r_k'}, \Pi_k) = \frac{\|(\Pi_k)_* \mathbf{v}_k'\|(B^{\Pi_k}_{\eta_0 \ell_k'}(q_k))}{\pi \eta_0^2 (\ell_k')^2} \to \frac{\|(\Pi_\infty)_* \mathbf{v}_\infty'\|(B^{\Pi_\infty}_{\eta_0 \ell_\infty'}(q_\infty))}{\pi \eta_0^2 (\ell_\infty')^2} = \nu,$$

where $q_k := \Pi_k(p_k')$ for $k \in \mathbb{N} \cup \{\infty\}$. Hence, $n(\Psi_k - p_k', B^2_{s_0^2 r_k'}, B^{\Pi_k}_{\eta_0 \ell_k'}) = \nu$ eventually. So the first equality in (4.5.11) holds eventually, giving the desired contradiction. $\square$

Lemma 4.5.6. *Assume that $\Psi \in C^\infty(\overline{B}_r^2(z), \mathcal{M}_{p,\ell})$ is a conformal immersion and Π is a 2-plane with $\Psi(z + r\cdot) \in \mathcal{R}^{\Pi}_{K_0, \delta_0}$ and $\frac{1}{2}\int_{B^2_r(z)} |\nabla \Psi|^2 \le E_0$. If $\int_{B^2_r(z)} |\mathbb{I}^\Psi|^4 \operatorname{vol}_\Psi$ and ℓ are sufficiently small, then $\Pi \circ \Psi$ is a diffeomorphism from $\overline{B}^2_{s_0^2 r}(z)$ onto its image.*

Proof. We can suppose that $z = 0$ and $r = 1$. Assume by contradiction that the claim does not hold, for a sequence of 2-planes $\Pi_k \to \Pi_\infty$ and immersions $\Psi_k : \overline{B}_1^2 \to \mathcal{M}_{p_k,\ell_k}$ with $\ell_k \to 0$ and second fundamental forms $\mathrm{I\!I}_k$ satisfying

$$\int_{B_1^2} |\mathrm{I\!I}_k|^4 \operatorname{vol}_{\Psi_k} \to 0. \tag{4.5.12}$$

Let $\lambda_k \in C^\infty(\overline{B}_1^2)$ be defined by $|\partial_1 \Psi_k| = |\partial_2 \Psi_k| =: e^{\lambda_k}$ and let $\mathrm{I\!I}_{p,\ell}$ and $\widetilde{\mathrm{I\!I}}_k$ denote the second fundamental forms of $\mathcal{M}_{p,\ell} \subseteq \mathbb{R}^Q$ and of the immersion Ψ_k in $\mathbb{R}^Q$ respectively, so that $\widetilde{\mathrm{I\!I}}_k = \mathrm{I\!I}_{p_k,\ell_k} + \mathrm{I\!I}_k$. Note that

$$\|\mathrm{I\!I}_{p_k,\ell_k}\|_{L^\infty} \le C(\mathcal{M})\ell_k \to 0, \tag{4.5.13}$$

so that

$$\int_{B_1^2} \left|\widetilde{\mathrm{I\!I}}_k\right|^4 \operatorname{vol}_{\Psi_k} \to 0. \tag{4.5.14}$$

With a slight abuse of notation, let us drop the dependence on k in the subsequent computations. We define the orthonormal frame

$$\widetilde{e}_1 := e^{-\lambda}\partial_1\Psi, \qquad \widetilde{e}_2 := e^{-\lambda}\partial_2\Psi \tag{4.5.15}$$

for the tangent space of the immersed surface Ψ. It is straightforward to check that the map $\widetilde{e}_1 \wedge \widetilde{e}_2 : \overline{B}_1^2 \to \Lambda_2\mathbb{R}^Q$ has $|\nabla(\widetilde{e}_1 \wedge \widetilde{e}_2)| = e^\lambda\left|\widetilde{\mathrm{I\!I}}\right|$, so

$$\int_{B_1^2} |\nabla(\widetilde{e}_1 \wedge \widetilde{e}_2)|^2 \, d\mathcal{L}^2 = \int_{B_1^2} e^{2\lambda}\left|\widetilde{\mathrm{I\!I}}\right|^2 d\mathcal{L}^2 = \int_{B_1^2} \left|\widetilde{\mathrm{I\!I}}\right|^2 \operatorname{vol}_\Psi \to 0 \tag{4.5.16}$$

by Hölder's inequality, since $\int_{B_1^2} \operatorname{vol}_\Psi \le E$. We identify the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^Q)$ of 2-planes in $\mathbb{R}^Q$ with a submanifold of the projectivization of $\Lambda_2\mathbb{R}^Q$, by means of Plücker's embedding. For k large enough [51, Lemma 5.1.4] applies and provides a rotated frame (e_1, e_2), given by

$$e_{\mathbb{C}} := e_1 + ie_2 = e^{i\theta}\widetilde{e}_{\mathbb{C}}, \qquad \widetilde{e}_{\mathbb{C}} := \widetilde{e}_1 + i\widetilde{e}_2, \tag{4.5.17}$$

for a suitable real function $\theta \in W^{1,2}(B_1^2)$ minimizing $\int_{B_1^2} |\nabla\theta + \widetilde{e}_1 \cdot \nabla\widetilde{e}_2|^2$ (in particular, θ and $e_{\mathbb{C}}$ are smooth functions on $\overline{B}_1^2$) and with $\|\nabla e_{\mathbb{C}}\|_{L^2}^2$ becoming arbitrarily small as $k \to \infty$. We will assume in the sequel that $\|\nabla e_{\mathbb{C}}\|_{L^2}^2 \le 1$. Observe that, whenever $\alpha, \beta \in C^1(\overline{B}_1^2)$,

$$\begin{aligned}
\partial_1\alpha\partial_2\beta - \partial_2\alpha\partial_1\beta &= \frac{1}{4}(\partial_1\alpha + \partial_2\beta)^2 + \frac{1}{4}(\partial_2\alpha - \partial_1\beta)^2 - \frac{1}{4}(\partial_1\alpha - \partial_2\beta)^2 - \frac{1}{4}(\partial_2\alpha + \partial_1\beta)^2 \\
&= |\partial_z(\alpha + i\beta)|^2 - |\partial_{\overline{z}}(\alpha + i\beta)|^2.
\end{aligned}$$

Hence, since $\widetilde{e}_1 + i\widetilde{e}_2 = 2e^{-\lambda}\partial_{\overline{z}}\Psi$ and $\partial_z\Psi \cdot \partial_z\Psi = \partial_{\overline{z}}\Psi \cdot \partial_{\overline{z}}\Psi = 0$ by conformality, we get

$$\begin{aligned}
&-(\partial_1\widetilde{e}_1 \cdot \partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1 \cdot \partial_1\widetilde{e}_2) \\
&= 4\partial_{\overline{z}}(e^{-\lambda}\partial_{\overline{z}}\Psi) \cdot \partial_z(e^{-\lambda}\partial_z\Psi) - 4\partial_{\overline{z}}(e^{-\lambda}\partial_z\Psi) \cdot \partial_z(e^{-\lambda}\partial_{\overline{z}}\Psi) \\
&= 4e^{-2\lambda}(\partial^2_{\overline{z}\overline{z}}\Psi \cdot \partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi \cdot \partial^2_{\overline{z}z}\Psi - \partial_{\overline{z}}\lambda\partial_{\overline{z}}\Psi \cdot \partial^2_{zz}\Psi - \partial_z\lambda\partial_z\Psi \cdot \partial^2_{\overline{z}\overline{z}}\Psi) \\
&\quad + 2e^{-2\lambda}\partial_{\overline{z}}\lambda\partial_{\overline{z}}(\partial_z\Psi \cdot \partial_z\Psi) + 2e^{-2\lambda}\partial_z\lambda\partial_z(\partial_{\overline{z}}\Psi \cdot \partial_{\overline{z}}\Psi) \\
&= 4e^{-2\lambda}(\partial^2_{\overline{z}\overline{z}}\Psi \cdot \partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi \cdot \partial^2_{\overline{z}z}\Psi - \partial_{\overline{z}}\lambda\partial_{\overline{z}}\Psi \cdot \partial^2_{zz}\Psi - \partial_z\lambda\partial_z\Psi \cdot \partial^2_{\overline{z}\overline{z}}\Psi).
\end{aligned}$$

On the other hand we have

$$2e^{2\lambda}\partial_z\lambda = \partial_z(e^{2\lambda}) = \partial_z(2\partial_{\overline{z}}\Psi\cdot\partial_z\Psi) = \partial_{\overline{z}}(\partial_z\Psi\cdot\partial_z\Psi) + 2\partial_{\overline{z}}\Psi\cdot\partial^2_{zz}\Psi = 2\partial_{\overline{z}}\Psi\cdot\partial^2_{zz}\Psi,$$

$$\begin{aligned}\Delta(e^{2\lambda}) &= 4\partial^2_{\overline{z}z}(2\partial_{\overline{z}}\Psi\cdot\partial_z\Psi) = 8\partial_{\overline{z}}(\partial_{\overline{z}}\Psi\cdot\partial^2_{zz}\Psi) + 4\partial^2_{\overline{z}\overline{z}}(\partial_z\Psi\cdot\partial_z\Psi)\\ &= 8(\partial^2_{\overline{z}\overline{z}}\Psi\cdot\partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi\cdot\partial^2_{\overline{z}z}\Psi) + 4\partial^2_{zz}(\partial_{\overline{z}}\Psi\cdot\partial_{\overline{z}}\Psi)\\ &= 8(\partial^2_{\overline{z}\overline{z}}\Psi\cdot\partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi\cdot\partial^2_{\overline{z}z}\Psi),\end{aligned}$$

so we arrive at

$$\partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2 = -\frac{\Delta(e^{2\lambda})}{2e^{2\lambda}} + 8\partial_{\overline{z}}\lambda\partial_z\lambda = -\Delta\lambda. \tag{4.5.18}$$

Alternatively, since the projections of $\partial_j\widetilde{e}_1$ and $\partial_k\widetilde{e}_2$ onto the tangent space of the immersion Ψ are orthogonal (as the projection of $\partial_j\widetilde{e}_1$ is a multiple of $\widetilde{e}_2$ and the projection of $\partial_k\widetilde{e}_2$ is a multiple of $\widetilde{e}_1$), we have

$$\partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2 = e^{2\lambda}(\widetilde{\mathbb{I}}(\widetilde{e}_1,\widetilde{e}_1)\cdot\widetilde{\mathbb{I}}(\widetilde{e}_2,\widetilde{e}_2) - \widetilde{\mathbb{I}}(\widetilde{e}_1,\widetilde{e}_2)\cdot\widetilde{\mathbb{I}}(\widetilde{e}_1,\widetilde{e}_2)) = e^{2\lambda}K,$$

by Gauss' formula, where K denotes the Gaussian curvature of the immersed surface. But, by the well-known formula for the curvature of a conformal metric, we have $K = -e^{-2\lambda}\Delta\lambda$, which gives again (4.5.18). Moreover,

$$\begin{aligned}\partial_1 e_1\cdot\partial_2 e_2 - \partial_2 e_1\cdot\partial_1 e_2 &= |\partial_z e_{\mathbb{C}}|^2 - |\partial_{\overline{z}} e_{\mathbb{C}}|^2 = |\partial_z\widetilde{e}_{\mathbb{C}}|^2 - |\partial_{\overline{z}}\widetilde{e}_{\mathbb{C}}|^2\\ &= \partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2,\end{aligned}$$

since $\partial_z\widetilde{e}_{\mathbb{C}}\cdot\overline{\widetilde{e}_{\mathbb{C}}} = -\widetilde{e}_{\mathbb{C}}\cdot\partial_z\overline{\widetilde{e}_{\mathbb{C}}}$ and similarly for the partial derivatives with respect to $\overline{z}$. Thus, calling $\mu\in C^\infty(\overline{B}^2_1)$ the solution to

$$\begin{cases}-\Delta\mu = \partial_1 e_1\cdot\partial_2 e_2 - \partial_2 e_1\cdot\partial_1 e_2 & \text{on } B^2_1\\ \mu = 0 & \text{on } \partial B^2_1,\end{cases}$$

we obtain that $\lambda-\mu$ is harmonic and, by Wente's inequality,

$$\|\mu\|_{L^\infty} \le C(Q)\Big(\|\nabla e_1\|^2_{L^2} + \|\nabla e_2\|^2_{L^2}\Big) \le C(Q). \tag{4.5.19}$$

Since $\lambda < e^{2\lambda}$, for all $x\in\overline{B}^2_{3/4}$ we get

$$(\lambda-\mu)(x) = \fint_{B^2_{1/4}(x)}(\lambda-\mu) \le \fint_{B^2_{1/4}(x)} e^{2\lambda} + \|\mu\|_{L^\infty} \le \frac{E}{\mathcal{L}^2(B^2_{1/4})} + C(Q). \tag{4.5.20}$$

Together with (4.5.19), this gives an upper bound for λ on $B^2_{3/4}$, depending only on E, Q. Although this is sufficient for the present purposes, one can also get a lower bound for λ on $B^2_{s_0}$. Indeed, calling M the right-hand side of (4.5.20), we obtain that $M-(\lambda-\mu)$ is a nonnegative harmonic function on $B^2_{3/4}$. Moreover, the length of the curve $\Psi\big|_{\partial B^2_{s_0}}$ is

$$\int_{\partial B^2_{s_0}} e^\lambda \ge 2\pi\eta_0, \tag{4.5.21}$$

since the composition of $\Pi \circ \Psi|_{\partial B^2_{s_0}}$ with the radial projection onto $\partial B^{\Pi}_{\eta_0}$ (which does not increase the length) is surjective (being a generator of the fundamental group of $\partial B^{\Pi}_{\eta_0}$). Hence, there exists some $x \in \partial B^2_{s_0}$ such that $\lambda(x) \geq \log\left(s_0^{-1}\eta_0\right)$. We deduce that

$$\inf_{B^2_{s_0}} (M - (\lambda - \mu)) \leq M + C(Q) - \log(s_0^{-1}\eta_0) \tag{4.5.22}$$

and so, by Harnack's inequality, the supremum of $M - (\lambda - \mu)$ on $B^2_{s_0}$ is bounded by a constant depending only on E, s_0, η_0, Q. This, together with (4.5.20) and (4.5.19), gives

$$\|\lambda\|_{L^\infty(B^2_{s_0})} \leq C(E, s_0, \eta_0, Q). \tag{4.5.23}$$

The mean curvature of the immersion Ψ is $\widetilde{H} = \frac{1}{2e^{2\lambda}}(\widetilde{\mathrm{II}}(\partial_1\Psi, \partial_1\Psi) + \widetilde{\mathrm{II}}(\partial_2\Psi, \partial_2\Psi)) = -\frac{\Delta\Psi}{2e^{2\lambda}}$ (note that $\Delta\Psi$ is already orthogonal to the tangent space of the immersion, since $\partial_z\Psi \cdot \Delta\Psi = 4\partial_z\Psi \cdot \partial^2_{\bar{z}z}\Psi = 2\partial_{\bar{z}}(\partial_z\Psi \cdot \partial_z\Psi) = 0$). So we get

$$\begin{aligned}
&\int_{B^2_{3/4}} |\Delta\Psi_k|^4 \, d\mathcal{L}^2 = 16 \int_{B^2_{3/4}} \left|\widetilde{H}_k\right|^4 e^{6\lambda_k} \operatorname{vol}_{\Psi_k} \\
&\leq C(E, Q) \int_{B^2_{3/4}} \left|\widetilde{\mathrm{II}}_k\right|^4 \operatorname{vol}_{\Psi_k} \to 0.
\end{aligned} \tag{4.5.24}$$

Since $s_0 \leq \frac{1}{2}$, this implies that (Ψ_k) is a bounded sequence in $W^{2,4}(B^2_{s_0})$ (by Lemma A.2 applied to $\Psi_k(\frac{3}{4}\cdot)$), so by the compact embedding $W^{2,4}(B^2_{s_0}) \hookrightarrow C^1(\overline{B}^2_{s_0})$ we obtain a strong limit Ψ_∞ in $C^1(\overline{B}^2_{s_0})$, up to subsequences. Thus Ψ_∞ is weakly conformal and, by (4.5.24), it is also harmonic. Lemma 4.4.3 applies (with $\Psi_\infty(s_0\cdot)$ and $\mathrm{id}_{\mathbb{R}^2}$ in place of Ψ and φ) and gives that $\Pi_\infty \circ \Psi_\infty$ is a diffeomorphism from $\overline{B}^2_{s_0/2} \supseteq \overline{B}^2_{s_0^2}$ onto its image; hence, the same is eventually true for $\Pi_k \circ \Psi_k$, giving the desired contradiction. □

4.6 Multiplicity one in the limit

Theorem 4.6.1. *Assume $\Phi \in C^\infty(\overline{B}^2_r(z), \mathcal{M})$ is a conformal immersion, almost critical for (4.3.1) on $B^2_r(z)$ and satisfying*

- $\ell^{-1}(\Phi(z + r\cdot) - p) \in \mathcal{R}^{\Pi}_{K_0,\delta_0}$ *for some* $\sqrt{\sigma/\varepsilon_0''} < \ell < 1$ *and* $p \in \mathcal{M}$*,*
- $\displaystyle\frac{1}{2}\int_{B^2_r(z)} |\nabla\Phi|^2 \leq E_0\ell^2,$
- $\displaystyle\int_{\Phi^{-1}(B^Q_\ell(p))} \operatorname{vol}_\Phi \leq V\pi\ell^2$ *and* $\displaystyle\int_{\Phi^{-1}(B^Q_{\eta_0\ell}(p))} \operatorname{vol}_\Phi \leq V\pi(\eta_0\ell)^2,$
- $\displaystyle\sigma^4\log(\sigma^{-1})\int_{B^2_s} \left|\mathrm{II}^\Phi\right|^4 \operatorname{vol}_\Phi \leq \frac{\varepsilon_0''}{E_0}\int_{B^2_s} \operatorname{vol}_\Phi$ *for all* $0 < s \leq r$.

Then, if σ and ℓ are small enough (independently of each other), we have

$$n(\Phi - p, B^2_{s_0^2 r}(z), B^{\Pi}_{\eta_0\ell}) = 1.$$

Proof. Let $r_0 := r$, $p_0 := p$, $\ell_0 := \ell$, $\tau_0 := \sigma \ell_0^{-2}$ and $\Pi_0 := \Pi$. Note that

$$\Psi_0 := \ell^{-1}(\Phi - p) = \ell_0^{-1}(\Phi - p_0)$$

is almost critical for (4.3.3), with $\tau := \tau_0 \leq \varepsilon_0''$. Thus Lemma 4.5.3 applies to Ψ_0 (if ℓ is small enough), giving a new radius $\varepsilon_0' r_0 < r_1 < s_0 r_0$, a new point $p' \in \mathcal{M}_{p,\ell}$, a new scale $\varepsilon_0' < \ell' < \frac{1}{2}$ and a new 2-plane Π'. Setting $r_1 := r'$, $p_1 := p_0 + \ell_0 p'$, $\ell_1 := \ell' \ell_0$, $\tau_1 := \sigma/\ell_1$, $\Pi_1 := \Pi'$ and recalling (4.5.10), the map

$$\Psi_1 := (\ell')^{-1}(\Psi_0 - p') = \ell_1^{-1}(\Phi - p_1)$$

still satisfies the hypotheses of Lemma 4.5.3, with the parameters $r_1, \tau_1, p_1, \ell_1, \Pi_1$, provided that $\tau_1 \leq \varepsilon_0'$: indeed, note that (assuming $\tau_1 \leq \varepsilon_0' < 1$)

$$\begin{aligned}
&\tau_1^4 \log(\tau_1^{-1}) \int_{B^2_{r_1}(z)} \left|\mathbb{I}^{\Psi_1}\right|^4 \mathrm{vol}_{\Psi_1} \leq \tau_1^4 \log(\sigma^{-1}) \int_{B^2_{r_1}(z)} \left|\mathbb{I}^{\Psi_1}\right|^4 \mathrm{vol}_{\Psi_1} \\
&\quad = \ell_1^{-2} \sigma^4 \log(\sigma^{-1}) \int_{B^2_{r_1}(z)} \left|\mathbb{I}^{\Phi}\right|^4 \mathrm{vol}_\Phi \leq \frac{\varepsilon_0'' \ell_1^{-2}}{E_0} \int_{B^2_{r_1}(z)} \mathrm{vol}_\Phi = \frac{\varepsilon_0''}{2E_0} \int_{B^2_{r_1}(z)} |\nabla \Psi_1|^2 \leq \varepsilon_0''.
\end{aligned}$$

Hence, we can iterate and define $r_j, \tau_j, p_j, \ell_j, \Pi_j$, for $j = 0, 1, \dots,$ up to a maximum index $k \geq 1$ such that $\tau_j \leq \varepsilon_0'' \leq \varepsilon_0'$ for $1 \leq j < k$ and $\tau_k > \varepsilon_0''$: such k exists since $\tau_j = \ell_j^{-1}\sigma \geq 2^j \tau_0$. With the same computation as above, this implies

$$\int_{B^2_{r_k}(z)} \left|\mathbb{I}^{\Psi_k}\right|^4 \mathrm{vol}_{\Psi_k} \leq \frac{\varepsilon_0''}{\tau_k^4 \log(\sigma^{-1})} \leq \frac{1}{(\varepsilon_0'')^3 \log(\sigma^{-1})}. \tag{4.6.1}$$

If σ and ℓ are small enough, Lemma 4.5.6 applies to the map $\Psi_k := \ell_k^{-1}(\Psi - p_k)$, on the ball $B^2_{r_k}(z)$: indeed, note that $\ell_k \leq \ell$ and $\int_{B^2_{r_k}(z)} |\mathbb{I}^{\Psi_k}|^4 \mathrm{vol}_{\Psi_k}$ can be assumed arbitrarily small (by taking σ and ℓ small enough), by virtue of (4.6.1). This, together with Lemma A.1, gives

$$n(\Psi_k, B^2_{s_0^2 r_k}(z), B^{\Pi_k}_{\eta_0}) = 1.$$

Also, Lemma 4.5.5 applies for all $j = 0, \dots, k-1$, giving

$$\begin{aligned}
n(\Phi - p, B^2_{s_0^2 r}(z), B^{\Pi}_{\eta_0 \ell}) &= n(\Psi_0, B^2_{s_0^2 r_0}(z), B^{\Pi_0}_{\eta_0}) \\
&= n(\Psi_1, B^2_{s_0^2 r_1}(z), B^{\Pi_1}_{\eta_0}) \\
&= \cdots \\
&= n(\Psi_k, B^2_{s_0^2 r_k}(z), B^{\Pi_k}_{\eta_0}) \\
&= 1.
\end{aligned}$$

□

As in Section 4.3, assume now that $\Phi_k : \Sigma \to \mathcal{M}$ is a sequence of almost critical points for

$$\int_\Sigma \mathrm{vol}_{\Phi_k} + \sigma_k^4 \int_\Sigma |\mathbb{I}^{\Phi_k}|^4 \mathrm{vol}_{\Phi_k} \tag{4.6.2}$$

with controlled area, namely

$$\lambda \le \int_\Sigma \mathrm{vol}_{\Phi_k} \le \Lambda,$$

and with

$$\sigma_k \to 0, \qquad \sigma_k^4 \log(\sigma_k^{-1}) \int_\Sigma |\mathbb{I}^{\Phi_k}|^4 \,\mathrm{vol}_{\Phi_k} \to 0.$$

By the main result of the second chapter, up to subsequences the varifolds $\mathbf{v}_k$ induced by Φ_k converge to a parametrized stationary varifold.

As explained in Section 2.6, there could be bubbling points, and also the conformal structures induced by Φ_k could degenerate (in the space of conformal structures up to diffeomorphisms). In the remainder of the chapter, we will assume for simplicity that there is no bubbling and no degeneration of the conformal structure. Note that the arguments will apply also to the general case, working on appropriate domains different from Σ, as it was done in Section 2.6.

Up to precomposing Φ_k with suitable diffeomorphisms of Σ, we can thus assume that there exist metrics g_k of constant curvature (1, 0 or -1, depending on the genus of Σ) such that $\Phi_k : (\Sigma, g_k) \to \mathcal{M}$ is conformal, and such that g_k converges smoothly to a limiting Riemannian metric g_∞. The limiting varifold $\mathbf{v}_\infty$ is a parametrized stationary varifold of the form $(\Sigma_\infty, \Theta_\infty, N_\infty)$. By the regularity result of the previous chapter, which was already exploited in Section 4.5, $\Theta_\infty : \Sigma_\infty \to \mathcal{M}$ is a smooth branched minimal immersion and N_∞ is locally (a.e.) constant. Also, calling $\Phi_\infty \in W^{1,2}(\Sigma, \mathcal{M})$ the weak limit of Φ_k (up to subsequences), $\Theta_\infty = \Phi_\infty \circ \varphi_\infty^{-1}$ for some quasiconformal homeomorphism $\varphi_\infty : \Sigma \to \Sigma_\infty$, with respect to g_∞. Here Σ_∞ is a Riemann surface homeomorphic (by means of φ_∞) to Σ. In particular, Φ_∞ is continuous and N_∞ is a.e. constant (Σ_∞ being connected).

In local conformal coordinates for (Σ, g_∞), as in (4.3.5) we have

$$\mathrm{vol}_{\Phi_k} \rightharpoonup N_\infty |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty| \,\mathcal{L}^2 \ge \frac{1}{2} |\nabla \Phi_\infty|^2 \,\mathcal{L}^2. \tag{4.6.3}$$

Setting $\nu_k := \mathrm{vol}_{\Phi_k}$ and $\nu_\infty := N_\infty |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty| \,\mathcal{L}^2$ (in local conformal coordinates for Σ), by (4.6.3) we have $\nu_k \rightharpoonup \nu_\infty$. We can find a conformal disk $U \subset (\Sigma, g_\infty)$, which we identify with $B_1^2 \subset \mathbb{C}$ and fix in the sequel, such that $\nu_\infty(B_{1/2}^2) > 0$.

Definition 4.6.2. We denote by ν the constant value of N_∞. Also, we call T the set of bad points $z \in B_1^2$ which are not Lebesgue for $\nabla \Phi_\infty$, or such that $\nabla \Phi_\infty(z)$ does not have full rank, or such that

$$\max_{|x|=1} |\langle \nabla \Phi_\infty(z), x \rangle| > 2\nu \min_{|x|=1} |\langle \nabla \Phi_\infty(z), x \rangle|. \tag{4.6.4}$$

We have $\mathcal{L}^2(T) = 0$, since $\nabla \Theta_\infty$ has full rank a.e. by conformality (hence the same holds for Φ_∞ by the chain rule[8]) and since (4.6.4) implies $\nu |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty|(z) < \frac{1}{2} |\nabla \Phi_\infty|^2(z)$ (as it can be immediately verified using a singular value decomposition for $\nabla \Phi_\infty(z)$).

[8] See e.g. [56, Lemma 4.12] and [65, Lemma III.6.4].

Definition 4.6.3. We now specify $K'' := 2\nu$ and we set $E'' := \pi\nu((K'')^2 + 1)$. Finally, we choose $V > 0$ such that $V = \lfloor V \rfloor + \frac{1}{2}$ and

$$\|\mathbf{v}_\infty\|(\overline{B}_\ell^Q(p)) < V\pi\ell^2 \tag{4.6.5}$$

for all $\ell > 0$ and all $p \in \mathcal{M}$. Such V exists by the monotonicity formula satisfied by the stationary varifold $\mathbf{v}_\infty$. Note that now also the constants K_0, E_0, s_0, η_0, as well as ε_0, δ_0, ε_0' and ε_0'', are determined.

Theorem 4.6.4. *We have $N_\infty = 1$ a.e., or equivalently $\nu = 1$.*

Proof. Let $\mathcal{B}_k$ be the Borel set of points $z \in B_{1/2}^2$ such that

$$\sigma_k^4 \log(\sigma_k^{-1}) \int_{B_r^2(z)} \left|\mathrm{I\!I}^{\Phi_k}\right|^4 \mathrm{vol}_{\Phi_k} \geq \frac{\varepsilon_0''}{E_0} \int_{B_r^2(z)} \mathrm{vol}_{\Phi_k}$$

for some radius $0 < r < \frac{1}{2}$. By Besicovitch's covering lemma, we get a collection of points $z_i \in \mathcal{B}_k$ and radii $0 < r_i < \frac{1}{2}$ such that

$$\sigma_k^4 \log(\sigma_k^{-1}) \int_{B_{r_i}^2(z_i)} \left|\mathrm{I\!I}^{\Phi_k}\right|^4 \mathrm{vol}_{\Phi_k} \geq \frac{\varepsilon_0''}{E_0} \int_{B_{r_i}^2(z_i)} \mathrm{vol}_{\Phi_k}, \qquad \mathbf{1}_{\mathcal{B}_k} \leq \sum_i \mathbf{1}_{B_{r_i}^2(z_i)} \leq \mathfrak{N},$$

for some universal constant $\mathfrak{N}$. Thus we get

$$\begin{aligned} \nu_k(\mathcal{B}_k) &\leq \sum_i \mathrm{vol}_{\Phi_k}(B_{r_i}^2(z_i)) \leq \frac{E_0}{\varepsilon_0''} \sigma_k^4 \log(\sigma_k^{-1}) \sum_i \int_{B_{r_i}^2(z_i)} \left|\mathrm{I\!I}^{\Phi_k}\right|^4 \mathrm{vol}_{\Phi_k} \\ &\leq \frac{E_0 \mathfrak{N}}{\varepsilon_0''} \sigma_k^4 \log(\sigma_k^{-1}) \int_\Sigma \left|\mathrm{I\!I}^{\Phi_k}\right|^4 \mathrm{vol}_{\Phi_k} \to 0. \end{aligned}$$

Up to subsequences, we can assume that $\overline{B_{1/2}^2 \setminus \mathcal{B}_k}$ converges in the Hausdorff topology to some compact set $S \subseteq \overline{B}_{1/2}^2$. We remark that $\nu_\infty(S) > 0$: indeed, for any compact neighborhood F of S in B_1^2, we have $B_{1/2}^2 \setminus \mathcal{B}_k \subseteq F$ eventually and so

$$\nu_\infty(F) \geq \limsup_{k\to\infty} \nu_k(F) \geq \limsup_{k\to\infty} (\nu_k(B_{1/2}^2) - \nu_k(\mathcal{B}_k)) = \limsup_{k\to\infty} \nu_k(B_{1/2}^2) \geq \nu_\infty(B_{1/2}^2) > 0.$$

It follows from (4.6.3) that $\mathcal{L}^2(S) > 0$.

We now show that $N_\infty = 1$ a.e. on $S \setminus T$, which has positive Lebesgue measure. This will show that $\nu = 1$, as desired. Fix any $z \in S \setminus T$ and take a sequence $z_k \in B_{1/2}^2 \setminus \mathcal{B}_k$ with $z_k \to z$. Locally we can find conformal reparametrizations $\widetilde{\Phi}_k$ of $\Phi_k(z_k + \cdot)$, by means of diffeomorphisms converging smoothly to the identity on a small neighborhood of 0.[9] By weak convergence $\widetilde{\Phi}_k \rightharpoonup \Phi_\infty(z + \cdot)$ in $W^{1,2}$, for a.e. radius $r > 0$ we have

$$\widetilde{\Phi}_k(r\cdot) \to \Phi_\infty(z + r\cdot) \quad \text{in } C^0(\partial B_1^2 \cup \partial B_{s_0}^2 \cup \partial B_{s_0^2}^2) \tag{4.6.6}$$

[9]For instance, one can isometrically identify a neighborhood of z_k in (Σ, g_k) with a neighborhood of z in (Σ, g_∞), by means of the exponential map.

up to further subsequences.[10] Using Lemma A.4 and the fact that $z \notin T$, we can assume that r satisfies

$$|\Phi_\infty(z+rx) - \Phi_\infty(z) - \langle \nabla\Phi_\infty(z), rx\rangle| < \delta_0 \ell \quad \text{for } x \in \partial B_1^2 \cup \partial B_{s_0}^2 \cup \partial B_{s_0^2}^2, \tag{4.6.7}$$

$$\frac{1}{2}\int_{B_r^2(z)} |\nabla\Phi_\infty|^2 < (\pi r^2)|\nabla\Phi_\infty(z)|^2 \le \ell^2\pi((K'')^2+1), \tag{4.6.8}$$

with $\ell := r\min_{|x|=1}|\langle\nabla\Phi_\infty(z), x\rangle|$. Note that (4.6.7) will guarantee (below) that the first assumption in Theorem 4.6.1 holds for $\widetilde{\Phi}_k$. Setting $p := \Phi_\infty(z)$, note that (4.6.5) gives

$$\|\mathbf{v}_k\|(B_\ell^Q(p)) < V\pi\ell^2, \quad \|\mathbf{v}_k\|(B_{\eta_0\ell}^Q(p)) < V\pi(\eta_0\ell)^2$$

eventually, which trivially implies

$$\int_{\widetilde{\Phi}_k^{-1}(B_\ell^Q(p))} \mathrm{vol}_{\widetilde{\Phi}_k} < V\pi\ell^2, \quad \int_{\widetilde{\Phi}_k^{-1}(B_{\eta_0\ell}^Q(p))} \mathrm{vol}_{\widetilde{\Phi}_k} < V\pi(\eta_0\ell)^2. \tag{4.6.9}$$

Also, (4.6.3) and (4.6.8) give

$$\lim_{k\to\infty}\frac{1}{2}\int_{B_r^2}\left|\nabla\widetilde{\Phi}_k\right|^2 = \lim_{k\to\infty}\nu_k(B_r^2(z)) \le \frac{\nu}{2}\int_{B_r^2(z)}|\nabla\Phi_\infty|^2 < E''\ell^2.$$

Thanks to the fact that $z_k \notin \mathcal{B}_k$ and the above inequalities, eventually $\widetilde{\Phi}_k$ satisfies the hypotheses of Theorem 4.6.1 on the ball B_r^2, provided that r (and thus ℓ) is chosen small enough.[11] Setting $\Psi_k := \ell^{-1}(\widetilde{\Phi}_k - p)$, we infer that

$$n(\Psi_k, B_{s_0^2 r}^2, B_{\eta_0}^\Pi) = 1, \tag{4.6.10}$$

where Π is the 2-plane spanned by $\nabla\Phi_\infty(z)$.

Since r can be chosen arbitrarily small (possibly changing the subsequence guaranteeing (4.6.6)), the argument used in the proof of Theorem 2.5.3 shows that $N_\infty(z) = 1$.

Alternatively, (4.6.10) gives

$$\left|\frac{\|\Pi_*\mathbf{v}_k'\|(B_{\eta_0}^\Pi)}{\pi\eta_0^2} - 1\right| < \frac{1}{8},$$

where the varifold $\mathbf{v}_k'$ is induced by $\Psi_k\big|_{B_{s_0^2 r}^2}$ and converges to the varifold $\mathbf{v}_\infty'$ induced by $(\varphi_\infty(B_{s_0^2 r}^2(z)), \ell^{-1}(\Theta_\infty - p), \nu)$. Assuming without loss of generality that $\nabla\Theta_\infty(\varphi_\infty(z)) \neq 0$, $\Pi \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(B_{s_0^2 r}^2(z))$ onto its image (for r small enough). Hence,

[10] This can be obtained by applying Lemma A.5 to the weakly converging $\mathbb{R}^{3Q}$-valued maps

$$(\widetilde{\Phi}_k, \widetilde{\Phi}_k(s_0\cdot), \widetilde{\Phi}_k(s_0^2\cdot)) \rightharpoonup (\Phi_\infty(z+\cdot), \Phi_\infty(z+s_0\cdot), \Phi_\infty(z+s_0^2\cdot)).$$

[11] To be precise, in the definition of $\mathcal{B}_k$ one should use balls in a conformal chart for g_k.

at a.e. point of Π the varifold $\Pi_*\mathbf{v}_\infty$ has density either 0 or ν. Since $\Pi\circ\Phi_\infty(B^2_{s_0^2 r}(z))$ is a superset of $B^\Pi_{\eta_0\ell}(\Pi(p))$ (by Lemma A.1), it follows that

$$\frac{\|\Pi_*\mathbf{v}'_\infty\|(B^\Pi_{\eta_0})}{\pi\eta_0^2}=\nu.$$

The convergence $\frac{\|\Pi_*\mathbf{v}'_k\|(B^\Pi_{\eta_0})}{\pi\eta_0^2}\to\frac{\|\Pi_*\mathbf{v}'_\infty\|(B^\Pi_{\eta_0})}{\pi\eta_0^2}$ thus gives $|\nu-1|\le\frac{1}{8}$, and again we conclude that $\nu=1$. $\square$

Appendix

Lemma A.1. *Assume that $F\in C^0(\overline{B}^2_1,\mathbb{R}^2)$ satisfies*

$$|F(x)-\varphi(x)|\le\delta\qquad\text{for all } x\in\partial B^2_1 \tag{A.1}$$

for some $0<\delta<1$ and some homeomorphism $\varphi:\mathbb{R}^2\to\mathbb{R}^2$, with $\varphi(0)=0$ and $\min_{|x|=1}|\varphi(x)|\ge 1$. Then

$$F(B^2_1)\supseteq B^2_{1-\delta}. \tag{A.2}$$

Proof. It suffices to show that, for a fixed $y\in B^2_{1-\delta}$, the closed curve $\Gamma':=F\big|_{\partial B^2_1}$ is not contractible in $\mathbb{R}^2\setminus\{y\}$: once this is done, if we had $y\notin F(B^2_1)$, i.e. $y\notin F(\overline{B}^2_1)$, then F would provide a homotopy from Γ' to the constant curve $F(0)$ in $\mathbb{R}^2\setminus\{y\}$, yielding a contradiction.

Letting $\Gamma:=\varphi\big|_{\partial B^2_1}$ and $\gamma:=\Gamma'-\Gamma$, we have $|\gamma(x)|\le\delta$ for all $x\in\partial B^2_1$. Hence, Γ is homotopic to Γ' in $\mathbb{R}^2\setminus B^2_{1-\delta}\subseteq\mathbb{R}^2\setminus\{y\}$ by means of the homotopy

$$\Gamma+t\gamma,\qquad 0\le t\le 1.$$

So we are left to show that Γ is not contractible in $\mathbb{R}^2\setminus\{y\}$, i.e. that $\Gamma-y$ is not contractible in $\mathbb{R}^2\setminus\{0\}$. The curve $\Gamma-y$ is homotopic to Γ in $\mathbb{R}^2\setminus\{0\}$, by means of the homotopy

$$\Gamma-ty,\qquad 0\le t\le 1,$$

which avoids the origin since $|y|<1$. Finally, Γ is not contractible in $\mathbb{R}^2\setminus\{0\}$, since φ (once restricted to a homeomorphism of $\mathbb{R}^2\setminus\{0\}$) induces an automorphism of $\pi_1(\mathbb{R}^2\setminus\{0\})$ sending the class of the generator $\mathrm{id}_{\partial B^2_1}$ to the class of Γ. Hence, $\Gamma-y$ is not contractible in $\mathbb{R}^2\setminus\{0\}$, too, as desired. $\square$

Lemma A.2. *For a function $\Psi\in C^\infty(\overline{B}_1)$ and $0<\tau<1$ we have*

$$\|\Psi\|_{W^{2,4}(B^2_\tau)}\le C(\tau)(\|\Delta\Psi\|_{L^4(B^2_1)}+\|\nabla\Psi\|_{L^2(B^2_1)}+\|\Psi\|_{L^2(B^2_1)}).$$

Proof. Given two radii $0 < r < s \leq 1$, let us choose a cut-off function $\rho \in C_c^\infty(B_s^2)$ with $\rho = 1$ on B_r^2. Since $\rho\Psi \in C_c^\infty(\mathbb{R}^2)$, standard Calderón–Zygmund estimates give

$$\begin{aligned} \|\nabla^2\Psi\|_{L^p(B_r^2)} &\leq \|\nabla^2(\rho\Psi)\|_{L^p(\mathbb{R}^2)} \leq C(p)\|\Delta(\rho\Psi)\|_{L^p(\mathbb{R}^2)} \\ &\leq C(p,r,s)(\|\Delta\Psi\|_{L^p(B_s^2)} + \|\nabla\Psi\|_{L^p(B_s^2)} + \|\Psi\|_{L^p(B_s^2)}) \end{aligned} \tag{A.3}$$

for all $1 < p < \infty$. Setting $t := \frac{1+\tau}{2}$ and applying (A.3) with $p := 2$, $r := t$ and $s := 1$ we get

$$\|\nabla^2\Psi\|_{L^2(B_t^2)} \leq C(\tau)(\|\Delta\Psi\|_{L^2(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}),$$

hence $\|\Psi\|_{W^{2,2}(B_t^2)}$ is bounded by the desired quantity. Using Sobolev's embedding $W^{2,2}(B_t^2) \hookrightarrow W^{1,4}(B_t^2)$ and (A.3) with $p := 4$, $r := \tau$ and $s := t$, we obtain

$$\begin{aligned} \|\Psi\|_{W^{2,4}(B_\tau^2)} &\leq C(\tau)(\|\Delta\Psi\|_{L^4(B_t^2)} + \|\Psi\|_{W^{2,2}(B_t^2)}) \\ &\leq C(\tau)(\|\Delta\Psi\|_{L^4(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}). \end{aligned}$$ □

Lemma A.3. *Given a sequence $\psi_k : \mathbb{C} \to \mathbb{C}$ of K-quasiconformal homeomorphisms with the normalization conditions*

$$\psi_k(0) = 0, \qquad \psi_k(1) = 1,$$

there exists a K-quasiconformal homeomorphism $\psi_\infty : \mathbb{C} \to \mathbb{C}$ satisfying the same normalization condition and such that, up to subsequences, $\psi_k \to \psi_\infty$ and $\psi_k^{-1} \to \psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

Proof. Let $\mu_k \in \mathcal{E}_K$ be defined by $\partial_{\overline{z}}\psi_k = \mu_k\partial_z\psi_k$.[12] The existence and uniqueness of a K-quasiconformal homeomorphism satisfying this equation and the normalization conditions are shown in [56, Theorem 4.30].

Given $M > 0$, we consider the set $\mathcal{E}_K^M := \{\mu \in \mathcal{E}_K : \mu = 0 \text{ a.e. on } \mathbb{C} \setminus B_M^2\}$. If F^μ denotes the normal solution to the equation $\partial_{\overline{z}}F^\mu = \mu\partial_z F^\mu$ (in the sense of [56, Theorem 4.24]), then F^μ satisfies estimates (4.21) and (4.24) from [56]. Applying them with the points 0 and 1, we infer that also the map $f^\mu := F^\mu(1)^{-1}F^\mu$ satisfies estimates of the form

$$|f^\mu(z_1) - f^\mu(z_2)| \leq C|z_1 - z_2|^\alpha + C|z_1 - z_2|, \tag{A.4}$$

$$|z_1 - z_2| \leq C|f^\mu(z_1) - f^\mu(z_2)|^\alpha + C|f^\mu(z_1) - f^\mu(z_2)|, \tag{A.5}$$

with C and $0 < \alpha < 1$ depending only on K and M. Given a sequence of homeomorphisms $f_k : \mathbb{C} \to \mathbb{C}$ satisfying these estimates, the Ascoli–Arzelà theorem applies to f_k and f_k^{-1} and so we can extract a subsequence (not relabeled) such that

$$f_k \to f_\infty, \quad f_k^{-1} \to \widetilde{f}_\infty \quad \text{in } C^0_{loc}(\mathbb{C}).$$

[12] Actually, the coefficient μ_k is uniquely determined a.e., as $\partial_z\psi_k \neq 0$ a.e. (this follows from $\mathrm{id}_{\mathbb{C}} = \psi_k^{-1} \circ \psi_k$ and the chain rule [65, Lemma III.6.4], together with $|\partial_{\overline{z}}\psi_k| \leq |\partial_z\psi_k|$).

From $f_k^{-1} \circ f_k = f_k \circ f_k^{-1} = \mathrm{id}_{\mathbb{C}}$ we get $\widetilde{f}_\infty \circ f_\infty = f_\infty \circ \widetilde{f}_\infty = \mathrm{id}_{\mathbb{C}}$ and thus $f_\infty : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, with $\widetilde{f}_\infty = f_\infty^{-1}$. Also, since $f_k(z), f_k^{-1}(z) \to \infty$ uniformly as $z \to \infty$, we deduce that the canonical extensions $\widehat{f}_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ converge uniformly to $\widehat{f}_\infty$ and that the same holds for $\widehat{f}_k^{-1}$.

We now closely examine the proof of [56, Theorem 4.30]: let $\widetilde{\mu}_k \in \mathcal{E}_K^1$ be given by equation (4.25) in [56], with $\mu_k \mathbf{1}_{\mathbb{C} \setminus B_1^2}$ in place of μ, and

$$g_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \qquad g_k(z) := \widehat{f^{\widetilde{\mu}_k}}(z^{-1})^{-1}.$$

This map corresponds to the map f^{μ_1} in the aforementioned proof (with μ_k in place of μ). The lower bound (A.5), applied with $f^{\widetilde{\mu}_k}$ and $z_1 := z^{-1}$, $z_2 := 0$, shows that $|g_k(z)|$ is bounded above by some M', for all k and all $z \in \overline{B}_1^2$. Hence, defining $\mu_{k,2}$ as in equation (4.27) in [56] (with μ_k in place of μ), we get $\mu_{k,2} \in \mathcal{E}_K^{M'}$. Calling $h_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the associated quasiconformal homeomorphism, normalized so that $h_k(0) = 0$ and $h_k(1) = 1$, by the above argument (with $M := M'$) we obtain the uniform convergence

$$g_k \to g_\infty, \quad g_k^{-1} \to g_\infty^{-1}, \quad h_k \to h_\infty, \quad h_k^{-1} \to h_\infty^{-1}$$

up to subsequences, for suitable homeomorphisms g_∞ and h_∞ of the Riemann sphere $\widehat{\mathbb{C}}$. Setting $\psi_\infty := h_\infty \circ g_\infty\big|_{\mathbb{C}}$ and observing that $\psi_k = h_k \circ g_k\big|_{\mathbb{C}}$, we get the desired convergence $\psi_k \to \psi_\infty$ and $\psi_k^{-1} \to \psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

Finally, we show that ψ_∞ is a K-quasiconformal homeomorphism. Given an open rectangle $R \subset\subset \mathbb{C}$, [56, Lemma 4.12] gives

$$\mathcal{L}^2(\psi_k(R)) = \int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \geq \int_R (1 - k^2)|\partial_z \psi_k|^2 \geq (1 - k^2)k^{-2} \int_R |\partial_{\bar{z}} \psi_k|^2,$$

where $k := \frac{K-1}{K+1}$. Since $\limsup_{k\to\infty} \mathcal{L}^2(\psi_k(R)) \leq \mathcal{L}^2(\psi_\infty(\overline{R}))$, we deduce that ψ_k is bounded in $W^{1,2}(R)$; thus, ψ_∞ is the limit of ψ_k in the weak $W^{1,2}_{loc}(\mathbb{C})$-topology. Given $\rho, \psi^1, \psi^2 \in C_c^\infty(\mathbb{C})$, integration by parts shows that

$$\int_{\mathbb{C}} \rho(\partial_1 \psi^1 \partial_2 \psi^2 - \partial_2 \psi^1 \partial_1 \psi^2) = -\int_{\mathbb{C}} (\partial_1 \rho \psi^1 \partial_2 \psi^2 - \partial_2 \rho \psi^1 \partial_1 \psi^2). \tag{A.6}$$

By writing $\psi_k = \psi_k^1 + i\psi_k^2$, a standard density argument shows that (A.6) still holds with ψ^1, ψ^2 replaced by ψ_k^1, ψ_k^2, for $k \in \mathbb{N} \cup \{\infty\}$. Hence, observing that $|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2 = (\partial_1 \psi_k^1 \partial_2 \psi_k^2 - \partial_2 \psi_k^1 \partial_1 \psi_k^2)$, we get

$$\int_{\mathbb{C}} \rho(|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \to \int_{\mathbb{C}} \rho(|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2). \tag{A.7}$$

Defining the positive measures $\nu_k := (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2)\mathcal{L}^2$, up to further subsequences we can assume that $\nu_k \rightharpoonup \nu_\infty$ as Radon measures. For any rectangle R such that $\nu_\infty(\partial R) = 0$, approximating $\mathbf{1}_R$ from above and below with smooth functions and applying (A.7) we get

$$\int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \to \int_R (|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2).$$

By monotonicity of both sides, this actually holds for every rectangle R. On the other hand, by lower semicontinuity of the L^2-norm,

$$\begin{aligned}\int_R (1-k^2)|\partial_z\psi_\infty|^2 &\le \liminf_{k\to\infty}\int_R (1-k^2)|\partial_z\psi_k|^2 \le \lim_{k\to\infty}\int_R (|\partial_z\psi_k|^2-|\partial_{\bar z}\psi_k|^2)\\ &= \int_R (|\partial_z\psi_\infty|^2-|\partial_{\bar z}\psi_\infty|^2).\end{aligned}$$

Since R is arbitrary, we get $|\partial_{\bar z}\psi_\infty|\le k|\partial_z\psi_\infty|$ a.e., as desired. □

Corollary A.4. *Given a sequence $\varphi_k\in\mathcal{D}_K$, there exists $\varphi_\infty\in\mathcal{D}_K$ such that, up to subsequences, $\varphi_k\to\varphi_\infty$ and $\varphi_k^{-1}\to\varphi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.*

Proof. Let $\mu_k\in\mathcal{E}_K$ be defined by $\partial_{\bar z}\varphi_k=\mu_k\partial_z\varphi_k$ for all k and let $\psi_k:\mathbb{C}\to\mathbb{C}$ be the unique K-quasiconformal homeomorphism satisfying the same differential equation, as well as $\psi_k(0)=0$, $\psi_k(1)=1$ (see [56, Theorem 4.30]).

By Lemma A.3, up to subsequences there exists a K-quasiconformal homeomorphism ψ_∞ such that $\psi_k\to\psi_\infty$ and $\psi_k^{-1}\to\psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

By the chain rule (see [65, Lemma III.6.4]), the map $\psi_k\circ\varphi_k^{-1}:\mathbb{C}\to\mathbb{C}$ is a biholomorphism and fixes the origin, so it equals the multiplication by a nonzero complex number λ_k, i.e. $\psi_k=\lambda_k\varphi_k$. On the other hand,

$$|\lambda_k|=\min_{x\in\partial B_1^2}|\psi_k(x)|\to\min_{x\in\partial B_1^2}|\psi_\infty(x)|\in(0,\infty).$$

Hence, up to further subsequences we can suppose that $\lambda_k\to\lambda_\infty\in\mathbb{C}\setminus\{0\}$. The statement follows with $\varphi_\infty:=\lambda_\infty^{-1}\psi_\infty$. □

Remark A.5. In general, given $\varphi_k\in\mathcal{D}_K$ (for $k\in\mathbb{N}\cup\{\infty\}$) with $\varphi_k\to\varphi_\infty$ and $\varphi_k^{-1}\to\varphi_\infty^{-1}$ locally uniformly, it is not true that the corresponding Beltrami coefficients satisfy $\mu_k\rightharpoonup\mu_\infty$ in $L^\infty(\mathbb{C})$. For instance, let $\mu_0(z):=\frac12$ if $\Re(z)\in\bigcup_{n\in\mathbb{Z}}\left[n,n+\frac12\right)$ and $\mu_0(z):=-\frac12$ otherwise. Then the bi-Lipschitz homeomorphism $\psi_0:\mathbb{C}\to\mathbb{C}$ given by

$$\psi_0(x+iy):=\begin{cases} n+\frac95(x-n)+\frac35 iy = n+\frac65(z-n)+\frac35(\bar z-n) & n\le x\le n+\frac12\\ n+\frac45+\frac{x-n}{5}+\frac35 iy = n+\frac45+\frac25(z-n)-\frac15(\bar z-n) & n+\frac12\le x\le n+1\end{cases}$$

satisfies $\partial_{\bar z}\psi_0=\mu_0\partial_z\psi_0$, with the normalization $\psi_0(0)=0$ and $\psi_0(1)=1$. So $\mu_k:=\mu_0(2^k\cdot)$ and $\psi_k:=2^{-k}\psi_0(2^k\cdot)$ satisfy $\partial_{\bar z}\psi_k=\mu_k\partial_z\psi_k$ with the same normalization. Moreover, they converge uniformly to $\psi_\infty(x+iy)=x+\frac35 iy=\frac45 z+\frac15\bar z$, together with their inverses. The homeomorphism ψ_∞ satisfies $\partial_{\bar z}\psi_\infty=\mu_\infty\partial_z\psi_\infty$ with $\mu_\infty:=\frac14$, but $\mu_k\rightharpoonup 0$. Dividing each ψ_k by $\min_{|z|=1}|\psi_k(z)|$, we obtain a counterexample in the class $\mathcal{D}_3$.

5 Codimension two minimal submanifolds from Yang–Mills–Higgs

5.1 Introduction

As already mentioned in the first chapter, a "level set" approach for the variational construction of minimal hypersurfaces was born from the work of Modica–Mortola [80], Modica [79], and Sternberg [103]. Starting from a suggestion by De Giorgi [26], they highlighted a deep connection between minimizers $u_\varepsilon : \mathcal{M} \to \mathbb{R}$ of the Allen–Cahn functional

$$F_\varepsilon(v) := \int_{\mathcal{M}} \Big(\varepsilon |dv|^2 + \frac{1}{4\varepsilon}(1 - v^2)^2\Big),$$

and two-sided minimal hypersurfaces in $\mathcal{M}$, showing essentially that the functionals F_ε Γ-converge to ($\frac{4}{3}$ times) the perimeter functional on Caccioppoli sets. Several years later, Hutchinson and Tonegawa [55] initiated the asymptotic study of critical points v_ε of F_ε with bounded energy, without the energy-minimality assumption. They showed, in particular, that their energy measures concentrate along a stationary, integral $(m-1)$-varifold, given by the limit of the level sets $v_\varepsilon^{-1}(0)$.

These developments, together with the deep regularity work by Tonegawa and Wickramasekera on stable solutions [108], opened the doors to a fruitful min-max approach to the construction of minimal hypersurfaces, providing a PDE alternative to the rather involved discretized min-max procedure implemented by Almgren and Pitts.

The initial motivation for the content of this chapter is to find, in a similar vein, a natural way to construct minimal varieties of codimension two through PDE methods. Recently, other attempts in this direction have been made by Cheng [21] and the second-named author [102], based on the study of the Ginzburg–Landau functionals

$$F_\varepsilon(v) := \frac{1}{|\log \varepsilon|} \int_{\mathcal{M}} \Big(|dv|^2 + \frac{1}{4\varepsilon^2}(1 - |v|^2)^2\Big)$$

on complex-valued maps $v : \mathcal{M} \to \mathbb{C}$. As discussed in the first chapter, this approach produces nontrivial stationary *rectifiable* $(m-2)$-varifolds, but it is not yet known whether a varifold produced in this way is always integral, nor whether the energy measures of min-max critical points concentrate along its support in the case $b_1(\mathcal{M}) \neq 0$.

In the present chapter, we consider instead the self-dual Yang–Mills–Higgs energy

$$E(u,\nabla) := \int_{\mathcal{M}} \Big(|\nabla u|^2 + |F_\nabla|^2 + W(u)\Big) \tag{5.1.1}$$

and its rescalings (for $\varepsilon \in (0,1]$)

$$E_\varepsilon(u,\nabla) := \int_{\mathcal{M}} \Big(|\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \varepsilon^{-2} W(u)\Big), \tag{5.1.2}$$

for couples (u,∇) consisting of a section u of a given Hermitian line bundle $L \to \mathcal{M}$, and a metric connection ∇ on L. Here, the nonlinear potential $W : L \to \mathbb{R}$ is given by

$$W(u) := \frac{1}{4}(1-|u|^2)^2, \tag{5.1.3}$$

while $F_\nabla \in \Omega^2(\mathrm{End}(L))$ denotes the curvature of ∇.

For the trivial bundle $L = \mathbb{C} \times \mathbb{R}^2$ on the plane $\mathcal{M} = \mathbb{R}^2$, a detailed study of the functional (5.1.1) and its critical points can be found in the doctoral work of Taubes [105, 106]. In [106], all finite-energy critical points (u,∇) of (5.1.1) in the plane are shown to solve the first order system[1]

$$\nabla_{\partial_1} u \pm i \nabla_{\partial_2} u = 0; \quad *F_\nabla = \pm\frac{1}{2}(1-|u|^2) \tag{5.1.4}$$

known as the *vortex equations*—a two-dimensional counterpart of the instanton equations in four-dimensional Yang–Mills theory. In particular, all such solutions (u,∇) minimize energy among pairs (u,∇) with fixed vortex number

$$N := \frac{1}{2\pi}\int_{\mathbb{R}^2} *F_\nabla \in \mathbb{Z},$$

and carry energy exactly $E(u,\nabla) = 2\pi|N|$. In [105], Taubes shows moreover that there exist solutions of (5.1.4) with any prescribed zero set

$$u^{-1}(0) = \{z_1,\dots,z_N\} \subset \mathbb{R}^2,$$

which are unique up to gauge equivalence, so that [105] and [106] together give a complete classification of finite-energy critical points of (5.1.1) in the plane.

In [53], Hong, Jost, and Struwe initiate the study of the rescaled functionals (5.1.2) in the limit $\varepsilon \to 0$ for line bundles $L \to \Sigma$ over a closed Riemann surface Σ. The main result of [53] shows that, for solutions $(u_\varepsilon, \nabla_\varepsilon)$ of the rescaled vortex equations (given by replacing $\frac{1}{2}(1-|u|^2)$ with $\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)$ in (5.1.4)), the curvature $*\frac{1}{2\pi}F_{\nabla_\varepsilon}$ converges as $\varepsilon \to 0$ to a finite sum of Dirac masses of total mass $|\deg(L)|$, away from which ∇_ε converges to a flat connection ∇_0, and u_ε to a unit section u_0 with $\nabla_0 u_0 = 0$, up to change of gauge. While the authors of [53] focus on the vortex equations over Riemann surfaces, they suggest that the asymptotic analysis of the rescaled functionals E_ε may also yield interesting results in

[1] Here and elsewhere, we implicitly identify F_∇ with the two-form ω given by $F_\nabla(X,Y) = -i\omega(X,Y)$.

higher dimension, pointing to similarities with the Allen–Cahn functionals for scalar-valued functions.

In the present chapter, we develop the asymptotic analysis as $\varepsilon \to 0$ for critical points of E_ε associated to line bundles $L \to \mathcal{M}$ over Riemannian manifolds $\mathcal{M}^m$ of arbitrary dimension $m \geq 2$. The bulk of the work is devoted to the proof of the following theorem, which describes the limiting behavior as $\varepsilon \to 0$ of the energy measures

$$\mu_\varepsilon := \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \operatorname{vol}_g$$

and curvatures F_{∇_ε} for critical points $(u_\varepsilon, \nabla_\varepsilon)$ satisfying a uniform energy bound.

Theorem 5.1.1. *Let $L \to \mathcal{M}$ be a Hermitian line bundle over a closed, oriented Riemannian manifold $\mathcal{M}^m$ of dimension $m \geq 2$, and let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of critical points for E_ε satisfying a uniform energy bound*

$$E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda < \infty.$$

Then, as $\varepsilon \to 0$, the energy measures

$$\mu_\varepsilon := \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \operatorname{vol}_g$$

converge subsequentially, in duality with $C^0(\mathcal{M})$, to the weight measure μ of a stationary, integral $(m-2)$-varifold V. Also, for all $0 \leq \delta < 1$,

$$\operatorname{spt}(\mu) = \lim_{\varepsilon \to 0} \{|u_\varepsilon| \leq \delta\}$$

in the Hausdorff topology. The $(m-2)$-currents dual to the curvature forms $\frac{1}{2\pi} F_{\nabla_\varepsilon}$ converge subsequentially to an integral $(m-2)$-cycle Γ, with $|\Gamma| \leq \mu$.

As will be clear from the proofs, orientability will be assumed only to show the statement concerning the current Γ.

Remark 5.1.2. We warn the reader that, while the qualitative analysis of the Allen–Cahn functionals does not depend on the precise choice of the double-well potential W, the analysis of the abelian Yang–Mills–Higgs functionals (5.1.1)–(5.1.2) seems to depend *quite strongly* on the choice $W(u) = \frac{1}{4}(1 - |u|^2)^2$. Indeed, already in two dimensions, replacing W with a potential $W_\lambda(u) := \frac{\lambda}{4}(1 - |u|^2)^2$ for some $\lambda \neq 1$ yields a dramatically different qualitative behavior, breaking the symmetry which leads to the first-order equations (5.1.4), and introducing interactions between disjoint components of the zero set (see, e.g., [59, Chapters I–III]). This should serve as one indication that the analysis of the abelian Higgs model is somewhat more delicate than that of related semilinear scalar equations, in spite of the strong parallels.

Of course, the results of Theorem 5.1.1 would be of limited interest if nontrivial critical points $(u_\varepsilon, \nabla_\varepsilon)$ could be found only in a few special settings. After completing the proof of

Theorem 5.1.1, we therefore establish the following general existence result, showing that nontrivial families satisfying the hypotheses of Theorem 5.1.1 arise naturally on any line bundle.

Theorem 5.1.3. *For any Hermitian line bundle $L \to \mathcal{M}$ over an arbitrary closed base manifold $\mathcal{M}^m$, there exists a family $(u_\varepsilon, \nabla_\varepsilon)$ satisfying the hypotheses of Theorem 5.1.1, with nonempty zero sets $u_\varepsilon^{-1}(0) \neq \varnothing$. In particular, the energy μ_ε of these families concentrates (subsequentially) on a nontrivial stationary integral $(m-2)$-varifold V as $\varepsilon \to 0$.*

Remark 5.1.4. We remark that a very special class of minimizers for E_ε are given by solutions $(u_\varepsilon, \nabla_\varepsilon)$ of the first-order vortex equations in Kähler manifolds $(\mathcal{M}^{2m}, \omega_K)$ of higher dimension; these generalize the system (5.1.4) from the two-dimensional setting by replacing $*F_\nabla$ in (5.1.4) by the inner product $\langle F_\nabla, \omega_K\rangle$ with the Kähler form ω_K, and requiring additionally that $F_\nabla^{0,2} = 0$. As in the two-dimensional setting, solutions of this first-order system minimize the energy E_ε in appropriate line bundles on Kähler manifolds, and it was shown by Bradlow[2] [17] that the moduli space of solutions corresponds to the space of complex subvarieties in $\mathcal{M}$ (of complex codimension one) via the zero locus $(u_\varepsilon, \nabla_\varepsilon) \mapsto u_\varepsilon^{-1}(0)$.

In particular, the zero loci $u_\varepsilon^{-1}(0)$ in this case are already area-minimizing subvarieties, before passing to the limit $\varepsilon \to 0$. Note that the analysis of the vortex equations plays a key role in the study of Seiberg–Witten invariants of Kähler surfaces [110], and a similar analysis figures crucially into Taubes's work relating the Seiberg–Witten and Gromov–Witten invariants of symplectic four-manifolds [107]. For a concise introduction to the higher-dimensional vortex equations and connections to Seiberg–Witten theory, we refer the interested reader to the survey [39] by García-Prada.

As an application of Theorem 5.1.3, we obtain a new proof of the existence of stationary integral $(m-2)$-varifolds in an arbitrary Riemannian manifold—a result first proved by Almgren in 1965 [7] using his powerful, but rather involved geometric measure theory framework.

Organization of the chapter

In Section 5.2 we fix notation and record some basic properties satisfied by critical pairs (u, ∇) for the energies E_ε.

In Section 5.3, we record some useful Bochner identities for the gauge-invariant quantities $|u|^2$, $|F_\nabla|^2$, and $|\nabla u|^2$, and use them to establish an initial rough estimate on $\xi_\varepsilon := \varepsilon|F_\nabla| - \frac{(1-|u|^2)}{2\varepsilon}$, whose role should be compared to that of the *discrepancy function* in the Allen–Cahn setting. Under suitable assumptions on the curvature of $\mathcal{M}$, the fact that $\xi_\varepsilon \leq 0$ follows quickly from the aforementioned Bochner identities and the maximum

[2] The precise form of the energies considered by Bradlow in [17] differs slightly from the functionals E_ε considered here, but the analysis is essentially the same.

principle. Without the curvature assumptions, some nontrivial additional work is required to obtain the pointwise upper bound $\xi_\varepsilon \le C(\mathcal{M}, E_\varepsilon(u,\nabla))$. This estimate is the key ingredient to obtain the sharp $(m-2)$-monotonicity of the energy, and relies on the specific choice of coupling constants appearing in the *self-dual* Yang–Mills–Higgs functionals.

In Section 5.4 we derive the stationarity equation for inner variations, from which an obvious $(m-4)$-monotonicity property of the energy follows rather immediately. Using our rough initial bounds on ξ_ε from Section 5.3, we deduce an intermediate $(m-3)$-monotonicity; we use this to reach the pointwise bound $\xi_\varepsilon \le C(\mathcal{M}, E_\varepsilon(u,\nabla))$, from which we finally infer the sharp $(m-2)$-monotonicity.

In Section 5.5 we show that, similar to the Allen–Cahn setting, the energy density $e_\varepsilon(u,\nabla)$ decays exponentially away from the set $u^{-1}(0)$—more precisely, away from $\{|u|^2 \ge 1-\beta_D\}$ for some β_D independent of ε.

Section 5.6, which constitutes the main part of the chapter, contains an initial description of the limiting varifold, showing that it is stationary, $(m-2)$-rectifiable, and has a lower density bound on the support. Then we establish its integrality with a blow-up analysis, employing the estimates from the preceding sections to reduce the problem to a statement for entire planar solutions, already contained in the work of Jaffe and Taubes [59]. We then use this analysis to show that the level sets $u_\varepsilon^{-1}(0)$ converge to the support of V in the Hausdorff topology, and conclude the section with a discussion of the asymptotics for the curvature forms $\frac{1}{2\pi}F_{\nabla_\varepsilon}$.

In Section 5.7, we show that E_ε satisfies a variant of the Palais–Smale property on suitable function spaces, allowing us to produce critical points via classical min-max methods. We provide a variational construction to get nontrivial critical points satisfying the assumptions of our main theorem, with energy bounded from above and below, both for nontrivial and trivial line bundles.

Finally, the appendix addresses the issue of showing regularity of critical points, as obtained from Section 5.7, when they are read in a local or global Coulomb gauge.

5.2 The Yang–Mills–Higgs equations on $U(1)$ bundles

Let $\mathcal{M}$ be a closed, oriented Riemannian manifold, and let $L \to \mathcal{M}^m$ be a complex line bundle over $\mathcal{M}$, endowed with a Hermitian structure $\langle\cdot,\cdot\rangle$. Denote by $W : L \to \mathbb{R}$ the nonlinear potential

$$W(u) := \frac{1}{4}(1-|u|^2)^2.$$

For a Hermitian connection ∇ on L, a section $u \in \Gamma(L)$ and a parameter $\varepsilon > 0$, denote by $E_\varepsilon(u,\nabla)$ the scaled Yang–Mills–Higgs energy

$$E_\varepsilon(u,\nabla) := \int_{\mathcal{M}} \Big(|\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \varepsilon^{-2}W(u)\Big), \tag{5.2.1}$$

where F_∇ is the curvature of ∇. Throughout, we will identify the curvature F_∇ with a closed real two-form ω via

$$F_\nabla(X,Y)u = [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u = -i\omega(X,Y)u. \tag{5.2.2}$$

In computing inner products for two-forms, we follow the convention

$$|\omega|^2 = \sum_{1\le j<k\le m} \omega(e_j,e_k)^2 = \frac{1}{2}\sum_{j,k=1}^{m} \omega(e_j,e_k)^2 \tag{5.2.3}$$

with respect to a local orthonormal basis $\{e_j\}_{j=1}^m$ for $T\mathcal{M}$.

Note that E_ε enjoys the $U(1)$ gauge invariance

$$E_\varepsilon(u,\nabla) = E_\varepsilon(e^{i\theta}u, \nabla - id\theta),$$

for any (smooth) $\theta : \mathcal{M} \to \mathbb{R}$. More generally, we have

$$E_\varepsilon(u,\nabla) = E_\varepsilon(\varphi u, \nabla - i\varphi^*(d\theta)),$$

for any $\varphi : \mathcal{M} \to S^1$, identifying S^1 with the unit circle in $\mathbb{C}$.

It is easy to check that the smooth pair (u,∇) gives a critical point for the energy E_ε, with respect to smooth variations, if and only if it satisfies the system

$$\nabla^*\nabla u = \frac{1}{2\varepsilon^2}(1-|u|^2)u, \tag{5.2.4}$$

$$\varepsilon^2 d^*\omega = \langle \nabla u, iu\rangle. \tag{5.2.5}$$

We denote $\Delta_H = dd^* + d^*d$ the usual positive definite Hodge Laplacian on differential forms and note that, in our convention, the adjoint to $d : \Omega^1(\mathcal{M}) \to \Omega^2(\mathcal{M})$ is

$$(d^*\omega)(e_k) = -\sum_{j=1}^{m} (D_{e_j}\omega)(e_j,e_k).$$

Since the curvature form ω is closed, taking the exterior derivative of (5.2.5) gives

$$\begin{aligned}
\varepsilon^2(\Delta_H\omega)(e_j,e_k) &= (d\langle\nabla u, iu\rangle)(e_j,e_k)\\
&= \langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle - \langle i\nabla_{e_k}u, \nabla_{e_j}u\rangle\\
&\quad + \langle iu, F_\nabla(e_j,e_k)u\rangle\\
&= \psi(u)(e_j,e_k) - |u|^2\omega(e_j,e_k);
\end{aligned}$$

i.e.,

$$\varepsilon^2\Delta_H\omega = -|u|^2\omega + \psi(u), \tag{5.2.6}$$

where

$$\psi(u)(e_j,e_k) := 2\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle.$$

For future reference, we record the simple bound

$$|\psi(u)| \leq |\nabla u|^2. \tag{5.2.7}$$

To confirm (5.2.7), fix $x \in \mathcal{M}$ and note that the linear map $\nabla u(x) : T_x\mathcal{M} \to L_x$ has a kernel of dimension at least $m-2$. Take an orthonormal basis $\{e_j\}$ of $T_x\mathcal{M}$ such that $e_j \in \ker \nabla u(x)$ for $j > 2$. We compute at x that

$$|\psi(u)| = 2|\langle i\nabla_{e_1}u, \nabla_{e_2}u\rangle| \leq 2|\nabla_{e_1}u||\nabla_{e_2}u| \leq |\nabla_{e_1}u|^2 + |\nabla_{e_2}u|^2,$$

which gives (5.2.7).

5.3 Bochner identities and preliminary estimates

From the equations (5.2.6) and (5.2.4), we apply the standard Bochner–Weitzenböck formulas to obtain some identities which will play a central role in our analysis. For the curvature two-form ω, it will be useful to record the Bochner identity

$$\Delta\frac{1}{2}|\omega|^2 = |D\omega|^2 + \varepsilon^{-2}(|u|^2|\omega|^2 - \langle\psi(u), \omega\rangle) + \mathcal{R}_2(\omega, \omega), \tag{5.3.1}$$

where D is the Levi–Civita connection and $\mathcal{R}_2$ denotes the Weitzenböck curvature operator for two-forms on the base Riemannian manifold $\mathcal{M}$. For any $\delta > 0$ we have

$$(|\omega|^2+\delta^2)^{1/2}\Delta(|\omega|^2+\delta^2)^{1/2} + |D|\omega||^2 \geq \Delta\frac{1}{2}(|\omega|^2+\delta^2) = \Delta\frac{1}{2}|\omega|^2.$$

Since $|D|\omega||^2 \leq |D\omega|^2$, (5.3.1) implies

$$(|\omega|^2+\delta^2)^{1/2}\Delta(|\omega|^2+\delta^2)^{1/2} \geq \varepsilon^{-2}(|u|^2|\omega|^2 - \langle\psi(u), \omega\rangle) + \mathcal{R}_2(\omega, \omega).$$

Dividing by $(|\omega|^2+\delta^2)^{1/2}$ and letting $\delta \to 0$, we obtain

$$\Delta|\omega| \geq \varepsilon^{-2}(|u|^2|\omega| - |\psi(u)|) - |\mathcal{R}_2^-||\omega|, \tag{5.3.2}$$

in the distributional sense (and classically on $\{|\omega| > 0\}$). Note that, by (5.2.7), the relation (5.3.2) also gives us the cruder subequation

$$\Delta|\omega| \geq \varepsilon^{-2}|u|^2|\omega| - \varepsilon^{-2}|\nabla u|^2 - |\mathcal{R}_2^-||\omega|. \tag{5.3.3}$$

For the modulus $|u|^2$ of the Higgs field u, we record

$$\Delta\frac{1}{2}|u|^2 = |\nabla u|^2 - \frac{1}{2\varepsilon^2}(1-|u|^2)|u|^2, \tag{5.3.4}$$

and observe that a simple application of the maximum principle yields the pointwise bound

$$|u|^2 \leq 1 \quad \text{on } \mathcal{M}.$$

For the energy density $|\nabla u|^2$ of the Higgs field u, we see that

$$\begin{aligned}
\Delta\frac{1}{2}|\nabla u|^2 &= |\nabla^2 u|^2 - \langle \nabla(\nabla^*\nabla u), \nabla u\rangle + \langle d^*\omega, \langle iu, \nabla u\rangle\rangle \\
&\quad - 2\langle \omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u) \\
&= |\nabla^2 u|^2 - 2\langle \omega, \psi(u)\rangle + \frac{1}{\varepsilon^2}|\langle iu, \nabla u\rangle|^2 \\
&\quad - \frac{1}{2\varepsilon^2}(1-|u|^2)|\nabla u|^2 + \frac{1}{\varepsilon^2}|\langle u, \nabla u\rangle|^2 + \mathcal{R}_1(\nabla u, \nabla u) \\
&= |\nabla^2 u|^2 + \frac{1}{2\varepsilon^2}(3|u|^2 - 1)|\nabla u|^2 - 2\langle \omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u),
\end{aligned}$$

where at $p \in \mathcal{M}$ we let $\mathcal{R}_1(\nabla u, \nabla u) = \mathrm{Ric}(e_i, e_j)\langle \nabla_{e_i} u, \nabla_{e_j} u\rangle$ and $\nabla^2_{e_i,e_j} u = \nabla_{e_i}(\nabla_{e_j} u)$, for any local orthonormal frame $\{e_i\}_{i=1}^m$ with $De_i(p) = 0$.

Next, we introduce the function

$$\xi_\varepsilon := \varepsilon|F_\nabla| - \frac{1}{2\varepsilon}(1-|u|^2), \tag{5.3.5}$$

and combine (5.3.3) with (5.3.4) to see that

$$\begin{aligned}
\Delta\xi_\varepsilon &\geq \varepsilon^{-1}|u|^2|\omega| - \varepsilon^{-1}|\nabla u|^2 - \varepsilon|\mathcal{R}_2^-||\omega| + \varepsilon^{-1}|\nabla u|^2 - \frac{1}{2\varepsilon^3}(1-|u|^2)|u|^2 \\
&\geq \varepsilon^{-2}|u|^2\xi_\varepsilon - \varepsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|.
\end{aligned}$$

If $\mathcal{R}_2 > 0$, we can actually replace the term $-\varepsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|$ with $c\varepsilon|\omega|$, for some positive constant $c = c(\mathcal{M})$; from a simple application of the maximum principle, in this case we get $\xi_\varepsilon \leq 0$ everywhere on $\mathcal{M}$, and consequently (cf. [59, Theorem III.8.1])

$$\varepsilon^2|F_\nabla|^2 \leq \frac{W(u)}{\varepsilon^2} \text{ pointwise, provided } \mathcal{R}_2 > 0 \text{ on } \mathcal{M}. \tag{5.3.6}$$

This balancing of the Yang–Mills and potential terms, which should be compared with Modica's gradient estimate in the asymptotic analysis of the Allen–Cahn equations (cf. [55, Proposition 3.3]), will play a key role in our analysis, allowing us to upgrade the obvious $(m-4)$-monotonicity typical of Yang–Mills–Higgs problems to the much stronger $(m-2)$-monotonicity $\frac{d}{dr}(r^{2-m}\int_{B_r} e_\varepsilon(u,\nabla)) \geq 0$.

Remark 5.3.1. We remark that the analog of the identity $\Delta\xi_\varepsilon \geq \varepsilon^{-2}|u|^2\xi_\varepsilon - \varepsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|$—and, consequently, the sharp $(m-2)$-monotonicity result—fails for choices of coupling constants other than those corresponding to the *self-dual* Yang–Mills–Higgs functionals considered here.

Without the positive curvature assumption, we may still employ the subequation

$$\Delta\xi_\varepsilon \geq \frac{|u|^2}{\varepsilon^2}\xi_\varepsilon - C(\mathcal{M})\varepsilon|F_\nabla|, \tag{5.3.7}$$

to obtain strong estimates for the positive part ξ_ε^+ of ξ_ε. To begin, denote by $G(x,y)$ the nonnegative Green's function for the Laplacian on $\mathcal{M}$, unique up to additive constant, so that $\Delta_x G(x,y) = \frac{1}{\mathrm{vol}(\mathcal{M})} - \delta_y$, and set

$$h_\varepsilon(x) := \int_{\mathcal{M}} G(x,y)\varepsilon|F_\nabla|(y)\,dy \geq 0, \tag{5.3.8}$$

so that
$$\Delta h_\varepsilon(x) = \frac{1}{\mathrm{vol}(\mathcal{M})}\|\varepsilon F_\nabla\|_{L^1} - \varepsilon|F_\nabla|(x). \tag{5.3.9}$$

Taking C' to be the constant appearing in (5.3.7), for the difference $\xi_\varepsilon - C'h_\varepsilon$ we then have

$$\begin{aligned}\Delta(\xi_\varepsilon - C'h_\varepsilon) &\geq \frac{|u|^2}{\varepsilon^2}(\xi_\varepsilon - C'h_\varepsilon) + C'\frac{|u|^2}{\varepsilon^2}h_\varepsilon - C'\frac{\|\varepsilon F_\nabla\|_{L^1}}{\mathrm{vol}(\mathcal{M})}\\ &\geq \frac{|u|^2}{\varepsilon^2}(\xi_\varepsilon - C'h_\varepsilon) - C'\frac{\|\varepsilon F_\nabla\|_{L^1}}{\mathrm{vol}(\mathcal{M})}.\end{aligned} \tag{5.3.10}$$

Observe that the L^1 norm of $\xi_\varepsilon - C'h_\varepsilon$ is bounded by the energy:

$$\begin{aligned}\|\xi_\varepsilon - C'h_\varepsilon\|_{L^1} &\leq \|\xi_\varepsilon\|_{L^1} + C(\mathcal{M})\|h_\varepsilon\|_{L^1}\\ &\leq \|\xi_\varepsilon\|_{L^1} + C(\mathcal{M})\|\varepsilon F_\nabla\|_{L^1}\\ &\leq C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}.\end{aligned} \tag{5.3.11}$$

(Where the constant $C(\mathcal{M})$ may of course change from line to line.)

Integrating (5.3.10) against the positive part $\zeta := (\xi_\varepsilon - C'h_\varepsilon)^+$ and bounding $\|\varepsilon F_\nabla\|_{L^1} \leq C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}$, we get

$$\begin{aligned}\int_{\mathcal{M}}|d\zeta|^2 &\leq -\int_{\mathcal{M}}\frac{|u|^2}{\varepsilon^2}\zeta^2 - C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}\int_{\mathcal{M}}\zeta\\ &\leq -C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}\int_{\mathcal{M}}\zeta.\end{aligned}$$

Applying (5.3.11), this gives $\|d\zeta\|_{L^2} \leq C(\mathcal{M})E_\varepsilon(u,\nabla)$.

Thus, applying Moser iteration, namely integrating (5.3.10) against powers ζ^γ with increasing exponents $\gamma > 1$, we deduce that

$$\xi_\varepsilon - C'h_\varepsilon \leq \zeta \leq C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}. \tag{5.3.12}$$

As a simple application of (5.3.12), we note that by definition (5.3.8) of h_ε and the standard estimate (see, e.g., [12, Section 4.2])

$$G(x,y) \leq C(\mathcal{M})d(x,y)^{2-m}$$

if $m \geq 3$ (or $G(x,y) \leq -C(\mathcal{M})\log(d(x,y)) + C(\mathcal{M})$ if $m = 2$), we have the L^∞ estimate

$$\|h_\varepsilon\|_{L^\infty} \leq C(\mathcal{M})\|\varepsilon F_\nabla\|_{L^{m-1}}$$

(with 2 replacing $m-1$ when $m = 2$). If $m = 2$, this inequality and (5.3.12) give a pointwise bound

$$\|\xi_\varepsilon^+\|_{L^\infty} \leq C(\mathcal{M})\|\varepsilon F_\nabla\|_{L^2} + C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2} \leq C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}.$$

In the sequel, we assume $m \geq 3$ and aim for a similar pointwise bound. We have

$$\|h_\varepsilon\|_{L^\infty} \leq C(\mathcal{M})\|\varepsilon F_\nabla\|_{L^{m-1}} \leq C\varepsilon\|F_\nabla\|_{L^\infty}^{\frac{m-3}{m-1}}\|F_\nabla\|_{L^2}^{\frac{2}{m-1}}.$$

Using this in (5.3.12), we compute at a maximum point for $|F_\nabla|$ to see that

$$\|\varepsilon F_\nabla\|_{L^\infty} - \frac{1}{2\varepsilon}(1-|u|^2) = \xi_\varepsilon \leq C\|\varepsilon F_\nabla\|_{L^\infty}^{\frac{m-3}{m-1}} E_\varepsilon(u,\nabla)^{\frac{1}{m-1}} + CE_\varepsilon(u,\nabla)^{1/2},$$

and, by an application of Young's inequality, it follows that

$$(1-C\delta)\|\varepsilon F_\nabla\|_{L^\infty} \leq \frac{1}{2\varepsilon} + C\delta^{\frac{3-m}{2}} E_\varepsilon(u,\nabla)^{1/2}$$

for any $\delta \in (0,1)$. Taking $\delta = \varepsilon^{2/m}$, we arrive at the crude preliminary estimate

$$\begin{aligned}\|\varepsilon F_\nabla\|_{L^\infty} &\leq \frac{1}{1-C\varepsilon^{2/m}}\Big(\frac{1}{2\varepsilon} + C\varepsilon^{3/m}\varepsilon^{-1}E_\varepsilon(u,\nabla)^{1/2}\Big)\\ &\leq \frac{1}{2\varepsilon} + \frac{\alpha(\varepsilon)}{2\varepsilon}(1+E_\varepsilon(u,\nabla)^{1/2}),\end{aligned}$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, consider the function

$$f := \varepsilon|\omega| - \frac{1+\alpha(\varepsilon)(1+E_\varepsilon(u,\nabla)^{1/2})}{2\varepsilon}(1-|u|^2).$$

By virtue of the preceding estimate for $\|F_\nabla\|_{L^\infty}$, we then see that

$$f \leq \frac{1+\alpha(\varepsilon)(1+E_\varepsilon(u,\nabla)^{1/2})}{2\varepsilon}|u|^2$$

pointwise. Appealing once again to (5.3.4) and (5.3.3), we see that

$$\Delta f \geq \frac{|u|^2}{\varepsilon^2} f - C\varepsilon|F_\nabla|,$$

so at a point where f achieves its maximum we have

$$\frac{|u|^2}{\varepsilon^2} f \leq C\varepsilon|F_\nabla| \leq \frac{C(1+E_\varepsilon(u,\nabla)^{1/2})}{\varepsilon}.$$

On the other hand, we know that $|u|^2 \geq \frac{\varepsilon}{C(1+E_\varepsilon(u,\nabla)^{1/2})} f$ everywhere, so the preceding computations yield an estimate of the form

$$\frac{(\max f)^2}{\varepsilon} \leq \frac{C(\mathcal{M}, E_\varepsilon(u,\nabla))}{\varepsilon},$$

provided $\max f \geq 0$, and we deduce that $f \leq C(\mathcal{M}, E_\varepsilon(u,\nabla))$ everywhere. Putting all this together, we arrive at the following lemma.

Lemma 5.3.2. *Let (u,∇) solve (5.2.4) and (5.2.5) on a line bundle $L \to \mathcal{M}$, and suppose $E_\varepsilon(u,\nabla) \leq \Lambda$. Then there exist a constant $C(\mathcal{M},\Lambda)$ and a function $\alpha(\mathcal{M},\Lambda,\varepsilon)$, with $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that*

$$\xi_\varepsilon \leq \alpha(\varepsilon)\frac{(1-|u|^2)}{\varepsilon} + C. \tag{5.3.13}$$

In the next section, we will improve the rough preliminary estimate of Lemma 5.3.2 to a uniform pointwise bound of the form $\xi_\varepsilon \leq C(\mathcal{M},\Lambda)$, but this will require some additional effort.

5.4 Inner variations and improved monotonicity

In this section, we derive the inner variation equation for solutions of (5.2.4)–(5.2.5), and explore the scaling properties of the energy $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon)$ over balls of small radius. Under the assumption that the curvature operator $\mathcal{R}_2$ appearing in (5.3.3) is positive-definite (so that (5.3.6) holds), the analysis simplifies considerably, leading with little effort to the desired monotonicity of the $(m-2)$-energy density. Without this curvature assumption, more work is required, first building on the cruder estimates of the preceding section to obtain a uniform pointwise bound for ξ_ε.

Fixing notation, with respect to a local orthonormal basis $\{e_i\}$ for $T\mathcal{M}$, define the $(0,2)$-tensors $\nabla u^*\nabla u$ and $\omega^*\omega$ by

$$(\nabla u^*\nabla u)(e_i, e_j) := \langle \nabla_{e_i} u, \nabla_{e_j} u\rangle, \tag{5.4.1}$$

$$\omega^*\omega(e_i, e_j) := \sum_{k=1}^{m} \omega(e_i, e_k)\omega(e_j, e_k). \tag{5.4.2}$$

Note that $\operatorname{tr}(\nabla u^*\nabla u) = |\nabla u|^2$ and $\operatorname{tr}(\omega^*\omega) = 2|\omega|^2$. Denote by $e_\varepsilon(u, \nabla)$ the energy integrand

$$e_\varepsilon(u, \nabla) := |\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \frac{W(u)}{\varepsilon^2}.$$

The fact that $d\omega = 0$ reads

$$D\omega(e_i, e_j) = D_{e_i}\omega(\cdot, e_j) + D_{e_j}\omega(e_i, \cdot),$$

where D is the Levi–Civita connection of $\mathcal{M}$. Using this identity, it is straightforward to check that

$$\begin{aligned} de_\varepsilon(u, \nabla) &= 2\operatorname{div}(\nabla u^*\nabla u) + 2\langle \nabla u, \nabla^*\nabla u\rangle + d\frac{W(u)}{\varepsilon^2} \\ &\quad + 2\omega(\langle iu, \nabla u\rangle^\#, \cdot) + 2\varepsilon^2\operatorname{div}(\omega^*\omega) - 2\varepsilon^2\omega((d^*\omega)^\#, \cdot). \end{aligned}$$

In particular, defining the stress-energy tensor $T_\varepsilon(u, \nabla)$ by

$$T_\varepsilon(u, \nabla) := e_\varepsilon(u, \nabla)g - 2\nabla u^*\nabla u - 2\varepsilon^2\omega^*\omega, \tag{5.4.3}$$

for (u, ∇) solving (5.2.4) and (5.2.5) it follows that

$$\operatorname{div}(T_\varepsilon(u, \nabla)) = 0, \tag{5.4.4}$$

meaning that $\sum_i (D_{e_i}T_\varepsilon)(e_i, \cdot) = 0$. Integrating (5.4.4) against a vector field X on some domain $\Omega \subseteq \mathcal{M}$, we arrive at the usual inner-variation equation

$$\int_\Omega \langle T_\varepsilon(u, \nabla), DX\rangle = \int_{\partial\Omega} T_\varepsilon(u, \nabla)(X, \nu), \tag{5.4.5}$$

where we identify $T_\varepsilon(u,\nabla)$ with a $(1,1)$-tensor and denote by ν the outer unit normal to Ω. Taking $\Omega = B_r(p)$ to be a small geodesic ball of radius r about a point $p \in \mathcal{M}$, and taking $X = \operatorname{grad}(\frac{1}{2}d_p^2)$, where d_p is the distance function to p, (5.4.5) gives

$$\begin{aligned} r\int_{\partial B_r(p)} (e_\varepsilon(u,\nabla) - 2|\nabla_\nu u|^2 - 2\varepsilon^2|\iota_\nu\omega|^2) &= \int_{B_r(p)} \langle T_\varepsilon(u,\nabla), DX\rangle \\ &= \int_{B_r(p)} \langle T_\varepsilon(u), g\rangle + \int_{B_r(p)} \langle T_\varepsilon(u), DX - g\rangle \\ &= \int_{B_r(p)} (me_\varepsilon(u,\nabla) - 2|\nabla u|^2 - 4\varepsilon^2|F_\nabla|^2) \\ &\quad + \int_{B_r(p)} \langle T_\varepsilon(u), DX - g\rangle. \end{aligned}$$

Now, by the Hessian comparison theorem, we know that

$$|DX - g| \le C(\mathcal{M})d_p^2;$$

applying this in the relations above, we see that

$$\begin{aligned} r\int_{\partial B_r(p)} e_\varepsilon(u,\nabla) \ge{}& 2r\int_{\partial B_r(p)} (|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2) \\ &+ \int_{B_r(p)} \Big((m-2)|\nabla u|^2 + (m-4)\varepsilon^2|F_\nabla|^2 + m\frac{W(u)}{\varepsilon^2}\Big) \\ &- C'(\mathcal{M})r^2\int_{B_r(p)} e_\varepsilon(u,\nabla). \end{aligned}$$

Setting

$$f(p,r) := e^{C'r^2}\int_{B_r(p)} e_\varepsilon(u,\nabla), \tag{5.4.6}$$

it follows from the computations above (temporarily throwing out the additional nonnegative boundary terms) that

$$\frac{\partial f}{\partial r} \ge \frac{e^{C'r^2}}{r}\int_{B_r(p)} \Big((m-2)|\nabla u|^2 + (m-4)\varepsilon^2|F_\nabla|^2 + m\frac{W(u)}{\varepsilon^2}\Big). \tag{5.4.7}$$

At this point, one easily observes that the right-hand side of (5.4.7) is bounded below by $\frac{m-4}{r}f(p,r)$, to obtain the monotonicity of the $(m-4)$-energy density

$$\frac{\partial}{\partial r}(r^{4-m}f(p,r)) \ge 0.$$

For general Yang–Mills and Yang–Mills–Higgs problems, this codimension-four energy growth is well known to be sharp (cf., e.g., [99, 112]). For solutions of (5.2.4) and (5.2.5) on Hermitian line bundles, however, we show now that this can be improved to (near-) monotonicity of the $(m-2)$-density $r^{2-m}f(p,r)$ on small balls, which constitutes a key technical ingredient in the proof of Theorem 5.1.1.

To begin, we rearrange (5.4.7), to see that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq \frac{m-2}{r}f(r) + \frac{2e^{C'r^2}}{r}\int_{B_r(p)}\Big(\frac{W(u)}{\varepsilon^2} - \varepsilon^2|F_\nabla|^2\Big)\\ &= \frac{m-2}{r}f(r) - \frac{2e^{C'r^2}}{r}\int_{B_r(p)}\xi_\varepsilon\Big(\varepsilon|F_\nabla| + \frac{1}{2\varepsilon}(1-|u|^2)\Big),\end{aligned}$$

recalling the notation $\xi_\varepsilon := \varepsilon|F_\nabla| - \frac{1}{2\varepsilon}(1-|u|^2)$. Now, by Lemma 5.3.2, assuming $E_\varepsilon(u,\nabla) \leq \Lambda$, we have the pointwise bound

$$\begin{aligned}\xi_\varepsilon\Big(\varepsilon|F_\nabla| + \frac{1}{2\varepsilon}(1-|u|^2)\Big) &\leq 2\Big(C + \alpha(\varepsilon)\frac{1-|u|^2}{\varepsilon}\Big)e_\varepsilon(u,\nabla)^{1/2}\\ &\leq Ce_\varepsilon(u,\nabla)^{1/2} + C\alpha(\varepsilon)e_\varepsilon(u,\nabla).\end{aligned}$$

Applying this in our preceding computation for $\frac{\partial f}{\partial r}$, we deduce that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq \frac{m-2}{r}f(r) - \frac{e^{C'r^2}}{r}\int_{B_r(p)}Ce_\varepsilon(u,\nabla)^{1/2} - \alpha(\varepsilon)\frac{e^{C'r^2}}{r}\int_{B_r(p)}Ce_\varepsilon(u,\nabla)\\ &\geq \frac{m-2-C\alpha(\varepsilon)}{r}f(r) - \frac{e^{C'r^2}}{r}Cr^{m/2}\Big(\int_{B_r(p)}e_\varepsilon(u,\nabla)\Big)^{1/2}\\ &\geq \frac{m-2-C''\alpha(\varepsilon)}{r}f(r) - C''r^{m/2-1}f(r)^{1/2}\end{aligned}$$

for some constant $C''(\mathcal{M},\Lambda)$ and $0 < r < c(\mathcal{M})$. Taking ε sufficiently small, we arrive next at the following coarse estimate for the $(m-3)$-energy density, which we will then use to establish an improved bound for ξ_ε.

Lemma 5.4.1. *For $\varepsilon \leq \varepsilon_M(\mathcal{M},\Lambda)$ sufficiently small, we have a uniform bound*

$$\sup_{0<r<\mathrm{inj}(\mathcal{M})} r^{3-m}\int_{B_r(p)}e_\varepsilon(u,\nabla) \leq C(\mathcal{M},\Lambda). \tag{5.4.8}$$

Proof. The statement is trivial if $m = 2, 3$, so assume $m \geq 4$. In the preceding computation, take $\varepsilon \leq \varepsilon_M(\mathcal{M},\Lambda)$ sufficiently small that $C''\alpha(\varepsilon) < \frac{1}{2}$. Then the estimate gives

$$f'(r) \geq \frac{m-2-1/2}{r}f(r) - C''r^{m/2-1}f(r)^{1/2},$$

from which it follows that, for $0 < r < c(\mathcal{M})$,

$$\begin{aligned}\frac{d}{dr}(r^{3-m}f(r)) &\geq r^{3-m}f'(r) + (3-m)r^{2-m}f(r)\\ &\geq r^{2-m}\Big(\Big(m-\frac{5}{2}\Big)f(r) - Cr^{m/2}f(r)^{1/2} + (3-m)f(r)\Big)\\ &\geq r^{2-m}\Big(\frac{1}{2}f(r) - Cr^{m/2}f(r)^{1/2}\Big).\end{aligned}$$

If $r^{3-m}f(r)$ has a maximum in $(0, c(\mathcal{M}))$, it follows that $f(r) \leq Cr^{m/2}f(r)^{1/2}$ there, and therefore $r^{3-m}f(r) \leq Cr^3 \leq C$. Obviously the desired estimate holds at $r = 0$ and $r = c(\mathcal{M})$, so (5.4.8) follows. □

With Lemma 5.4.1 in hand, we can now improve the bounds of Lemma 5.3.2 to a uniform pointwise estimate, as follows.

Proposition 5.4.2. *Let (u,∇) solve (5.2.4)–(5.2.5) on a line bundle $L \to \mathcal{M}$, with the energy bound $E_\varepsilon(u,\nabla) \leq \Lambda$ and $\varepsilon \leq \varepsilon_M$. Then there is a constant $C(\mathcal{M},\Lambda)$ such that*

$$\xi_\varepsilon := \varepsilon|F_\nabla| - \frac{1}{2\varepsilon}(1-|u|^2) \leq C(\mathcal{M},\Lambda). \tag{5.4.9}$$

Proof. We can assume $m \geq 3$, as we already obtained the claim for $m = 2$ in Section 5.3. Recall from that section the function

$$h_\varepsilon(x) := \int_M G(x,y)\varepsilon|F_\nabla|(y)\,dy,$$

where G is the nonnegative Green's function on $\mathcal{M}$. As discussed in Section 5.3, we can deduce from (5.3.7) a pointwise estimate of the form

$$\xi_\varepsilon \leq C(\mathcal{M})h_\varepsilon + C(\mathcal{M})E_\varepsilon(u,\nabla)^{1/2}. \tag{5.4.10}$$

Thus, to arrive at the desired bound (5.4.9), it will suffice to establish a pointwise bound of the same form for h_ε.

To this end, recall again that $G(x,y) \leq C(\mathcal{M})d(x,y)^{2-m}$, so that by definition we have

$$\begin{aligned}
h_\varepsilon(x) &\leq C\int_{\mathcal{M}} d(x,y)^{2-m}\varepsilon|F_\nabla|(y)\,dy \\
&\leq C\int_{\mathcal{M}} d(x,y)^{2-m}e_\varepsilon(u,\nabla)^{1/2}(y)\,dy \\
&\leq C\int_{\mathcal{M}} (d(x,y)^{-m+1/2} + d(x,y)^{3-m+1/2}e_\varepsilon(u,\nabla))\,dy,
\end{aligned}$$

where the last line is a simple application of Young's inequality. Since the integral $\int_{\mathcal{M}} d(x,y)^{-m+1/2}\,dy$ is finite, it follows that

$$\begin{aligned}
h_\varepsilon(x) &\leq C(\mathcal{M}) + C(\mathcal{M})\Lambda + C(\mathcal{M})\int_0^{\mathrm{inj}(\mathcal{M})} r^{3-m+1/2}\Big(\int_{\partial B_r(x)} e_\varepsilon(u,\nabla)\Big)\,dr \\
&= C(\mathcal{M},\Lambda) + C(\mathcal{M})\int_0^{\mathrm{inj}(\mathcal{M})} \frac{d}{dr}\Big(r^{-m+7/2}\int_{B_r(x)} e_\varepsilon(u,\nabla)\Big)\,dr \\
&\quad + (m-7/2)C(\mathcal{M})\int_0^{\mathrm{inj}(\mathcal{M})} r^{3-m-1/2}\Big(\int_{B_r(x)} e_\varepsilon(u,\nabla)\Big)\,dr \\
&\leq C(\mathcal{M},\Lambda) + C(\mathcal{M})\int_0^{\mathrm{inj}(\mathcal{M})} r^{3-m-1/2}\Big(\int_{B_r(x)} e_\varepsilon(u,\nabla)\Big)\,dr.
\end{aligned}$$

On the other hand, by Lemma 5.4.1, we know that $r^{3-m}\int_{B_r(x)} e_\varepsilon(u,\nabla) \leq C(\mathcal{M},\Lambda)$ for every r, so we see finally that

$$h_\varepsilon(x) \leq C(\mathcal{M},\Lambda) + C(\mathcal{M},\Lambda)\int_0^{\mathrm{inj}(\mathcal{M})} r^{-1/2}\,dr \leq C(\mathcal{M},\Lambda),$$

as desired. □

Applying (5.4.9) in our original computation for $f'(r)$, we see now that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq \frac{m-2}{r} f(r) - \frac{2e^{C'r^2}}{r} \int_{B_r(p)} \xi_\varepsilon \Big(\varepsilon |F_\nabla| + \frac{1}{2\varepsilon}(1-|u|^2) \Big) \\ &\geq \frac{m-2}{r} f(r) - \frac{2e^{C'r^2}}{r} \int_{B_r(p)} C(\mathcal{M}, \Lambda) e_\varepsilon(u, \nabla)^{1/2} \\ &\geq \frac{m-2}{r} f(r) - C(\mathcal{M}, \Lambda) r^{\frac{m-2}{2}} f(r)^{1/2}.\end{aligned}$$

In fact, bringing in the extra boundary terms that we have been neglecting, and applying Young's inequality to the term $r^{\frac{m-2}{2}} f(r)^{1/2}$, we see that

$$\begin{aligned}\frac{\partial f}{\partial r} \geq\ & 2e^{C'r^2} \int_{\partial B_r(p)} (|\nabla_\nu u|^2 + \varepsilon^2 |\iota_\nu F_\nabla|^2) \\ &+ \frac{m-2}{r} f(r) - C r^{\frac{m-2}{2}} f(r)^{1/2} \\ \geq\ & 2e^{C'r^2} \int_{\partial B_r(p)} (|\nabla_\nu u|^2 + \varepsilon^2 |\iota_\nu F_\nabla|^2) \\ &+ \frac{m-2}{r} f(r) - C f(r) - C r^{m-2}.\end{aligned}$$

With this differential inequality in place, a straightforward computation leads us finally to one of our key technical theorems, the monotonicity formula for the $(m-2)$-density.

Theorem 5.4.3. *Let (u, ∇) solve (5.2.4)–(5.2.5) on a Hermitian line bundle $L \to \mathcal{M}$, with an energy bound $E_\varepsilon(u, \nabla) \leq \Lambda$. Then there exist positive constants $\varepsilon_M(\mathcal{M}, \Lambda)$ and $C_M(\mathcal{M}, \Lambda)$ such that the normalized energy density*

$$\widetilde{E}_\varepsilon(x, r) := e^{C_M r} r^{2-m} \int_{B_r(x)} e_\varepsilon(u, \nabla) \tag{5.4.11}$$

satisfies

$$\widetilde{E}_\varepsilon'(r) \geq 2r^{2-m} \int_{\partial B_r(x)} (|\nabla_\nu u|^2 + \varepsilon^2 |\iota_\nu F_\nabla|^2) - C_M, \tag{5.4.12}$$

for $0 < r < \mathrm{inj}(\mathcal{M})$ and $\varepsilon \leq \varepsilon_M$.

As a simple corollary of the monotonicity result (together with a pointwise bound for $|\nabla u|$ derived in the following section), we deduce that (u, ∇) must have positive $(m-2)$-energy density wherever $|u|$ is bounded away from 1.

Corollary 5.4.4 (clearing-out). *Let (u, ∇) solve (5.2.4)–(5.2.5) on a line bundle $L \to \mathcal{M}$, with $E_\varepsilon(u, \nabla) \leq \Lambda$ and $\varepsilon \leq \varepsilon_M$. Given $0 < \delta < 1$, if*

$$r^{2-m} \int_{B_r(x)} e_\varepsilon(u, \nabla) \leq \eta(\mathcal{M}, \Lambda, \delta)$$

with $x \in \mathcal{M}$ and $\varepsilon < r < \mathrm{inj}(\mathcal{M})$, then we must have $|u(x)| > 1 - \delta$.

Proof. For $\varepsilon \le \varepsilon_M$, Theorem 5.4.3 gives

$$\varepsilon^{2-m}\int_{B_\varepsilon(x)} e_\varepsilon(u,\nabla) \le C(\mathcal{M},\Lambda)\eta + C(\mathcal{M},\Lambda)r.$$

The gradient bound (5.5.3) in Proposition 5.5.1 of the following section gives $|d|u|| \le C\varepsilon^{-1}$. Hence, if $|u(x)| \le 1-\delta$ then $|u(y)| < 1-\frac{\delta}{2}$ on $B_{\varepsilon\delta/(2C)}(x)$, so that $1-|u(y)|^2 \ge 1-|u(y)| > \frac{\delta}{2}$. We deduce that

$$\frac{\delta^2}{16}\operatorname{vol}(B_{\varepsilon\delta/(2C)}(x)) \le \int_{B_\varepsilon(x)} W(u) \le \varepsilon^2\int_{B_\varepsilon(x)} e_\varepsilon(u,\nabla) \le C\varepsilon^m(\eta+r).$$

Since $\operatorname{vol}(B_{\varepsilon\delta/(2C)}(x))$ is bounded below by $c(\mathcal{M},\Lambda,\delta)\varepsilon^m$, we can choose $\widetilde{\eta}(\mathcal{M},\Lambda,\delta) \le \operatorname{inj}(\mathcal{M})$ so small that we get a contradiction if $r,\eta \le \widetilde{\eta}$. On the other hand, if $r > \widetilde{\eta}$ then

$$\widetilde{\eta}^{2-m}\int_{B_{\widetilde{\eta}}(x)} e_\varepsilon(u,\nabla) \le \Big(\frac{\operatorname{inj}(\mathcal{M})}{\widetilde{\eta}}\Big)^{m-2}\eta.$$

Hence, setting $\eta := \Big(\frac{\widetilde{\eta}}{\operatorname{inj}(\mathcal{M})}\Big)^{m-2}\widetilde{\eta} \le \widetilde{\eta}$, we can reduce to the previous case (replacing r with $\widetilde{\eta}$), reaching again a contradiction. □

5.5 Decay away from the zero set

Again, let (u,∇) solve (5.2.4)–(5.2.5) on a line bundle $L \to \mathcal{M}$, with the energy bound $E_\varepsilon(u,\nabla) \le \Lambda$. In the preceding section, we obtained the pointwise estimate

$$|F_\nabla| \le \frac{1}{2\varepsilon^2}(1-|u|^2) + \frac{1}{\varepsilon}C(\mathcal{M},\Lambda) \tag{5.5.1}$$

when $\varepsilon \le \varepsilon_M$. As a first step toward establishing strong decay of the energy away from the zero set of u, we show in the following proposition that the full energy density $e_\varepsilon(u,\nabla)$ is controlled by the potential $\frac{W(u)}{\varepsilon^2}$.

Proposition 5.5.1. *For (u,∇) as above, we have the pointwise estimates*

$$\varepsilon^2|F_\nabla|^2 \le C(\mathcal{M},\Lambda)\frac{W(u)}{\varepsilon^2} + C(\mathcal{M},\Lambda)\varepsilon \tag{5.5.2}$$

and

$$|\nabla u|^2 \le C(\mathcal{M},\Lambda)\frac{W(u)}{\varepsilon^2} + C(\mathcal{M},\Lambda)\varepsilon^2, \tag{5.5.3}$$

provided $\varepsilon \le \varepsilon_D$, for some $\varepsilon_D = \varepsilon_D(\mathcal{M},\Lambda)$.

Proof. To begin, let $C_1 = C_1(\mathcal{M},\Lambda)$ be the constant from (5.5.1), and consider the function

$$f := \varepsilon|F_\nabla| - \frac{1+2C_1\varepsilon}{2\varepsilon}(1-|u|^2) = \xi_\varepsilon - C_1 + C_1|u|^2.$$

Similar to the proof of Lemma 5.3.2, observe that $C_1|u|^2 \ge f$ pointwise, by (5.5.1), while the computations from Section 5.3 give

$$\Delta f \ge \frac{|u|^2}{\varepsilon^2}f - C'(\mathcal{M})\varepsilon|F_\nabla|.$$

By (5.5.1) we have $|F_\nabla| \le \frac{1}{2\varepsilon^2} + \frac{C_1}{\varepsilon}$, so at a positive maximum for f it follows that

$$0 \ge \frac{|u|^2}{\varepsilon^2} f - C'\varepsilon |F_\nabla| \ge \frac{f^2}{C_1\varepsilon^2} - \frac{C(\mathcal{M},\Lambda)}{\varepsilon},$$

so that

$$(\max f)^2 \le C\varepsilon$$

(provided $\max f \ge 0$), and consequently $f \le C\varepsilon^{1/2}$ everywhere. As a consequence, at any point, we have either $f < 0$, in which case

$$\varepsilon^2 |F_\nabla|^2 \le (1 + 2C_1\varepsilon)^2 \frac{W(u)}{\varepsilon^2},$$

or $f \ge 0$, in which case

$$\begin{aligned}\varepsilon^2 |F_\nabla|^2 &\le 2f^2 + 2(1 + 2C_1\varepsilon)^2 \frac{W(u)}{\varepsilon^2}\\ &\le C\varepsilon + 2(1 + 2C_1\varepsilon)^2 \frac{W(u)}{\varepsilon^2}.\end{aligned}$$

In either scenario, we obtain a bound of the desired form (5.5.2).

To bound $|\nabla u|^2$, recall from Section 5.3 the identity

$$\Delta \frac{1}{2} |\nabla u|^2 = |\nabla^2 u|^2 + \frac{1}{2\varepsilon^2}(3|u|^2 - 1)|\nabla u|^2 - 2\langle \omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u). \tag{5.5.4}$$

In view of the estimate (5.5.1) for $|F_\nabla| = |\omega|$ and (5.2.7), we can estimate the term $2\langle \omega, \psi(u)\rangle$ from above by

$$2|F_\nabla||\nabla u|^2 \le \frac{1}{\varepsilon^2}(1 - |u|^2)|\nabla u|^2 + \frac{C}{\varepsilon}|\nabla u|^2,$$

to obtain the existence of $C_2(\mathcal{M},\Lambda)$ such that

$$\Delta \frac{1}{2}|\nabla u|^2 \ge |\nabla^2 u|^2 + \frac{1}{2\varepsilon^2}(5|u|^2 - 3)|\nabla u|^2 - \frac{C_2}{\varepsilon}|\nabla u|^2.$$

For $\Delta|\nabla u|$, this then gives

$$\Delta|\nabla u| \ge \frac{1}{2\varepsilon^2}(5|u|^2 - 3)|\nabla u| - \frac{C_2}{\varepsilon}|\nabla u|. \tag{5.5.5}$$

Recalling once again the equation (5.3.4) for $\Delta\frac{1}{2}|u|^2$, we define

$$w := |\nabla u| - \frac{1}{\varepsilon}(1 - |u|^2),$$

and observe that

$$\begin{aligned}\Delta w &\ge \frac{1}{2\varepsilon^2}(5|u|^2 - 3)|\nabla u| - \frac{C_2}{\varepsilon}|\nabla u|\\ &\quad + \frac{2}{\varepsilon}|\nabla u|^2 - \frac{1}{\varepsilon^3}|u|^2(1 - |u|^2)\\ &= \frac{|u|^2}{\varepsilon^2} w + |\nabla u|\Big(\frac{2}{\varepsilon}|\nabla u| - \frac{3}{2}\frac{(1 - |u|^2)}{\varepsilon^2} - \frac{C_2}{\varepsilon}\Big)\\ &= \frac{|u|^2}{\varepsilon^2} w + \frac{|\nabla u|}{\varepsilon}\Big(2w + \frac{1}{2\varepsilon}(1 - |u|^2) - C_2\Big).\end{aligned}$$

We then have

$$\Delta w \geq \frac{|u|^2}{\varepsilon^2} w + \frac{1}{\varepsilon}\Big(w + \frac{1}{\varepsilon}(1-|u|^2)\Big)\Big(2w + \frac{1}{2\varepsilon}(1-|u|^2) - C_2\Big). \tag{5.5.6}$$

If w has a positive maximum, it follows that

$$2w + \frac{1}{2\varepsilon}(1-|u|^2) \leq C_2$$

at this maximum point; in particular, we deduce then that

$$|u|^2 \geq 1 - 2C_2\varepsilon$$

at this point, and see from (5.5.6) that here

$$0 \geq \frac{1-2C_2\varepsilon}{\varepsilon^2} w - \frac{1}{\varepsilon}\Big(w + \frac{1}{\varepsilon}(1-|u|^2)\Big)C_2 \geq \frac{1-3C_2\varepsilon}{\varepsilon^2} w - 2\frac{C_2^2}{\varepsilon}.$$

If $\varepsilon \leq \varepsilon_D(\mathcal{M},\Lambda)$ is small enough, it follows that $\max w \leq C\varepsilon$; as a consequence, we check that

$$|\nabla u|^2 \leq C\frac{W(u)}{\varepsilon^2} + C\varepsilon^2,$$

completing the proof of (5.5.3). □

As a simple consequence of the estimates in Proposition 5.5.1, we obtain the following corollary.

Corollary 5.5.2. *There exist constants* $0 < \beta_D(\mathcal{M},\Lambda) < 1$ *and* $C(\mathcal{M},\Lambda)$ *such that, for* (u,∇) *as above, we have*

$$\Delta\frac{1}{2}(1-|u|^2) \geq \frac{1}{4\varepsilon^2}(1-|u|^2) - C\varepsilon^2 \tag{5.5.7}$$

on the set $Z_{\beta_D}(u) := \{|u|^2 \geq 1-\beta_D\}$.

Proof. By the formula (5.3.4) for $\Delta\frac{1}{2}|u|^2$, we know that

$$\Delta\frac{1}{2}(1-|u|^2) = \frac{1}{2\varepsilon^2}|u|^2(1-|u|^2) - |\nabla u|^2.$$

Combining this with the estimate (5.5.3) for $|\nabla u|^2$, we then deduce the existence of a constant $\widehat{C} = \widehat{C}(\mathcal{M},\Lambda)$ such that

$$\Delta\frac{1}{2}(1-|u|^2) \geq |u|^2\frac{1}{2\varepsilon^2}(1-|u|^2) - \widehat{C}\frac{(1-|u|^2)^2}{2\varepsilon^2} - C\varepsilon^2.$$

By taking $\beta_D = \beta_D(\mathcal{M},\Lambda) > 0$ sufficiently small, we can arrange that

$$|u|^2 - \widehat{C}(1-|u|^2) \geq 1 - \beta_D - \widehat{C}\beta_D \geq \frac{1}{2}$$

on $\{|u|^2 \geq 1-\beta_D\}$, from which the claimed estimate follows. □

Next, we employ the result of Corollary 5.5.2 to show that the quantity $(1-|u|^2)$ vanishes rapidly away from $Z_{\beta_D}(u)$ (compare [59, Sections III.7–III.8]).

Proposition 5.5.3. *Let (u, ∇) be as before, with $\varepsilon \le \varepsilon_D$, and define the set*

$$Z_{\beta_D} := \{x \in \mathcal{M} : |u(x)|^2 \le 1 - \beta_D\},$$

where $\beta_D(\mathcal{M}, \Lambda)$ is the constant provided by Corollary 5.5.2. Defining $r : \mathcal{M} \to [0, \infty)$ by

$$r(p) := \operatorname{dist}(p, Z_\beta),$$

we have an estimate of the form

$$(1 - |u|^2)(p) \le Ce^{-a_D r(p)/\varepsilon} + C\varepsilon^4 \tag{5.5.8}$$

for some $C = C(\mathcal{M}, \Lambda)$ and $a_D = a_D(\mathcal{M}) > 0$.

Proof. Fix a point $p \in \mathcal{M}$, and let $r = r(p) = \operatorname{dist}(p, Z_\beta)$ as above. We can clearly assume $r(p) < \frac{1}{2}\operatorname{inj}(\mathcal{M})$. On the ball $B_r(p)$, for some constant $a = a_D > 0$ to be chosen later, consider the function

$$\varphi(x) := e^{(a/\varepsilon)(d_p(x)^2 + \varepsilon^2)^{1/2}},$$

where $d_p(x) := \operatorname{dist}(p, x)$. A straightforward computation then gives

$$\begin{aligned}
\Delta\varphi &= \frac{a}{\varepsilon}\varphi\left(\frac{(a/\varepsilon)d_p^2}{d_p^2 + \varepsilon^2} - \frac{d_p^2}{(d_p^2 + \varepsilon^2)^{3/2}}\right) \\
&\quad + \frac{a}{2\varepsilon}\varphi\frac{\Delta d_p^2}{(d_p^2 + \varepsilon^2)^{1/2}} \\
&\le \frac{a^2}{\varepsilon^2}\varphi + \frac{a}{2\varepsilon}\varphi\frac{\Delta d_p^2}{(d_p^2 + \varepsilon^2)^{1/2}} \\
&\le \frac{a^2 + C_1 a}{\varepsilon^2}\varphi
\end{aligned}$$

for some $C_1 = C_1(\mathcal{M})$. Now, fix some constant $c_2 > 0$ to be chosen later, and let

$$f := \frac{1}{2}(1 - |u|^2) - c_2\varphi.$$

Combining the preceding computation with (5.5.7), we see that, on $B_r(p)$,

$$\begin{aligned}
\Delta f &\ge \frac{1}{4\varepsilon^2}(1 - |u|^2) - C(\mathcal{M}, \Lambda)\varepsilon^2 - \frac{a^2 + C_1 a}{\varepsilon^2}c_2\varphi \\
&= \frac{1}{2\varepsilon^2}f + \frac{1 - 2a^2 - 2C_1 a}{2\varepsilon^2}c_2\varphi - C(\mathcal{M}, \Lambda)\varepsilon^2.
\end{aligned}$$

Choosing $a = a_D(\mathcal{M}) > 0$ sufficiently small, we can arrange that $2a^2 + 2C_1 a \le 1$, so that the above computation gives

$$\Delta f \ge \frac{f}{2\varepsilon^2} - C\varepsilon^2. \tag{5.5.9}$$

On the boundary of the ball $\partial B_r(p)$, it follows from definition of $r = r(p)$ that $|u|^2 \ge 1 - \beta_D$, and therefore

$$f(x) \le \frac{\beta_D}{2} - c_2\varphi \le \frac{\beta_D}{2} - c_2 e^{ar/\varepsilon} \quad \text{on } \partial B_r(p).$$

Taking $c_2 := \beta_D e^{-ar/\varepsilon}$, it then follows that $f < 0$ on $\partial B_r(p)$, so we can apply the maximum principle with (5.5.9) to deduce that

$$f \leq C\varepsilon^4 \quad \text{in } B_r(p).$$

Evaluating at p, this gives

$$C\varepsilon^4 \geq f(p) = \frac{1}{2}(1-|u|^2)(p) - \beta_D e^{-ar(p)/\varepsilon} e^a,$$

so that

$$(1-|u|^2)(p) \leq C(\mathcal{M},\Lambda)e^{-ar(p)/\varepsilon} + C(\mathcal{M},\Lambda)\varepsilon^4,$$

as desired. □

Combining these estimates with those of Proposition 5.5.1, we arrive immediately at the following decay estimate for the energy integrand $e_\varepsilon(u,\nabla)$.

Corollary 5.5.4. *Defining Z_{β_D} and $r(p) = \operatorname{dist}(p, Z_{\beta_D})$ as in Proposition 5.5.3, there exist $a_D(\mathcal{M}) > 0$ and $C_D(\mathcal{M},\Lambda)$ such that*

$$e_\varepsilon(u,\nabla)(p) \leq C_D \frac{e^{-a_D r(p)/\varepsilon}}{\varepsilon^2} + C_D\varepsilon. \tag{5.5.10}$$

5.6 The energy-concentration varifold

This section is devoted to the proof of the main result of the chapter, which we recall now.

Theorem 5.6.1. *Let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of solutions to (5.2.4)–(5.2.5) satisfying a uniform energy bound $E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) \leq \Lambda$ as $\varepsilon \to 0$. Then, as $\varepsilon \to 0$, the energy measures*

$$\mu_\varepsilon := \frac{1}{2\pi} e_\varepsilon(u_\varepsilon,\nabla_\varepsilon)\ \mathrm{vol}_g$$

converge subsequentially, in duality with $C^0(\mathcal{M})$, to the weight measure of a stationary, integral $(m-2)$-varifold V. Also, for all $0 \leq \delta < 1$,

$$\operatorname{spt}(|V|) = \lim_{\varepsilon\to 0}\{|u_\varepsilon| \leq \delta\}$$

in the Hausdorff topology. The $(m-2)$-currents dual to the curvature forms $\frac{1}{2\pi}\omega_\varepsilon$ converge subsequentially to an integral $(m-2)$-cycle Γ, with $|\Gamma| \leq \mu$.

Convergence to a stationary rectifiable varifold

Let $(u_\varepsilon, \nabla_\varepsilon)$ be as in Theorem 5.6.1, and pass to a subsequence $\varepsilon_j \to 0$ such that the energy measures μ_{ε_j} converge weakly-* to a limiting measure μ, in duality with $C^0(\mathcal{M})$.

Note that, for $0 < r < R < \operatorname{inj}(\mathcal{M})$, Theorem 5.4.3 yields

$$\begin{aligned} e^{CR}R^{2-m}\mu(\overline{B}_R(x)) + CR &\geq \limsup_{\varepsilon\to 0} e^{CR}R^{2-m}\mu_\varepsilon(\overline{B}_R(x)) + CR \\ &\geq \liminf_{\varepsilon\to 0} e^{Cr}r^{2-m}\mu_\varepsilon(B_r(x)) + Cr \\ &\geq e^{Cr}r^{2-m}\mu(B_r(x)) + Cr \end{aligned}$$

with $C = C_M$, so approximating R with smaller radii we deduce

$$e^{CR}R^{2-m}\mu(B_R(x)) + CR \geq e^{Cr}r^{2-m}\mu(B_r(x)) + Cr, \tag{5.6.1}$$

and in particular the $(m-2)$-density

$$\Theta_{m-2}(\mu, x) := \lim_{r\to 0}(\omega_{m-2}r^{m-2})^{-1}\mu(B_r(x))$$

is defined. As a first step toward the proof of Theorem 5.6.1, we show that this density is bounded from above and below on the support $\operatorname{spt}(\mu)$.

Proposition 5.6.2. *There exists a constant $0 < C = C(\mathcal{M}, \Lambda) < \infty$ such that*

$$C^{-1} \leq r^{2-m}\mu(B_r(x)) \leq C \quad \textit{for } x \in \operatorname{spt}(\mu),\ 0 < r < \operatorname{inj}(\mathcal{M}), \tag{5.6.2}$$

and thus $C^{-1} \leq \Theta_{m-2}(\mu, x) \leq C$ for all $x \in \operatorname{spt}(\mu)$.

Proof. The upper bound follows from (5.6.1), which gives (when $R = \operatorname{inj}(\mathcal{M})$)

$$\begin{aligned} r^{2-m}\mu(B_r(x)) &\leq e^{C_M r}r^{2-m}\mu(B_r(x)) + C_M r \\ &\leq C(\mathcal{M},\Lambda)\mu(\mathcal{M}) + C(\mathcal{M},\Lambda)\operatorname{inj}(\mathcal{M}) \\ &\leq C(\mathcal{M},\Lambda). \end{aligned}$$

To see the lower bound, let $\beta_D = \beta_D(\mathcal{M}, \Lambda) \in (0, 1)$ be the constant given by Corollary 5.5.4, and again set

$$Z_\beta(u_\varepsilon) := \{x \in \mathcal{M} : |u_\varepsilon(x)|^2 \leq 1 - \beta\}.$$

Let Σ be the set of all limits $x = \lim_\varepsilon x_\varepsilon$, with $x_\varepsilon \in Z_{\beta_D}(u_\varepsilon)$; that is, take

$$\Sigma := \bigcap_{\eta>0} \overline{\bigcup_{0<\varepsilon<\eta} Z_{\beta_D}(u_\varepsilon)}.$$

We then claim that

$$\operatorname{spt}(\mu) \subseteq \Sigma \tag{5.6.3}$$

and

$$\mu(B_r(x)) \geq c(\mathcal{M},\Lambda)r^{m-2} \quad \text{for } x \in \Sigma,\ 0 < r < \operatorname{inj}(\mathcal{M}). \tag{5.6.4}$$

Once both (5.6.3) and (5.6.4) are established, the lower bound in (5.6.2) follows immediately.

To establish (5.6.3), fix some $p \in \mathcal{M} \setminus \Sigma$; by definition of Σ, there must exist $\delta = \delta(p) > 0$ such that

$$\operatorname{dist}(p, Z_{\beta_D}(u_\varepsilon)) \geq 2\delta$$

for all ε sufficiently small. Applying Corollary 5.5.4 for all $x \in B_\delta(p)$, we deduce that

$$\begin{aligned}\mu(B_\delta(p)) &\leq \liminf_{\varepsilon\to 0} \frac{1}{2\pi}\int_{B_\delta(p)} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\\ &\leq \lim_{\varepsilon\to 0}\int_{B_\delta(p)} (C\varepsilon^{-2}e^{-a\delta/\varepsilon} + C\varepsilon)\\ &= 0.\end{aligned}$$

In particular, $p \in \mathcal{M} \setminus \operatorname{spt}(\mu)$, confirming (5.6.3).

To see (5.6.4), let $x \in \Sigma$. Note that, by definition of Σ, there exist points $x_\varepsilon \in Z_{\beta_D}(u_\varepsilon)$ with $x_\varepsilon \to x$ as $\varepsilon \to 0$ (along a subsequence). We then see that

$$|u_\varepsilon(x_\varepsilon)|^2 \leq 1 - \beta_D$$

and Corollary 5.4.4 gives $c(\mathcal{M}, \Lambda)$ such that

$$\mu_\varepsilon(B_r(x_\varepsilon)) \geq c(\mathcal{M}, \Lambda) r^{m-2}$$

for $\varepsilon < r < \operatorname{inj}(\mathcal{M})$. Since for any $\delta > 0$ we have $B_r(x_\varepsilon) \subseteq \overline{B}_{r+\delta}(x)$ eventually, it follows that $\mu(\overline{B}_{r+\delta}(x)) \geq cr^{m-2}$, hence

$$\mu(B_r(x)) \geq cr^{m-2}$$

for $0 < r < \operatorname{inj}(\mathcal{M})$, which is (5.6.4). □

With Proposition 5.6.2 in place, we will invoke a result by Ambrosio and Soner [10] to conclude that the limiting measure $\mu = \lim_{\varepsilon\to 0}\mu_\varepsilon$ coincides with the weight measure of a stationary, rectifiable $(m-2)$-varifold. Recall from Section 5.4 the stress-energy tensors

$$T_\varepsilon = e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)g - 2\nabla_\varepsilon u_\varepsilon^* \nabla_\varepsilon u_\varepsilon - 2\varepsilon^2 F_{\nabla_\varepsilon}^* F_{\nabla_\varepsilon}.$$

We record first the following lemma; in its statement, we canonically identify (and pair with each other) tensors of rank $(2,0)$, $(1,1)$, and $(0,2)$, using the underlying metric g.

Lemma 5.6.3. *As $\varepsilon \to 0$, the tensors T_ε converge (subsequentially) as* $\operatorname{Sym}(T\mathcal{M})$*-valued measures, in duality with $C^0(\mathcal{M}, \operatorname{Sym}(T\mathcal{M}))$, to a limit T satisfying*

$$\langle T, DX\rangle = 0 \quad \textit{for all vector fields } X \in C^1(\mathcal{M}, T\mathcal{M}), \tag{5.6.5}$$

$$\frac{1}{2\pi}\langle T, \varphi g\rangle \geq (m-2)\langle \mu, \varphi\rangle \quad \textit{for every } 0 \leq \varphi \in C^0(\mathcal{M}), \tag{5.6.6}$$

and

$$-\int_{\mathcal{M}} |X|^2\, d\mu \leq \frac{1}{2\pi}\langle T, X\otimes X\rangle \leq \int_{\mathcal{M}} |X|^2\, d\mu \quad \textit{for all } X \in C^0(\mathcal{M}, T\mathcal{M}). \tag{5.6.7}$$

Proof. For each $\varepsilon > 0$, note that, by definition of T_ε, for every continuous vector field $X \in C^0(\mathcal{M}, T\mathcal{M})$ we have

$$\int_{\mathcal{M}} \langle T_\varepsilon, X \otimes X\rangle = \int_{\mathcal{M}} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)|X|^2 - \int_{\mathcal{M}} 2|(\nabla_\varepsilon)_X u_\varepsilon|^2 - \int_{\mathcal{M}} 2\varepsilon^2 |\iota_X F_{\nabla_\varepsilon}|^2.$$

Evaluating (5.2.3) in an orthonormal basis such that X is a multiple of e_1, we see that $|\iota_X F_{\nabla_\varepsilon}|^2 \le |F_{\nabla_\varepsilon}|^2 |X|^2$, while $|(\nabla_\varepsilon)_X u_\varepsilon|^2 \le |\nabla_\varepsilon u_\varepsilon|^2 |X|^2$. We deduce that

$$-\int_{\mathcal{M}} |X|^2 e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le \int_{\mathcal{M}} \langle T_\varepsilon, X \otimes X\rangle \le \int_M e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)|X|^2. \tag{5.6.8}$$

As an immediate consequence, we see that the uniform energy bound $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le \Lambda$ gives a uniform bound on $\|T_\varepsilon\|_{(C^0)^*}$ as $\varepsilon \to 0$, so we can indeed extract a weak-* subsequential limit $T \in C^0(\mathcal{M}, \mathrm{Sym}(T\mathcal{M}))^*$, for which (5.6.7) follows from (5.6.8).

The stationarity condition (5.6.5) for the limit T follows from (5.4.5). It remains to establish the trace inequality (5.6.6). For this, we simply compute, for nonnegative $\varphi \in C^0(\mathcal{M})$,

$$\begin{aligned}
\int_{\mathcal{M}} \langle T_\varepsilon, \varphi g\rangle &= \int_{\mathcal{M}} \varphi(n e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) - 2|\nabla_\varepsilon u_\varepsilon|^2 - 4\varepsilon^2 |F_{\nabla_\varepsilon}|^2) \\
&= \int_{\mathcal{M}} (m-2)\varphi e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) + 2\int_{\mathcal{M}} \varphi\Big(\frac{W(u_\varepsilon)}{\varepsilon^2} - \varepsilon^2 |F_{\nabla_\varepsilon}|^2\Big) \\
&\ge 2\pi(m-2)\langle \mu_\varepsilon, \varphi\rangle - 4\pi \int_{\mathcal{M}} \varphi e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2}\Big(\varepsilon|F_{\nabla_\varepsilon}| - \frac{(1-|u_\varepsilon|^2)}{2\varepsilon}\Big)^+.
\end{aligned}$$

Recalling from Proposition 5.4.2 that

$$\varepsilon|F_{\nabla_\varepsilon}| - \frac{(1-|u_\varepsilon|^2)}{2\varepsilon} \le C(\mathcal{M}, \Lambda),$$

we then see that

$$\langle T, \varphi g\rangle = \lim_{\varepsilon\to 0} \int_{\mathcal{M}} \langle T_\varepsilon, \varphi g\rangle \ge 2\pi(m-2)\langle \mu, \varphi\rangle - C \lim_{\varepsilon \to 0} \int_{\mathcal{M}} \varphi e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2}.$$

In particular, (5.6.6) will follow once we show that $\lim_{\varepsilon\to 0} \int_{\mathcal{M}} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} = 0$.

But this is straightforward: from Proposition 5.6.2 we know that for $0 < \delta < \mathrm{inj}(\mathcal{M})$ we have

$$\mu(B_\delta(x)) \ge c(\mathcal{M}, \Lambda)\delta^{m-2} \quad \text{for } x \in \Sigma = \mathrm{spt}(\mu).$$

Since $\mathrm{vol}(B_{5\delta}(x)) \le C(\mathcal{M})\delta^m$, a simple Vitali covering argument then implies that the δ-neighborhood $B_\delta(\Sigma)$ of Σ satisfies a volume bound

$$\mathrm{vol}(B_\delta(\Sigma)) \le C(\mathcal{M}, \Lambda)\delta^2.$$

With this estimate in hand, we then see that

$$\begin{aligned}
\int_{\mathcal{M}} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} &= \int_{B_\delta(\Sigma)} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} + \int_{\mathcal{M}\setminus B_\delta(\Sigma)} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} \\
&\le \mathrm{vol}(B_\delta(\Sigma))^{1/2}\Lambda^{1/2} + C(\mathcal{M})\mu_\varepsilon(\mathcal{M} \setminus B_\delta(\Sigma))^{1/2}.
\end{aligned}$$

Fixing δ and taking the limit as $\varepsilon \to 0$, we have $\mu_\varepsilon(\mathcal{M} \setminus B_\delta(\Sigma)) \to 0$. Since $\mathrm{vol}(B_\delta(\Sigma)) \le C\delta^2$, we find that

$$\limsup_{\varepsilon \to 0} \int_{\mathcal{M}} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} \le C\delta\Lambda^{1/2}.$$

Finally, taking $\delta \to 0$, we conclude that $\int_M e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} \to 0$ as $\varepsilon \to 0$, completing the proof. □

Estimate (5.6.7) says that $|T|$ is absolutely continuous with respect to μ, so by the Radon–Nikodym theorem we can write the limiting $\mathrm{Sym}(T\mathcal{M})$-valued measure T from Lemma 5.6.3 as

$$\frac{1}{2\pi}\langle T, S\rangle = \int_{\mathcal{M}} \langle P(x), S(x)\rangle \, d\mu(x) \tag{5.6.9}$$

for some L^∞ (with respect to μ) section $P : \mathcal{M} \to \mathrm{Sym}(T\mathcal{M})$. Moreover, it follows from (5.6.6) and (5.6.7) that $-g \le P(x) \le g$ and $\mathrm{tr}(P(x)) \ge m-2$ at μ-a.e. $x \in \mathcal{M}$, so that $\frac{1}{2\pi}T$ defines in a natural way a generalized $(m-2)$-varifold in the sense of Ambrosio and Soner, namely a Radon measure on the bundle

$$A_{m,n-2}(\mathcal{M}) := \{S \in \mathrm{Sym}(T\mathcal{M}) : -ng \le S \le g,\, \mathrm{tr}(S) \ge m-2\}. \tag{5.6.10}$$

We refer the reader to [10, Section 3]. Note that in [10] the authors work in the Euclidean space and require the trace to be equal to $m-2$ in (5.6.10); however, the main result on generalized varifolds, namely [10, Theorem 3.8], still holds in our setting.

Hence, in view of the stationarity condition (5.6.5) and the density bounds of Proposition 5.6.2, we can apply [10, Theorem 3.8(c)] to conclude that $\frac{1}{2\pi}T$ can be identified with a stationary, rectifiable $(m-2)$-varifold with weight measure μ (so, in particular, $\mathrm{spt}(\mu)$ is $(m-2)$-rectifiable), and that $P(x)$ is given μ-a.e. by the orthogonal projection onto the $(m-2)$-subspace $T_x\,\mathrm{spt}(\mu) \subset T_x\mathcal{M}$. We collect this information in the following statement.

Proposition 5.6.4. *For a family $(u_\varepsilon, \nabla_\varepsilon)$ satisfying the hypotheses of Theorem 5.6.1, after passing to a subsequence, there exists a stationary, rectifiable $(m-2)$-varifold $V = v(\Sigma^{m-2}, \theta)$ such that*

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathcal{M}} \langle T_\varepsilon(u_\varepsilon, \nabla_\varepsilon), S\rangle = \int_\Sigma \theta(x)\langle T_x\Sigma, S(x)\rangle \, d\mathcal{H}^{m-2} \tag{5.6.11}$$

for every continuous section $S \in C^0(\mathcal{M}, \mathrm{Sym}(T\mathcal{M}))$. The energy measure μ is given by $\mu = \theta\mathcal{H}^{m-2} \llcorner \Sigma$. Also, we can choose $\Sigma := \mathrm{spt}(\mu)$ and $\theta(x) := \Theta_{m-2}(\mu, x)$.

Integrality of the limit varifold and convergence of level sets

We now show that the varifold V is integer rectifiable. Given $x \in \mathrm{spt}(\mu)$ and $s > 0$, we define $\mathcal{M}_{x,s}$ to be the ball of radius $s^{-1}\,\mathrm{inj}(\mathcal{M})$ in the Euclidean space $(T_x\mathcal{M}, g_x)$ and define $\iota_{x,s} : \mathcal{M}_{x,s} \to \mathcal{M}$ by $\iota_{x,s}(y) := \exp_x(sy)$. We endow $\mathcal{M}_{x,s}$ with the smooth metric $g_{x,s} := s^{-2}\iota_{x,s}^* g$, which converges locally smoothly to the Euclidean metric g_x as $s \to 0$.

By rectifiability, for μ-a.e. x the dilated varifolds $V_{x,s} := (\iota_{x,s}^{-1})_*(V \llcorner B_{\mathrm{inj}(\mathcal{M})}(x))$ in $\mathcal{M}_{x,s}$ satisfy

$$V_{x,s} \rightharpoonup v(T_x\Sigma, \Theta_{m-2}(x)) \tag{5.6.12}$$

as $s \to 0$, in duality with $C_c(T_x\mathcal{M})$. Fix $x \in \mathrm{spt}(\mu)$ such that (5.6.12) holds. The integrality of V will follow once we prove that $\Theta = \Theta_{m-2}(\mu, x)$ is an integer.

We can identify $(T_x\mathcal{M}, g_x)$ with $\mathbb{R}^m$ by a linear isometry such that $T_x\Sigma = \{0\} \times \mathbb{R}^{m-2}$. We also call $\mu_{x,s}$ the mass measure of $V_{x,s}$; equivalently,

$$\mu_{x,s} := s^{2-m}(\iota_{x,s}^{-1})_*(\mu \llcorner B_{\mathrm{inj}(\mathcal{M})}(x)).$$

With a diagonal selection, changing our sequence $\varepsilon \to 0$ accordingly, we can find scales $s_\varepsilon \to 0$ such that we have the convergence of Radon measures

$$\lim_{\varepsilon\to 0} \widehat{\mu}_\varepsilon = \lim_{s\to 0} \mu_{x,s} = \Theta\mathcal{H}^{m-2} \llcorner T_x\Sigma,$$

where $(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)$ is the pullback of $(u_{s_\varepsilon\varepsilon}, \nabla_{s_\varepsilon\varepsilon})$ by means of ι_{x,s_ε}, and $\widehat{\mu}_\varepsilon$ is the associated energy measure. Note that $(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)$ is stationary for E_ε in the line bundle $\iota_{x,s_\varepsilon}^* L$, with respect to the base metric g_{x,s_ε}. We introduce the notation

$$e_\varepsilon^T(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) := \sum_{i=3}^m (|(\nabla_\varepsilon)_{\partial_i}\widehat{u}_\varepsilon|^2 + \varepsilon^2|\iota_{\partial_i}F_{\widehat{\nabla}_\varepsilon}|^2).$$

Balls will be denoted by $\mathcal{B}_r(y)$ or $B_r^m(y)$, depending on whether they are with respect to g_{x,s_ε} or $g_{\mathbb{R}^m}$, respectively. The volume $|E|$ of a set E will be always understood with respect to the Euclidean metric.

The next proposition, which exploits quantitatively the monotonicity formula, is similar to an estimate in the proof of [69, Lemma 2.1].

Proposition 5.6.5. *As $\varepsilon \to 0$ we have*

$$\lim_{\varepsilon\to 0} \int_{B_2^2 \times B_2^{m-2}} e_\varepsilon^T(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) = 0.$$

Proof. Let C_M be the constant in Theorem 5.4.3. We first note that, given $y \in \{0\} \times \mathbb{R}^{m-2}$,

$$\lim_{\varepsilon\to 0} \widehat{\mu}_\varepsilon(\mathcal{B}_r(y)) = \Theta\omega_{m-2}r^{m-2};$$

indeed, for any $\eta > 0$, $B_{r-\eta}^m(y) \subseteq \mathcal{B}_r(y) \subseteq B_{r+\eta}^m(y)$ eventually. Setting $y_\varepsilon := \iota_{x,s_\varepsilon}(y) \in \mathcal{M}$, we deduce that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(e^{C_M s_\varepsilon r}(s_\varepsilon r)^{2-m}\mu_{s_\varepsilon\varepsilon}(B_{s_\varepsilon r}(y_\varepsilon)) + C_M s_\varepsilon r) \\
&= \lim_{\varepsilon\to 0}(e^{C_M s_\varepsilon r} r^{2-m}\widehat{\mu}_\varepsilon(\mathcal{B}_r(y)) + C_M s_\varepsilon r) \\
&= \Theta\omega_{m-2}.
\end{aligned} \tag{5.6.13}$$

Pick $3 \le i \le m$ and fix $R > 0$. Choosing $y := -2Re_i$, we can apply (5.4.12) between the radii $s_\varepsilon R$ and $3s_\varepsilon R$ to obtain that

$$\begin{aligned}\int_{B_{3s_\varepsilon R}(p_i)\setminus B_{s_\varepsilon R}(p_i)} & d_{p_i}^{2-m}(|\nabla_{\nu_{R,i}} u_{s_\varepsilon\varepsilon}|^2 + s_\varepsilon^2\varepsilon^2|\iota_{\nu_{R,i}}F_{\nabla_{s_\varepsilon\varepsilon}}|^2) \\ &\le (e^{C_M(3s_\varepsilon R)}(3s_\varepsilon R)^{2-m}\mu_{s_\varepsilon\varepsilon}(B_{3s_\varepsilon R}(p_i)) + C_M(3s_\varepsilon R)) \\ &\quad - (e^{C_M(s_\varepsilon R)}(s_\varepsilon R)^{2-m}\mu_{s_\varepsilon\varepsilon}(B_{s_\varepsilon R}(p_i)) + C_M(s_\varepsilon R)),\end{aligned}$$

where $p_i := \iota_{x,s_\varepsilon}(-2Re_i)$ and $\nu_{R,i} := \operatorname{grad} d_{p_i}$. Now (5.6.13) and the comparability of g_{x,s_ε} with $g_{\mathbb{R}^m}$ give

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}_{3R}(-2Re_i)\setminus\mathcal{B}_R(-2Re_i)}(|\nabla_{\widetilde{\nu}_{R,i}}\widehat{u}_\varepsilon|^2 + \varepsilon^2|\iota_{\widetilde{\nu}_{R,i}}F_{\widehat{\nabla}_\varepsilon}|^2) = 0,$$

where $\widetilde{\nu}_{R,i}$ is the gradient of the distance function d_{-2Re_i}, both with respect to the metric g_{x,s_ε}. Since eventually $\mathcal{B}_{3R}(-2Re_i)\setminus\mathcal{B}_R(-2Re_i)$ includes $B_2^2\times B_2^{m-2}$ for R big enough, we get

$$\lim_{\varepsilon\to 0}\int_{B_2^2\times B_2^{m-2}}(|\nabla_{\widetilde{\nu}_{R,i}}\widehat{u}_\varepsilon|^2 + \varepsilon^2|\iota_{\widetilde{\nu}_{R,i}}F_{\widehat{\nabla}_\varepsilon}|^2) = 0. \tag{5.6.14}$$

By monotonicity, as $\varepsilon \to 0$ we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{B_2^2\times B_2^{m-2}} e_\varepsilon(\widehat{u}_\varepsilon,\widehat{\nabla}_\varepsilon) &\le \limsup_{\varepsilon\to 0} s_\varepsilon^{2-m}\int_{B_{5s_\varepsilon}(x)} e_{s_\varepsilon\varepsilon}(u_{s_\varepsilon\varepsilon},\nabla_{s_\varepsilon\varepsilon}) \\ &\le C(\mathcal{M},\Lambda).\end{aligned} \tag{5.6.15}$$

The smooth convergence $g_{x,s_\varepsilon}\to g_{\mathbb{R}^m}$ gives $\widetilde{\nu}_{R,i}(y)\to Y_{R,i}(y) := \frac{y+2Re_i}{|y+2Re_i|}$ uniformly on $B_2^2\times B_2^{m-2}$. Hence, the bound (5.6.15) and (5.6.14) give

$$\lim_{\varepsilon\to 0}\int_{B_2^2\times B_2^{m-2}}(|\nabla_{Y_{R,i}}\widehat{u}_\varepsilon|^2 + \varepsilon^2|\iota_{Y_{R,i}}F_{\widehat{\nabla}_\varepsilon}|^2) = 0. \tag{5.6.16}$$

Now $Y_{R,i}\to e_i = \partial_i$ as $R\to\infty$, and the statement follows from (5.6.16) and the uniform bound (5.6.15). □

We now state the main technical result of the section, which will be shown later. Fix a cut-off function $\chi \in C_c^\infty(B_2^2)$ with $\chi(z) = 1$ for $|z| \le \frac{3}{2}$ and $0 \le \chi \le 1$, and let $\widehat{\chi}(z,t) := \chi(z)$.

Proposition 5.6.6. *There exists $F_\varepsilon \subseteq B_1^{m-2}$ with $|F_\varepsilon| \ge \frac{1}{4}|B_1^{m-2}|$ such that*

$$\sup_{t\in F_\varepsilon}\operatorname{dist}\Big(\int_{\mathbb{R}^2\times\{t\}}\chi(z)e_\varepsilon(\widehat{u}_\varepsilon,\widehat{\nabla}_\varepsilon)(z,t), 2\pi\mathbb{N}\Big)\to 0 \quad \text{as } \varepsilon\to 0. \tag{5.6.17}$$

Before giving the proof, let us see how this implies the integrality of V.

Proof of Theorem 5.6.1. As $\varepsilon \to 0$, we have both (5.6.17) and

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2\times B_1^{m-2}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2\times B_1^{m-2}} \widehat{\chi}\, d\widehat{\mu}_\varepsilon = \omega_{m-2}\Theta, \tag{5.6.18}$$

$$\int_{\mathbb{R}^2\times B_2^{m-2}} |d\widehat{\chi}|\, d\widehat{\mu}_\varepsilon \le C\widehat{\mu}_\varepsilon((B_2^2 \setminus B_1^2) \times B_1^{m-2}) \to 0, \tag{5.6.19}$$

as $\widehat{\mu}_\varepsilon \rightharpoonup \Theta\mathcal{H}^{m-2} \llcorner \{0\} \times \mathbb{R}^{m-2}$.

In view of (5.6.15) and (5.6.19), for any vector field $(Y^3, \dots, Y^m) \in C_c^\infty(B_2^{m-2}, \mathbb{R}^{m-2})$ we can integrate (5.4.4) against $\widehat{\chi}(\sum_{i=3}^m Y^i \partial_i)$ and obtain, in the Euclidean metric,

$$\Big| \int_{\mathbb{R}^2\times B_2^{m-2}} \widehat{\chi} \langle T_\varepsilon(u_\varepsilon, \nabla_\varepsilon), dY^i \otimes \partial_i \rangle \Big| \le \lambda_\varepsilon(\|Y\|_{L^\infty} + \|DY\|_{L^\infty})$$

for some sequence $\lambda_\varepsilon \to 0$, thanks to the smooth convergence $g_{x,s_\varepsilon} \to g_{\mathbb{R}^m}$.

Invoking Proposition 5.6.5 and noting that $\|Y\|_{L^\infty} \le 2\|DY\|_{L^\infty}$, we can conclude that the nonnegative function $f_\varepsilon(t) := \frac{1}{2\pi}\int_{\mathbb{R}^2\times\{t\}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)$ satisfies

$$\Big| \int_{B_2^{m-2}} f_\varepsilon \operatorname{div}(Y) \Big| \le \lambda_\varepsilon \|DY\|_{L^\infty}$$

for a possibly different sequence $\lambda_\varepsilon \to 0$. Applying the Hahn–Banach theorem to the subspace $\{DY \mid Y \in C_c^\infty(B_2^{m-2}, \mathbb{R}^{m-2})\} \subseteq C_0(B_2^{m-2}, \mathbb{R}^{m-2} \otimes \mathbb{R}^{m-2})$ (C_0 denoting the closure of C_c), we can find real measures $(\nu_\varepsilon)_j^i$ such that

$$\partial_j f_\varepsilon = \sum_{i=3}^m \partial_i (\nu_\varepsilon)_j^i \quad \text{for all } j = 3, \dots, n$$

as distributions and $|(\nu_\varepsilon)_j^i|(B_2^{m-2}) \to 0$. Allard's strong constancy lemma [4, Theorem 1.(4)] gives then

$$\Big\| f_\varepsilon - \frac{1}{\omega_{m-2}} \int_{B_1^{m-2}} f_\varepsilon \Big\|_{L^1(B_1^{m-2})} \to 0.$$

Since the sets F_ε of Proposition 5.6.6 have positive measure, there clearly exists $t_\varepsilon \in F_\varepsilon$ such that

$$\Big| f_\varepsilon(t_\varepsilon) - \frac{1}{\omega_{m-2}} \int_{B_1^{m-2}} f_\varepsilon \Big| \le \frac{1}{|F_\varepsilon|} \Big\| f_\varepsilon - \frac{1}{\omega_{m-2}} \int_{B_1^{m-2}} f_\varepsilon \Big\|_{L^1(B_1^{m-2})} \to 0.$$

Recalling (5.6.17), we deduce that

$$\operatorname{dist}\Big(\frac{1}{\omega_{m-2}} \int_{B_1^{m-2}} f_\varepsilon, 2\pi\mathbb{N} \Big) \to 0.$$

Hence, by (5.6.18), we get $\operatorname{dist}(\Theta, \mathbb{N}) = 0$, which concludes the proof that V is integral. $\square$

Proof of Proposition 5.6.6. Taking into account Proposition 5.6.5, the classical Hardy–Littlewood weak-(1,1) maximal estimate (applied to the function $t \mapsto \int_{B_2^2\times\{t\}} e_\varepsilon^T(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)$) gives

$$\frac{1}{r^{m-2}} \int_{B_2^2\times B_r^{m-2}(t)} e_\varepsilon^T(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) \le C(m) \int_{B_2^2\times B_2^{m-2}} e_\varepsilon^T(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) \to 0 \tag{5.6.20}$$

for all $t \in B_1^{m-2} \setminus E_1^\varepsilon$ and $0 < r < 1$, where E_1^ε is a Borel set with $|E_1^\varepsilon| \le \frac{1}{4}|B_1^{m-2}|$. Similarly, (5.6.15) and (5.6.19) give

$$\frac{1}{r^{m-2}}\widehat{\mu}_\varepsilon(B_2^2 \times B_r^{m-2}(t)) \le C(\mathcal{M}, \Lambda), \tag{5.6.21}$$

$$\frac{1}{r^{m-2}}\widehat{\mu}_\varepsilon((B_2^2 \setminus B_1^2) \times B_r^{m-2}(t)) \le C(m)\widehat{\mu}_\varepsilon((B_2^2 \setminus B_1^2) \times B_2^{m-2}) \to 0 \tag{5.6.22}$$

for $t \in B_1^{m-2} \setminus (E_2^\varepsilon \cup E_3^\varepsilon)$ and $0 < r < 1$, with $|E_2^\varepsilon|, |E_3^\varepsilon| \le \frac{1}{4}|B_1^{m-2}|$.

Pick any $t^\varepsilon \in B_1^{m-2} \setminus (E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon)$ and, for $0 < r < 1$, define

$$\mathcal{V}^\varepsilon(r) := \{z \in B_1^2 : \operatorname{dist}((z, t^\varepsilon),\, Z_{\beta_D/2}(\widehat{u}_\varepsilon)) < r\}$$

(with the Euclidean distance), where $Z_{\beta_D/2}(\widehat{u}_\varepsilon) = \{|\widehat{u}_\varepsilon|^2 \le 1 - \beta_D/2\}$. In other words, $\mathcal{V}^\varepsilon$ is the t^ε-slice of the neighborhood $B_r^m(Z_{\beta_D/2}(\widehat{u}_\varepsilon))$.

We claim that, for $0 < r < \frac{1}{2}$, $\mathcal{V}^\varepsilon(r)$ satisfies a uniform area bound

$$|\mathcal{V}^\varepsilon(r)| \le C(\mathcal{M}, \Lambda)r^2, \tag{5.6.23}$$

provided $\varepsilon < r$ and ε is small enough. Indeed, $\mathcal{V}^\varepsilon(r) \times \{t^\varepsilon\}$ is covered by the balls $B_r^m(y)$ with $y \in (B_{3/2}^2 \times B_r^{m-2}(t^\varepsilon)) \cap Z_{\beta_D/2}(\widehat{u}_\varepsilon)$. Vitali's covering lemma gives a disjoint collection $\{B_r^m(y_j) \mid j \in J\}$ such that $\mathcal{V}^\varepsilon(r) \times \{t^\varepsilon\} \subseteq \bigcup_j B_{5r}^m(y_j)$. By Corollary 5.4.4, we have a bound on the cardinality $|J|$:

$$\widehat{\mu}_\varepsilon(B_2^2 \times B_{2r}^{m-2}(t^\varepsilon)) \ge \sum_{j\in J} \widehat{\mu}_\varepsilon(B_r^m(y_j)) \ge \sum_{j\in J} \widehat{\mu}_\varepsilon(\mathcal{B}_{r/2}(y_j)) \ge c(\mathcal{M}, \Lambda)r^{m-2}|J|$$

(since $\frac{1}{4}g_{\mathbb{R}^m} \le g_{x,s_\varepsilon} \le 4g_{\mathbb{R}^m}$ for ε sufficiently small). Using also (5.6.21), we get $|J| \le C(\mathcal{M}, \Lambda)$. Hence, writing $y_j = (z_j, t_j)$, we obtain

$$|\mathcal{V}^\varepsilon(r)| \le \sum_{j\in J} |B_{5r}^2(z_j)| \le 25\pi|J|r^2 \le C(\mathcal{M}, \Lambda)r^2,$$

confirming (5.6.23).

Given $R > 0$, let $\{z_1^\varepsilon, \dots, z_{N(R,\varepsilon)}^\varepsilon\}$ be a maximal subset of $\mathcal{V}^\varepsilon(R\varepsilon)$ with $|z_k^\varepsilon - z_\ell^\varepsilon| \ge 2\varepsilon$. Since $\bigcup_k (B_1^2 \cap B_\varepsilon^2(z_k)) \subseteq \mathcal{V}^\varepsilon((R+1)\varepsilon)$ and the balls $B_\varepsilon^2(z_k)$ are disjoint, (5.6.23) gives a uniform bound on $N(R, \varepsilon)$ independent of ε (eventually), so up to subsequences we can assume that $N(R) = N(R, \varepsilon)$ is constant and that $\varepsilon^{-1}|z_k^\varepsilon - z_\ell^\varepsilon|$ has a limit $r_{k\ell}$ as $\varepsilon \to 0$, for each k, l.

We say that $k \sim \ell$ if $r_{k\ell} < \infty$; this is evidently an equivalence relation (as $r_{km} \le r_{k\ell} + r_{\ell m}$), so we can pick a set of representatives $\{k_1, \dots, k_P\}$ of the distinct equivalence classes $[k_1], \dots, [k_P]$ and conclude that

$$\mathcal{V}^\varepsilon(R\varepsilon) \subseteq \bigcup_{j=1}^P B^2_{S\varepsilon}(z^\varepsilon_{k_j})$$

eventually, for any fixed $S \ge S_0(R) := \max\{\sum_{\ell \in [k_j]} r_{k_j \ell} + 2 \mid j = 1, \dots, P\}$.

Fix such an S which is also bigger than the constants C in (5.6.21) and a_D^{-1}, C_D in Corollary 5.5.4. For any fixed $\delta > 0$, (5.6.20) and (5.6.21) show that, for ε sufficiently small, Proposition 5.6.7 below applies to $\widehat{u}_\varepsilon(z^\varepsilon_{k_j} + \varepsilon \cdot, t^\varepsilon + \varepsilon \cdot)$ (with $\beta := \beta_D$). Writing $K = K(\beta_D, \delta, S) > S$, note that the balls $B^2_{K\varepsilon}(z_{k_j})$ are eventually disjoint and included in $\{\chi = 1\}$. Hence, Proposition 5.6.7 and (5.6.22) give

$$\begin{aligned}
\operatorname{dist}\Big(\int_{\mathbb{R}^2 \times \{t^\varepsilon\}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon), 2\pi\mathbb{N}\Big) &\le P\delta + \int_{B^2_2 \setminus \bigcup_{j=1}^P B^2_{K\varepsilon}(z^\varepsilon_{k_j})} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(\cdot, t^\varepsilon) \\
&\le P\delta + \int_{B^2_2 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(\cdot, t^\varepsilon) \\
&\le (P+1)\delta + \int_{B^2_1 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(\cdot, t^\varepsilon)
\end{aligned}$$

(for ε sufficiently small). Choosing $\delta = \delta(R) \le \frac{1}{(P+1)R}$, we arrive at the estimate

$$\operatorname{dist}\Big(\int_{\mathbb{R}^2 \times \{t^\varepsilon\}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon), 2\pi\mathbb{N}\Big) \le \frac{1}{R} + \int_{B^2_1 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(\cdot, t^\varepsilon).$$

To conclude the proof, it suffices to show that

$$\lim_{R \to 0} \limsup_{\varepsilon \to 0} \int_{B^2_1 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(\cdot, t^\varepsilon) \to 0. \tag{5.6.24}$$

Once we have this, we infer that

$$\liminf_{\varepsilon \to 0} \operatorname{dist}\Big(\int_{\mathbb{R}^2 \times \{t^\varepsilon\}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon), 2\pi\mathbb{N}\Big) = 0$$

for the original sequence (t^ε). Noting that the choice of t^ε in $F_\varepsilon := B^{m-2}_1 \setminus E^\varepsilon_1 \cup E^\varepsilon_2 \cup E^\varepsilon_3$ was arbitrary, we get

$$\liminf_{\varepsilon \to 0} \sup_{t \in F_\varepsilon} \operatorname{dist}\Big(\int_{\mathbb{R}^2 \times \{t\}} \widehat{\chi} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon), 2\pi\mathbb{N}\Big) = 0.$$

Since the argument applies to an arbitrary subsequence $\varepsilon_j \to 0$, the proposition then follows.

To show (5.6.24), note that for $z \in B^2_1$ the distance of $\iota_{x,s_\varepsilon}((z, t^\varepsilon))$ to the set $Z_{\beta_D/2}(u_{s_\varepsilon \varepsilon})$ is (eventually) bounded below by $\frac{s_\varepsilon}{2}\min\{1, r_\varepsilon(z)\}$, where $r_\varepsilon(z)$ is the (Euclidean) distance of

(z, t^ε) to $Z_{\beta_D/2}(\widehat{u}_\varepsilon)$). Since $Z_{\beta_D/2}(u_{s_\varepsilon \varepsilon}) \supseteq Z_{\beta_D}(u_{s_\varepsilon \varepsilon})$, for any $R > 1$ Corollary 5.5.4 gives

$$\begin{aligned}
\int_{B_1^2 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon) &\le C\varepsilon^{-2} \int_{B_1^2 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} e^{-a_D r_\varepsilon(z)/(2\varepsilon)} + C\varepsilon^{-2} e^{-a_D/(2\varepsilon)} + C s_\varepsilon \varepsilon \\
&= C\varepsilon^{-3} \int_{B_1^2 \setminus \mathcal{V}^\varepsilon(R\varepsilon)} \int_{r_\varepsilon(z)}^\infty \frac{a_D}{2} e^{-a_D r/(2\varepsilon)} \, dr \, dz + C\varepsilon^{-2} e^{-a_D/(2\varepsilon)} + C s_\varepsilon \varepsilon \\
&= C\varepsilon^{-3} \int_{R\varepsilon}^\infty \frac{a_D}{2} e^{-a_D r/(2\varepsilon)} |\mathcal{V}^\varepsilon(r)| \, dr + C\varepsilon^{-2} e^{-a_D/(2\varepsilon)} + C s_\varepsilon \varepsilon \\
&\le C\varepsilon^{-3} \int_{R\varepsilon}^\infty e^{-a_D r/(2\varepsilon)} r^2 \, dr + C\varepsilon \\
&= C \int_R^\infty e^{-a_D t/2} t^2 \, dt + C\varepsilon,
\end{aligned}$$

where we used Fubini's theorem in the second equality. The statement follows. □

The following key technical proposition, used in the proof of Proposition 5.6.6, relies ultimately on the quantization phenomenon for the energy of entire solutions in the plane, presented in [59, Chapter III]. For the reader's convenience, we give a self-contained proof, including the relevant arguments from [59].

Proposition 5.6.7. *Given $0 < \beta, \delta < \frac{1}{2}$ and $S > 1$, there exist $K(\beta, \delta, S) > S$ and $0 < \kappa(\beta, \delta, S, m) < K^{-1}$ such that the following is true. Assume (u, ∇) is smooth and solves (5.2.4) and (5.2.5), with $|u| \le 1$ and $\varepsilon = 1$, on a line bundle L over a cylinder (Q, g), with $Q = B^2_{\kappa^{-1}} \times B^{m-2}_{\kappa^{-1}}$. If we have*

$$Z_{\beta/2}(u) \cap (B^2_{\kappa^{-1}} \times \{0\}) \subseteq \overline{B}^2_S \times \{0\}, \tag{5.6.25}$$

the energy bounds

$$e_1(u, \nabla) \le S, \tag{5.6.26}$$

$$\sum_{i=3}^m \int_{B^2_{\kappa^{-1}} \times B^{m-2}_r} (|\nabla_{\partial_i} u|^2 + |\iota_{\partial_i} F_\nabla|^2) \le \kappa r^{m-2} \quad \text{for all } 0 < r < \kappa^{-1}, \tag{5.6.27}$$

as well as the decay

$$e_1(u, \nabla)(p) \le S e^{-S^{-1} r} + \kappa \quad \text{whenever } \mathcal{B}_r(p) \subset\subset Q \setminus Z_\beta, \tag{5.6.28}$$

and $\|g - g_{\mathbb{R}^m}\|_{C^2} \le \kappa$, then

$$\Big| \int_{B^2_K \times \{0\}} e_1(u, \nabla) - 2\pi |p| \Big| < \delta,$$

where p is the degree of $\frac{u}{|u|}(S\cdot, 0)$, as a map from the circle to itself.

Proof. To begin with, fix a real number $K(\beta, \delta, S) > S$ so big that

$$\int_K^\infty (2\pi r) S e^{-S^{-1}(r - S)} < \delta. \tag{5.6.29}$$

Arguing by contradiction, assume there exists a sequence $\kappa_j \to 0$ such that the statement admits a counterexample (u_j, ∇_j) (for $\kappa = \kappa_j$) for a (necessarily trivial) line bundle L_j over $Q_j = B^2_{\kappa_j^{-1}} \times B^{m-2}_{\kappa_j^{-1}}$, with respect to a metric $g = g_j$ satisfying $\|g - g_{\mathbb{R}^m}\|_{C^2} \le \kappa_j$. Fixing a trivialization of L_j over Q_j, we can write $\nabla_j = d - iA_j$ for some real one-form A_j.

By virtue of the uniform pointwise estimate (5.6.28) for $e_1(u_j, \nabla_j) \ge |d|u_j||^2$, we see that the functions $|u_j|$ are locally equi-Lipschitz. In particular, we can apply the Arzelà–Ascoli theorem to extract a subsequence $|u_j|$ converging in C^0_{loc} to a continuous function $\rho_\infty : \mathbb{R}^m \to \mathbb{R}$.

Since $|\partial_k|u_j|| \le |(\nabla_j)_{\partial_k} u_j|$ for all k, (5.6.27) implies that ρ_∞ depends only on the first two variables. Moreover, (5.6.25) gives $\rho_\infty^2 \ge 1 - \frac{\beta}{2} > 1 - \beta$ outside $B^2_S \times \mathbb{R}^{m-2}$. In particular, setting

$$R_j := \max\Big\{r \le \kappa_j^{-1} : (B^2_r \setminus B^2_S) \times B^{m-2}_1 \subseteq \Big\{|u_j| > \frac{1}{2}\Big\}\Big\},$$

we have $R_j \to \infty$. Let $w_j := \frac{u_j}{|u_j|}$ on $\{|u_j| > \frac{1}{2}\}$.

The degree p_j is uniformly bounded as, for $r \ge S$ and $t \in \mathbb{R}^{m-2}$,

$$2\pi p_j = \int_{\partial B^2_r \times \{t\}} w_j^*(d\theta) = \int_{B^2_r \times \{t\}} dA_j + \int_{\partial B^2_r \times \{t\}} (w_j^*(d\theta) - A_j)$$

for j sufficiently large, so averaging over $S < r < 2S$ and $t \in B^{m-2}_1$ we get

$$\begin{aligned} 2\pi|p_j| &\le C(S) \int_{B^2_{2S} \times B^{m-2}_1} |dA_j| + C(S) \int_{(B^2_{2S} \setminus B^2_S) \times B^{m-2}_1} |w_j^*(d\theta) - A_j| \\ &\le C(\beta, S) \Big(\int_{B^2_{2S} \times B^{m-2}_1} e_1(u_j, A_j) \Big)^{1/2}, \end{aligned}$$

as $|u_j||w_j^*(d\theta) - A_j| \le |\nabla_j u_j|$. Thus, up to subsequences we can assume $p_j = p$ is constant.

We now claim that, up to change of gauge, $(u_j, A_j) \to (u_\infty, A_\infty)$ subsequentially in $C^1_{loc}(\mathbb{R}^2 \times B^{m-2}_1)$. Let $\widetilde{u}_j = e^{i\theta_j} u_j$ and $\widetilde{A}_j = A_j + d\theta_j$ be the section and the connection in the Coulomb gauge on the domain $(\overline{B}^m_{5S}, g_j)$, with $\widetilde{A}_j(\nu) = 0$ on the boundary (as described in the Appendix). Note that B^m_{5S} includes the cylinder $Q' := B^2_{4S} \times B^{m-2}_1$, and observe that, on $Q'' := (B^2_{4S} \setminus B^2_S) \times B^{m-2}_1$, $\widetilde{u}_j$ has the form

$$\widetilde{u}_j(re^{i\theta}, t) = |u_j| e^{ip\theta + i\psi_j}$$

for a unique real function ψ_j with $0 \le \psi_j(2S, 0) < 2\pi$.

Hence, $u_j = |u_j| e^{i(p\theta + \psi_j - \theta_j)}$ on Q'' and we can extend $\psi_j - \theta_j$ uniquely to a function $\sigma_j : (B^2_{R_j} \setminus B^2_S) \times B^{m-2}_1 \to \mathbb{R}$ so that $u_j = |u_j| e^{ip\theta + i\sigma_j}$ holds true on all the domain of σ_j. Finally, we replace (u_j, A_j) with $(e^{i\tau_j} u_j, A_j + d\tau_j)$, where

$$\tau_j(z, t) := \begin{cases} \theta_j - \chi(|z|)\psi_j & |z| < 4S \\ -\sigma_j & |z| > 3S \end{cases}$$

for a fixed smooth function $\chi : [0, \infty) \to [0, 1]$ such that $\chi = 0$ on $[0, 2S]$ and $\chi = 1$ on $[3S, \infty)$. Observe that, in the cylinder $Q' = B^2_{4S} \times B^{m-2}_1$, the new couple equals

$$(\widetilde{u}_j e^{-\chi(|z|)\psi_j}, \widetilde{A}_j - d(\chi(|z|)\psi_j)).$$

The function ψ_j obeys uniform local $W^{2,q}$ bounds, on (the interior of) Q'', for all $1 \le q < \infty$, thanks to the Coulomb gauge specification (per Proposition A.1 in the Appendix). Hence, the new couple (u_j, A_j) has uniform local $W^{2,q}$ bounds on Q'.

Moreover, in the exterior annular region $\mathcal{A}_j := (B^2_{R_j} \setminus \overline{B}^2_{3S}) \times B^{m-2}_1$, we have that $u_j(re^{i\theta}, t) = |u_j|e^{pi\theta}$ and we can obtain local $W^{2,q}$ bounds noting that

$$pd\theta - A_j = |u_j|^{-2}\langle \nabla_j u_j, iu_j\rangle.$$

Indeed, since the right-hand side is bounded by $2e_1(u_j, \nabla_j)^{1/2} \le 2S^{1/2}$ and $pd\theta$ is a fixed smooth one-form, we immediately obtain uniform L^∞ bounds for A_j locally in $\mathcal{A}_j$. Next, note that the identity (5.3.4) applies to give us an estimate

$$|\Delta|u_j|^2| \le Ce_1(u_j, \nabla_j) + C \le CS$$

in $\mathcal{A}_j$, from which it follows that the modulus $|u_j|$ satisfies uniform $W^{2,q}$ bounds for every $q \in (1, \infty)$ locally in $\mathcal{A}_j$. Multiplying (5.2.4) by $e^{-pi\theta}$ and taking the imaginary part gives

$$|u_j|d^*(pd\theta - A_j) = 2\langle d|u_j|, pd\theta - A_j\rangle,$$

from which it follows that d^*A_j satisfies uniform L^∞ bounds locally in $\mathcal{A}_j$ as well; together with the obvious pointwise bound $|dA_j| \le e_1(u_j, \nabla_j)^{1/2} \le S^{1/2}$, this in particular yields uniform bounds on the full derivative $\|DA_j\|_{L^q}$ for every $q \in (1, \infty)$ on fixed compact subsets of $\mathcal{A}_j$ (this follows, e.g., from [58, Lemma 4.7] and a cut-off argument).

Finally, writing (5.2.5) as

$$\Delta_H A_j = dd^*A_j + |u_j|^2(pd\theta - A_j),$$

the preceding chain of identities and estimates give a uniform L^q bound on the right-hand side over any fixed compact subset of $\mathcal{A}_j$, for any $q \in (1, \infty)$; in particular, this gives us the desired uniform local $W^{2,q}$ bounds for A_j (while we already have the desired $W^{2,q}$ bounds for $u_j = |u_j|e^{pi\theta}$).

Thanks to the compact embedding $W^{2,q} \hookrightarrow C^1$ on bounded regular domains (for $q > m$), we obtain a limit couple (u_∞, A_∞) on $\mathbb{R}^2 \times B^{m-2}_1$, as claimed, which solves (5.2.4) and (5.2.5) with respect to the flat metric. Also, $|u_\infty| = \rho_\infty$ and

$$(\nabla_\infty)_{\partial_k} u_\infty = 0, \quad \iota_{\partial_k} dA_\infty = 0 \quad \text{for } k = 3, \ldots, m. \tag{5.6.30}$$

The second part of (5.6.30) implies that we can find a function $\alpha \in C^1(\mathbb{R}^2 \times B^{m-2}_1)$ with $\alpha(z, 0) = 0$ and $\partial_k \alpha = (A_\infty)_k$, for all $z \in \mathbb{R}^2$ and all $k \ge 3$. Set $\widetilde{u}_\infty := e^{-i\alpha}u_\infty$ and $\widetilde{A}_\infty := A_\infty - d\alpha$, so that

$$(\widetilde{A}_\infty)_k = 0, \quad \partial_k(\widetilde{A}_\infty)_\ell = \partial_k(A_\infty)_\ell - \partial^2_{k\ell}\alpha = \partial_\ell(A_\infty - d\alpha)_k = 0$$

for all $k = 3, \dots, m$ and $\ell = 1, \dots, m$ (using again the second part of (5.6.30)). The first part gives instead $\partial_k \widetilde{u}_\infty = 0$ for $k = 3, \dots, m$. Hence, $(\widetilde{u}_\infty, \widetilde{A}_\infty)$ depends only on the first two variables and therefore corresponds to a planar solution of (5.2.4) and (5.2.5).

Also, from (5.6.28) we deduce that

$$e_1(\widetilde{u}_\infty, \widetilde{A}_\infty)(z,t) = e_1(u_\infty, A_\infty)(z,t) = \lim_{j\to\infty} e_1(u_j, A_j)(z,t) \le S e^{-S^{-1}(|z|-S)} \tag{5.6.31}$$

for $|z| > S$, as eventually $\overline{B}^m_{|z|-S}(z,t) \cap Z_\beta(u_j) = \emptyset$.

Integrating (5.4.4) on $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ against the position vector field we get

$$\int_{\mathbb{R}^2} |d\widetilde{A}_\infty|^2 = \int_{\mathbb{R}^2} W(\widetilde{u}_\infty).$$

Thanks to the decay of $e_1(\widetilde{u}_\infty, \widetilde{A}_\infty)$, we can repeat the proof of (5.3.6): starting from

$$\Delta \widetilde{\xi}_\infty \ge |\widetilde{u}_\infty|^2 \widetilde{\xi}_\infty, \text{ with } \widetilde{\xi}_\infty := |d\widetilde{A}_\infty| - \frac{1 - |\widetilde{u}_\infty|^2}{2},$$

and applying the maximum principle, we deduce that the decaying function $\widetilde{\xi}_\infty$ is nonpositive. We then obtain $|d\widetilde{A}_\infty| \le \sqrt{W(\widetilde{u}_\infty)}$, so we must have $|d\widetilde{A}_\infty| = \sqrt{W(\widetilde{u}_\infty)}$ everywhere (cf. [59, Section III.10]).

Observe that, by (5.3.4) and the strong maximum principle, $|\widetilde{u}_\infty| < 1$ (unless $|\widetilde{u}_\infty| = 1$ everywhere, in which case $|d\widetilde{A}_\infty| = \sqrt{W(\widetilde{u}_\infty)} = 0$ and $|\widetilde{\nabla}_\infty \widetilde{u}_\infty| = 0$ by (5.3.4), thus $e_1(\widetilde{u}_\infty, \widetilde{A}_\infty) = 0$ and $p = 0$; so the statement of the proposition holds eventually, contradiction). As a consequence, $|*d\widetilde{A}_\infty| = W(\widetilde{u}_\infty) > 0$ and we get either $\frac{1-|\widetilde{u}_\infty|^2}{2} = *d\widetilde{A}_\infty$ everywhere or $\frac{1-|\widetilde{u}_\infty|^2}{2} = -*d\widetilde{A}_\infty$ everywhere. Thus, integrating by parts and using (5.2.4), as well as the decay of $|p d\theta - \widetilde{A}_\infty|$,

$$\begin{aligned}\int_{\mathbb{R}^2} e_1(\widetilde{u}_\infty, \widetilde{A}_\infty) &= \int_{\mathbb{R}^2} (|\widetilde{\nabla}_\infty \widetilde{u}_\infty|^2 + 2W(\widetilde{u}_\infty)) = \int_{\mathbb{R}^2} (\langle \widetilde{\nabla}^*_\infty \widetilde{\nabla}_\infty \widetilde{u}_\infty, \widetilde{u}_\infty \rangle + 2W(\widetilde{u}_\infty)) \\ &= \int_{\mathbb{R}^2} \frac{1 - |\widetilde{u}_\infty|^2}{2} = \pm \int_{\mathbb{R}^2} d\widetilde{A}_\infty = \pm \lim_{r\to\infty} \int_{\partial B^2_r} \widetilde{A}_\infty = \pm \lim_{r\to\infty} \int_{\partial B^2_r} p\,d\theta = \pm 2\pi p.\end{aligned}$$

Hence, the energy of the two-dimensional solution $(\widetilde{u}_\infty, \widetilde{A}_\infty)$ is $2\pi|p|$. Our choice of K, namely (5.6.29), together with (5.6.31), then ensures that

$$\operatorname{dist}\Big(\int_{B^2_K \times \{0\}} e_1(u_\infty, A_\infty), 2\pi\mathbb{N}\Big) < \delta.$$

As a consequence, this must hold eventually also for (u_j, A_j), giving the desired contradiction. □

Remark 5.6.8. As a consequence, one also finds that

$$\int_{B^2_K \times \{0\}} e_1(u, \nabla) < \delta$$

if $|u| > 0$ everywhere on the cylinder Q. Indeed, if $|u| > 0$ everywhere, then the degree p in the statement of Proposition 5.6.7 clearly must vanish.

We are now able to address the statement on the convergence of level sets.

Proposition 5.6.9. *For any* $0 \le \delta < 1$ *we have* $\mathrm{spt}(\mu) = \lim_{\varepsilon\to 0}\{|u_\varepsilon| \le \delta\}$, *in the Hausdorff topology.*

Proof. If $x = \lim_{\varepsilon\to 0} x_\varepsilon$, for points $x_\varepsilon \in \{|u_\varepsilon| \le \delta\}$ defined along a subsequence, then the same argument used in the proof of Proposition 5.6.2 shows that $x \in \mathrm{spt}(\mu)$. Hence, for all $\eta > 0$, eventually $\{|u_\varepsilon| \le \delta\}$ is included in the η-neighborhood of $\mathrm{spt}(\mu)$.

To conclude the proof, it suffices to show that the converse inclusion $\mathrm{spt}(\mu) \subseteq B_\eta(\{u_\varepsilon = 0\})$ holds eventually. Arguing by contradiction, assume that there are points $p_\varepsilon \in \mathrm{spt}(\mu)$ whose distance from $\{u_\varepsilon = 0\}$ is at least η, along some subsequence (not relabeled). Up to further subsequences, let $p_\varepsilon \to p_0 \in \mathrm{spt}(\mu)$.

Since μ is $(m-2)$-rectifiable, there exists a point $q \in \mathrm{spt}(\mu)$ with $\mathrm{dist}(p_0, q) < \frac{\eta}{2}$, and such that μ blows up to $\Theta_{m-2}(\mu, q)\mathcal{H}^{m-2} \llcorner T_q\Sigma$ at q. Observe that eventually we have

$$\mathrm{dist}(q, \{u_\varepsilon = 0\}) \ge \frac{\eta}{2}. \tag{5.6.32}$$

Now, repeating all the preceding blow-up analysis at q, in view of Remark 5.6.8 we can improve (5.6.17) to the uniform convergence

$$\int_{\mathbb{R}^2\times\{t\}} \chi(z) e_\varepsilon(\widehat{u}_\varepsilon, \widehat{\nabla}_\varepsilon)(z,t) \to 0$$

for $t \in F_\varepsilon$, which implies that $\Theta_{m-2}(\mu, q) = 0$. However, since $q \in \mathrm{spt}(\mu)$, this is impossible, by Proposition 5.6.2. □

Limiting behavior of the curvature

As before, we identify the curvature F_{∇_ε} with a closed two-form ω_ε by $F_{\nabla_\varepsilon}(X,Y) = -i\omega_\varepsilon(X,Y)$. Recall that the cohomology class $[\frac{1}{2\pi}\omega_\varepsilon]$ represents the (rational) first Chern class $c_1(L) \in H^2(\mathcal{M};\mathbb{R})$ of the complex line bundle $L \to \mathcal{M}$.

Theorem 5.6.10. *Let* $(u_\varepsilon, \nabla_\varepsilon)$ *be a family as in Theorem 5.6.1. The curvature forms* $\frac{1}{2\pi}\omega_\varepsilon$ *can be identified with* $(m-2)$*-currents that converge (weakly), as* $\varepsilon \to 0$*, to an integer rectifiable cycle* Γ *which is Poincaré dual to* $c_1(L)$*, and whose mass measure* $|\Gamma|$ *satisfies* $|\Gamma| \le \mu$.

Proof. Recall from Section 5.2 that

$$d\langle \nabla_\varepsilon u_\varepsilon, i u_\varepsilon\rangle = \psi(u_\varepsilon) - |u_\varepsilon|^2\omega_\varepsilon,$$

where $\psi(u_\varepsilon) = \langle 2i\nabla u_\varepsilon, \nabla_\varepsilon u_\varepsilon\rangle$ is a two-form satisfying $|\psi(u_\varepsilon)| \le |\nabla_\varepsilon u_\varepsilon|^2$ pointwise. In particular, denoting by $J(u_\varepsilon, \nabla_\varepsilon)$ the two-form

$$J(u_\varepsilon, \nabla_\varepsilon) := \psi(u_\varepsilon) + (1 - |u_\varepsilon|^2)\omega_\varepsilon,$$

we can rewrite this identity as

$$J(u_\varepsilon, \nabla_\varepsilon) - \omega_\varepsilon = d\langle \nabla_\varepsilon u_\varepsilon, iu_\varepsilon \rangle, \tag{5.6.33}$$

and observe that

$$|J(u_\varepsilon, \nabla_\varepsilon)| \le |\nabla_\varepsilon u_\varepsilon|^2 + \varepsilon^2 |\omega_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2 = e_\varepsilon(u_\varepsilon, \nabla_\varepsilon). \tag{5.6.34}$$

The dual $(m-2)$-currents given by

$$\langle \Gamma_\varepsilon, \zeta \rangle := \frac{1}{2\pi} \int_{\mathcal{M}} J(u_\varepsilon, \nabla_\varepsilon) \wedge \zeta,$$

for any $(m-2)$-form $\zeta \in \Omega^{m-2}(\mathcal{M})$, are thus bounded in mass by $\frac{1}{2\pi}\Lambda$. (Here we compute the mass with the ℓ^2 norm on exterior algebras; for the limit current, by rectifiability this will coincide with the usual mass, dual to the comass.) Up to subsequences, we can take a weak limit Γ. The bound $|\Gamma_\varepsilon| \le \mu_\varepsilon$ implies that also $|\Gamma| \le \mu$.

From (5.6.33) and integration by parts we get

$$\int_{\mathcal{M}} \omega_\varepsilon \wedge \zeta = \int_{\mathcal{M}} J(u_\varepsilon, \nabla_\varepsilon) \wedge \zeta - \int_{\mathcal{M}} \langle \nabla_\varepsilon u_\varepsilon, iu_\varepsilon \rangle \wedge d\zeta.$$

Since (as discussed in the proof of Proposition 5.6.2)

$$\int_{\mathcal{M}} |\langle \nabla_\varepsilon u_\varepsilon, iu_\varepsilon \rangle| \le \int_{\mathcal{M}} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2} \to 0$$

as $\varepsilon \to 0$, it follows that

$$\langle \Gamma, \zeta \rangle = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathcal{M}} J(u_\varepsilon, \nabla_\varepsilon) \wedge \zeta = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathcal{M}} \omega_\varepsilon \wedge \zeta \tag{5.6.35}$$

for every smooth $(m-2)$-form $\zeta \in \Omega^{m-2}(\mathcal{M})$.

Since the two-forms ω_ε are closed, for any $\xi \in \Omega^{m-3}(\mathcal{M})$ we have

$$\langle \partial\Gamma, \xi \rangle = \langle \Gamma, d\xi \rangle = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathcal{M}} \omega_\varepsilon \wedge d\xi = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathcal{M}} d(\omega_\varepsilon \wedge \xi) = 0,$$

so Γ is a cycle. Since μ is $(m-2)$-rectifiable, Γ must be a rectifiable $(m-2)$-current: this can be seen by blow-up, applying [64, Proposition 7.3.5]. By (5.6.35), Γ is Poincaré dual to $c_1(L)$.

To complete the proof, it remains to show that Γ has integer multiplicity. By means of a diagonal selection of a subsequence, as in the previous subsection, we can deduce integrality at those points $p \in \operatorname{spt}(\mu)$ where μ and Γ blow up respectively to $\Theta_{m-2}(\mu, p)\mathcal{H}^{m-2} \llcorner T_p\Sigma$ and a multiple of $[T_p\Sigma]$, using the following lemma. Note that its hypotheses are verified thanks to Corollary 5.5.4 and the fact that $Z_{\beta_D}(u_\varepsilon)$ necessarily converges to a subset of $T_p\Sigma$ in the local Hausdorff topology, after rescaling (see the proof of Proposition 5.6.2).

Since μ is $(m-2)$-rectifiable, we deduce that the limiting current Γ has integer multiplicity $\mathcal{H}^{m-2}$-a.e. on its support, as claimed. □

Lemma 5.6.11. *On the Euclidean ball B_4^m, let $(u_\varepsilon, \nabla_\varepsilon)$ be a sequence of sections and connections in a trivial line bundle $L \to B_4^m$ (not necessarily satisfying any equation) for which $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda$, $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \to 0$ in $C^0_{loc}(B_4^m \setminus P)$ and $*\omega_\varepsilon \to \theta_1[P]$ in $\mathcal{D}_{m-2}(B_4^m)$, where $P = \{0\} \times \mathbb{R}^{m-2}$. Then $\theta_1 \in 2\pi\mathbb{Z}$.*

Proof. To begin, fix a test function $\varphi \in C_c^1(B_1^2 \times B_1^{m-2})$ of the form $\varphi(x^1, \ldots, x^m) = \psi(x^1, x^2)\eta(x^3, \ldots, x^m)$, with $\psi(x^1, x^2) = 1$ for $|(x^1, x^2)| \leq \frac{1}{2}$. In the sequel, we shall omit the domain of integration when it equals $\mathbb{R}^m$. By assumption, we then have

$$\theta_1 \int_P \eta dx^3 \wedge \cdots \wedge dx^m = \lim_{\varepsilon \to 0} \int \varphi \omega_\varepsilon \wedge dx^3 \wedge \cdots \wedge dx^m.$$

Fixing trivializations of L over B_2^m, we write $\nabla_\varepsilon = d - iA_\varepsilon$ for some one-forms A_ε, so that $\omega_\varepsilon = dA_\varepsilon$, and the right-hand term in the preceding limit becomes

$$\begin{aligned}\int \omega_\varepsilon \wedge (\varphi dx^3 \wedge \cdots \wedge dx^m) &= \int d(\varphi A_\varepsilon \wedge dx^3 \wedge \cdots \wedge dx^m) \\ &\quad + \int A_\varepsilon \wedge d\varphi \wedge dx^3 \wedge \cdots \wedge dx^m \\ &= \int \eta |u_\varepsilon|^2 A_\varepsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m \\ &\quad + \int \eta(1 - |u_\varepsilon|^2) A_\varepsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m.\end{aligned}$$

On B_2^m we can choose our trivializations so that $d^* A_\varepsilon = 0$, and $A_\varepsilon(\nu) = 0$ on ∂B_2^m (see the Appendix). We then have the L^2 control

$$\int_{B_2^m} |A_\varepsilon|^2 \leq C \int_{B_2^m} |dA_\varepsilon|^2 \leq C\varepsilon^{-2}\Lambda \tag{5.6.36}$$

(see, e.g., [58, Theorem 4.8]), and consequently

$$\begin{aligned}\left| \int \eta(1 - |u_\varepsilon|^2) A_\varepsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m \right| &\leq C \|1 - |u_\varepsilon|^2\|_{C^0(\mathrm{spt}(\eta d\psi))} \|A_\varepsilon\|_{L^1(B_2^m)} \\ &\leq C\Lambda^{1/2} \|\varepsilon^{-1}(1 - |u_\varepsilon|^2)\|_{C^0(\mathrm{spt}(\eta d\psi))} \\ &\leq C\Lambda^{1/2} \|e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\|^{1/2}_{C^0(\mathrm{spt}(\eta d\psi))} \\ &\to 0\end{aligned}$$

as $\varepsilon \to 0$, where we have used the fact that $d\psi(x^1, x^2) = 0$ for $|(x^1, x^2)| \leq \frac{1}{2}$, and the assumption that $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \to 0$ in $C^0_{loc}(B_2^m \setminus P)$.

Combining our computations thus far, we have arrived at the identity

$$\theta_1 \int_P \eta dx^3 \wedge \cdots \wedge dx^m = \lim_{\varepsilon \to 0} \int \eta |u_\varepsilon|^2 A_\varepsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m.$$

Noting next that

$$||u_\varepsilon|^2 A_\varepsilon - \langle du_\varepsilon, iu_\varepsilon \rangle| = |\langle \nabla_\varepsilon u_\varepsilon, iu_\varepsilon \rangle| \leq e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)^{1/2},$$

and using again the hypothesis that $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \to 0$ uniformly on $\mathrm{spt}(\eta d\psi)$, the preceding identity yields

$$\begin{aligned}\theta_1 \int_P \eta dx^3 \wedge \cdots \wedge dx^m &= \lim_{\varepsilon\to 0} \int \eta \langle du_\varepsilon, iu_\varepsilon\rangle \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m \\ &= \lim_{\varepsilon\to 0} \int \eta |u_\varepsilon|^2 (u_\varepsilon/|u_\varepsilon|)^*(d\theta) \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m \\ &= \lim_{\varepsilon\to 0} \int \eta (u_\varepsilon/|u_\varepsilon|)^*(d\theta) \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^m.\end{aligned}$$

Finally, since the one-form $(u_\varepsilon/|u_\varepsilon|)^*(d\theta)$ is closed on $\{u_\varepsilon \neq 0\}$ and $d\eta\wedge dx^3\wedge\cdots\wedge dx^m = 0$, integrating by parts on $(\mathbb{R}^2 \setminus B^2_{1/2}) \times \mathbb{R}^{m-2}$ we see that

$$\begin{aligned}\int \eta (u_\varepsilon/|u_\varepsilon|)^*(d\theta) \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^n &= \int_{\mathbb{R}^{m-2}} \eta(t) \int_{\partial B^2_{1/2}\times\{t\}} (u_\varepsilon/|u_\varepsilon|)^*(d\theta)\, dt \\ &= 2\pi \deg(u_\varepsilon, P) \int_P \eta,\end{aligned}$$

where $\deg(u_\varepsilon, P)$ stands for the degree of $(u_\varepsilon/|u_\varepsilon|)(\frac{1}{2}e^{i\theta}, 0)$. The statement follows. □

5.7 Examples from variational constructions

The goal of this section is to show that, for every closed manifold $\mathcal{M}$ and every line bundle $L \to \mathcal{M}$ endowed with a Hermitian metric, there exist critical couples $(u_\varepsilon, \nabla_\varepsilon)$ for the Yang–Mills–Higgs functional E_ε, for ε small enough, in such a way that

$$0 < \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) < \infty. \tag{5.7.1}$$

This will be easier when the line bundle is nontrivial, as in this case we can just take $(u_\varepsilon, \nabla_\varepsilon)$ to be a global minimizer for E_ε. The upper and lower bounds in (5.7.1) have the following immediate consequence—proved previously by Almgren [7] using GMT methods.

Corollary 5.7.1. *Every closed Riemannian manifold $(\mathcal{M}^m, g)$ supports a nontrivial stationary, integral $(m-2)$-varifold.*

Proof. We can always equip $\mathcal{M}$ with the trivial line bundle $L := \mathbb{C} \times \mathcal{M}$. As shown in the next subsection, there exists a sequence of critical couples $(u_\varepsilon, \nabla_\varepsilon)$ satisfying (5.7.1). The statement now follows from Theorem 5.6.1. □

Min-max families for the trivial line bundle

In this section we will show how min-max methods may be applied to the functionals E_ε to produce nontrivial critical points in the trivial bundle $L = \mathbb{C} \times \mathcal{M}$ on an arbitrary closed manifold $\mathcal{M}$ of dimension $m \geq 2$. The min-max construction that we consider here is based on two-parameter families parametrized by the unit disk, similar to the constructions employed in [21] and [102] for the Ginzburg–Landau functionals—with several technical adjustments to account for the gauge-invariance and other features particular to the Yang–Mills–Higgs energies.

One can show that the families we consider induce a nontrivial class in $\pi_2(\mathcal{M})$ for the quotient

$$\mathcal{M} := \{(u, \nabla) \mid 0 \not\equiv u \in \Gamma(L),\ \nabla \text{ a Hermitian connection}\}/\{\text{gauge transformations}\},$$

and the analysis that follows can be reformulated in terms of min-max methods applied directly to $\mathcal{M}$, which can be given the structure of a Banach manifold.

Without loss of generality, we assume henceforth that $\mathcal{M}$ is connected. In some proofs we will also implicitly assume that $m = \dim(\mathcal{M}) \geq 3$, leaving the obvious changes for $m = 2$ to the reader.

Definition 5.7.2. Fix $m = \dim(\mathcal{M}) < p < \infty$. In what follows, $\widehat{X}$ will denote the Banach space of couples (u, A), where $u \in L^p(\mathcal{M}, \mathbb{C})$ and $A \in \Omega^1(\mathcal{M}, \mathbb{R})$, both of class $W^{1,2}$, with the norm

$$\|(u, A)\| := \|u\|_{L^p} + \|du\|_{L^2} + \|A\|_{L^2} + \|DA\|_{L^2}.$$

Denote by $X := \{(u, A) \in \widehat{X} : d^*A = 0\}$ the subspace consisting of those couples for which the connection form A is co-closed.

Note that, for $(u, A) \in X$, the full covariant derivative $\int_{\mathcal{M}} |DA|^2$ is bounded by $C(\mathcal{M}) \int_{\mathcal{M}}(|A|^2 + |dA|^2)$: see, e.g., [58, Theorem 4.8] for a proof.

Definition 5.7.3. Given a form $A \in \Omega^1(\mathcal{M}, \mathbb{R})$ in L^2, we denote by $h(A)$ the harmonic part of its Hodge decomposition, or equivalently the orthogonal projection of A onto the (finite-dimensional) space $\mathcal{H}^1(\mathcal{M})$ of harmonic one-forms.

Remark 5.7.4. Selection of a Coulomb gauge gives a continuous retraction $\mathcal{R} : \widehat{X} \to X$: namely, given a couple $(u, A) \in \widehat{X}$, consider the unique solution $\theta \in W^{2,2}(\mathcal{M}, \mathbb{R})$ to the equation

$$\Delta\theta = d^*A,$$

with $\int_{\mathcal{M}} \theta = 0$, and set

$$\mathcal{R}((u, A)) := (e^{i\theta}u, A + d\theta).$$

Note that the continuity of $(u, A) \mapsto d(e^{i\theta}u) = e^{i\theta}(du + iud\theta)$, from $\widehat{X}$ to L^2, follows from the fact that $L^p \cdot L^{2^*} \subseteq L^2$, where $2^* = \frac{2m}{m-2}$.

Throughout this section, $W(u) = f(|u|)$ will be a smooth radial function given by $W(u) = \frac{(1-|u|^2)^2}{4}$ for $|u| \leq 3/2$, and satisfying $W(u), W'(u)[u] > 0$ for all $|u| > 1$. For technical reasons, we also find it convenient to require that

$$W(u) = |u|^p \quad \text{for } |u| \geq 2, \tag{G}$$

which evidently gives the additional estimates $|u|f'(|u|)+|u|^2f''(|u|)\leq C|u|^p$ for $|u|\geq 2$, for some constant C. For future use, observe also that the potential $W(u)$ then satisfies a simple bound of the form

$$(1-|u|)^2 \leq CW(u). \tag{5.7.2}$$

Proposition 5.7.5. *The functional E_ε is of class C^1 on $\widehat{X}$. Moreover, a couple (u,A) is critical in $\widehat{X}$ for E_ε if and only if $\mathcal{R}((u,A))$ is critical in X. Critical points are smooth up to change of gauge.*

Proof. Given a point $(u,A)\in\widehat{X}$ and a pair $(v,B)\in\widehat{X}$ with $\|(v,B)\|_{\widehat{X}}\leq 1$, direct computation gives

$$\begin{aligned} E_\varepsilon(u+v,A+B) = E_\varepsilon(u,A) &+ 2\int_{\mathcal{M}}\langle du-iuA, dv-ivA-iuB\rangle \\ &+2\varepsilon^2\int_{\mathcal{M}}\langle dA,dB\rangle+\varepsilon^{-2}\int_{\mathcal{M}}W'(u)[v]+O(\|(v,B)\|^2_{\widehat{X}}), \end{aligned}$$

where we are using the fact that $\widehat{X}\cdot\widehat{X}\subseteq L^m\cdot L^{2^*}\subseteq L^2$ to see that

$$\|vA\|^2_{L^2}+\|uB\|^2_{L^2}+\|vB\|^2_{L^2}+E_\varepsilon(u,A)^{1/2}\|vB\|_{L^2}=O(\|(v,B)\|^2_{\widehat{X}}),$$

and we invoke our assumptions on the structure of W to see that

$$\int_{\mathcal{M}}(W(u+v)-W(u))=\int_M W'(u)[v]+O(\|(v,B)\|^2_{\widehat{X}})$$

for fixed $(u,A)\in\widehat{X}$. It follows immediately that E_ε is C^1 on $\widehat{X}$, with differential

$$dE_\varepsilon(u,A)[v,B]=\int_{\mathcal{M}}(2\langle du-iuA,dv-ivA-iuB\rangle+2\varepsilon^2\langle dA,dB\rangle+\varepsilon^{-2}W'(u)[v]).$$

To confirm the second statement, assume without loss of generality that v and B are smooth, and observe that

$$\mathcal{R}((u+tv,A+tB))=(e^{ti\psi}\widetilde{u}+te^{i\theta+ti\psi}v,\widetilde{A}+tB+td\psi),$$

where $(\widetilde{u},\widetilde{A}):=\mathcal{R}((u,A))=(e^{i\theta}u,A+d\theta)$ and ψ solves $\Delta\psi=d^*B$. This easily gives

$$\mathcal{R}((u+tv,A+tB))=\mathcal{R}((u,A))+t(e^{i\theta}v+i\psi\widetilde{u},B+d\psi)+o(t)\quad\text{in } X$$

and, using the gauge invariance $E_\varepsilon=E_\varepsilon\circ\mathcal{R}$, we deduce that

$$dE_\varepsilon(u,A)[v,B]=dE_\varepsilon(\widetilde{u},\widetilde{A})[e^{i\theta}v+i\psi\widetilde{u},B+d\psi]. \tag{5.7.3}$$

It follows that if $(\widetilde{u},\widetilde{A})$ is critical for E_ε in X then (u,A) is critical for E_ε in $\widehat{X}$, as claimed. The converse is similar.

Finally, if (u, A) is critical for E_ε (in either $\widehat{X}$ or X), then applying the above formula for the differential with $v = (|u|-1)^+ u/|u| \in W^{1,2}$ and $B = 0$ we get

$$\begin{aligned} 0 &= \int_{\mathcal{M}} 2\langle (d-iA)u, (d-iA)v\rangle + \varepsilon^{-2}\int_{\mathcal{M}} W'(u)[v] \\ &\geq \varepsilon^{-2}\int_{\mathcal{M}} |u|^{-1}(|u|-1)^+ W'(u)[u], \end{aligned}$$

where we used the fact that $\langle u \otimes d((|u|-1)^+/|u|), \nabla u\rangle$ equals $|u|^{-1}|d|u||^2 \geq 0$ a.e. on $\{|u| > 1\}$ and vanishes elsewhere. Since $W'(u)[u] > 0$ on $\{|u| > 1\}$ by our assumption on W, we deduce that $|u| \leq 1$. Together with Proposition A.1 and Remark A.3 in the Appendix, this implies that (u, A) is smooth in an appropriate (Coulomb) gauge. □

We next show that the functionals E_ε satisfy a suitable variant of the Palais–Smale condition on X, giving compactness of critical sequences for E_ε after an appropriate change of gauge. (Cf. [61] for similar results in the Seiberg–Witten setting.)

Proposition 5.7.6. *The functional E_ε satisfies the following form of the Palais–Smale condition: every sequence (u_j, A_j) in X with bounded energy and $dE_\varepsilon(u_j, A_j) \to 0$ in X^* admits a subsequence converging strongly in X to a critical couple (u_∞, A_∞), up to possibly replacing (u_j, A_j) with*

$$v_j \cdot (u_j, A_j) := (v_j u_j, A_j + v_j^*(d\theta))$$

for suitable smooth harmonic functions $v_j : \mathcal{M} \to S^1$.

Proof. First, we show that the boundedness of $E_\varepsilon(u_j, A_j)$ implies the boundedness of the sequence in X, up to a change of gauge as in the statement. The assumption (G) on the potential W gives

$$\int_{\mathcal{M}} |u_j|^p \leq C + \int_{\mathcal{M}} W(u_j) \leq C + E_\varepsilon(u_j, A_j) \leq C, \tag{5.7.4}$$

that is, u_j is uniformly bounded in L^p.

Denote by $\Lambda \subseteq \mathcal{H}^1(\mathcal{M})$ the lattice in the space of harmonic one-forms given by

$$\begin{aligned} \Lambda &:= \{-v_j^*(d\theta) \mid v_j : \mathcal{M} \to S^1 \text{ harmonic}\} \\ &= \Big\{h \in \mathcal{H}^1(\mathcal{M}) : \int_\gamma h \in 2\pi\mathbb{Z} \text{ for every } \gamma \in C^1(S^1, \mathcal{M})\Big\}, \end{aligned}$$

and let $\lambda_j \in \Lambda$ be a closest integral harmonic one-form to $h(A_j)$ (with respect to the L^2 norm, say, on $\mathcal{H}^1(\mathcal{M})$). Then $\lambda_j = -v_j^*(d\theta)$ for a suitable harmonic map $v_j : \mathcal{M} \to S^1$, and

$$\|\lambda_j - h(A_j)\|_{L^2} \leq C(\mathcal{M}).$$

Replacing (u_j, A_j) with the change of gauge $(v_j u_j, A_j - \lambda_j) \in X$, we can then assume that $h(A_j)$ is bounded.

By standard Hodge theory we can write

$$A_j = h(A_j) + d^*\xi_j$$

for some closed $\xi_j \in W^{2,2}$ satisfying $\Delta_H \xi_j = dA_j$ and $\|d^*\xi_j\|_{W^{1,2}} \le C(\mathcal{M})\|dA_j\|_{L^2}$. Thus, given the energy bound $E_\varepsilon(u_j, A_j) \le C$, we see that

$$\|A_j\|_{W^{1,2}}^2 \le C + 2\|d^*\xi_j\|_{W^{1,2}}^2 \le C + C\|dA_j\|_{L^2}^2 \le C,$$

whereby A_j is bounded in $W^{1,2}$ and, consequently, in L^{2^*}. As a consequence, we see next that

$$\begin{aligned}\|du_j\|_{L^2}^2 &\le 2\int_{\mathcal{M}} |du_j - iu_jA_j|^2 + 2\int_{\mathcal{M}} |u_jA_j|^2 \\ &\le C + C\|u_j\|_{L^p}^2 \|A_j\|_{L^{2^*}}^2 \\ &\le C + C\|u_j\|_{L^p}^p;\end{aligned}$$

taking into account (5.7.4), we infer then that $\|du_j\|_{L^2}$ is also bounded as $j \to \infty$.

We have therefore shown that (u_j, A_j) is uniformly bounded in X as $j \to \infty$, so passing to subsequences we can assume that (u_j, A_j) converges pointwise a.e. and weakly (in X) to a limiting couple (u_∞, A_∞).

In particular, defining r by

$$\frac{1}{r} := \frac{1}{2} - \frac{1}{q} > \frac{1}{2} - \frac{1}{m} = \frac{1}{2^*},$$

where $m < q < p$ is an arbitrary fixed exponent, it follows from the compactness of the embedding $W^{1,2} \hookrightarrow L^r$ that

$$A_j \to A_\infty \text{ strongly in } L^r.$$

Moreover, the boundedness of u_j in L^p and the pointwise convergence to u_∞ give

$$u_j \to u_\infty \text{ strongly in } L^q. \tag{5.7.5}$$

By definition of r, this implies in particular that

$$\lim_{j,k\to\infty} u_jA_k = u_\infty A_\infty \text{ strongly in } L^2.$$

Next, compute

$$\begin{aligned}dE_\varepsilon(u_j, A_j)[u_j - u_k, A_j - A_k] = &\int_{\mathcal{M}} 2\langle (d - iA_j)u_j, (d - iA_j)(u_j - u_k) - iu_j(A_j - A_k)\rangle \\ &+ \int_{\mathcal{M}} (2\varepsilon^2\langle dA_j, d(A_j - A_k)\rangle + \varepsilon^{-2}W'(u_j)[u_j - u_k]),\end{aligned}$$

and observe that, due to the L^2 convergence $u_jA_k \to u_\infty A_\infty$, the right-hand side equals

$$\int_{\mathcal{M}} (2\langle (d - iA_j)u_j, d(u_j - u_k)\rangle + 2\varepsilon^2\langle dA_j, d(A_j - A_k)\rangle + \varepsilon^{-2}W'(u_j)[u_j - u_k]) + o(1)$$

as $j,k\to\infty$. For the difference

$$D_{j,k} := dE_\varepsilon(u_j,A_j)[u_j-u_k,A_j-A_k] - dE_\varepsilon(u_k,A_k)[u_j-u_k,A_j-A_k],$$

we then see that

$$D_{j,k} = \int_{\mathcal{M}} (2|d(u_j-u_k)|^2 + 2\varepsilon^2|d(A_j-A_k)|^2 + \varepsilon^{-2}(W'(u_j)-W'(u_k))[u_j-u_k]) + o(1)$$

as $j,k\to\infty$.

Now, by our assumption (G) on the structure of $W(u)$, it is not difficult to check (see, e.g., [48, Corollary 1]) that the zeroth order term in our computation for $D_{j,k}$ satisfies a lower bound

$$(W'(u_j)-W'(u_k))[u_j-u_k] \geq C^{-1}|u_j-u_k|^p - C|u_j-u_k|$$

for some constant $C>0$. In particular, it follows now from the preceding computations and the L^1 convergence $u_j\to u_\infty$ that

$$D_{j,k} \geq \int_{\mathcal{M}} (2|d(u_j-u_k)|^2 + 2\varepsilon^2|d(A_j-A_k)|^2 + C^{-1}\varepsilon^{-2}|u_j-u_k|^p) + o(1)$$

as $j,k\to\infty$. On the other hand, since $dE_\varepsilon(u_j,A_j)\to 0$ and (u_j-u_k,A_j-A_k) is bounded in X, we know also that

$$D_{j,k}\to 0 \quad \text{as } j,k\to\infty,$$

and it then follows that (u_j,A_j) is Cauchy in X. In particular, (u_j,A_j) converges strongly to (u_∞,A_∞), which necessarily satisfies

$$dE_\varepsilon(u_\infty,A_\infty) = \lim_{j\to\infty} dE_\varepsilon(u_j,A_j) = 0. \qquad \square$$

Having confirmed that the energies E_ε satisfy a Palais–Smale condition, we now argue in roughly the same spirit as [21, 102] to produce nontrivial critical points via min-max methods. To begin, note that the space X splits as $\mathbb{C}\oplus Y$, where $\mathbb{C}$ is identified with the set of constant couples $(\alpha,0)$ and

$$Y := \Big\{(u,A)\in X : \int_{\mathcal{M}} u = 0\Big\}.$$

Definition 5.7.7. Let Γ denote the set of continuous families of couples $F:\overline{D}\to X$ parametrized by the closed unit disk $\overline{D}$, with

$$F(e^{i\theta}) = (e^{i\theta},0)$$

for all $\theta\in\mathbb{R}$. Equivalently, under the above identification $\mathbb{C}\subset X$, we require $F|_{\partial D}=\mathrm{id}$. We denote by $\omega_\varepsilon(\mathcal{M})$ the "width" of Γ with respect to the energy E_ε, namely

$$\omega_\varepsilon(\mathcal{M}) := \inf_{F\in\Gamma}\max_{y\in\overline{D}} E_\varepsilon(F(y)).$$

Thanks to Proposition 5.7.6, we can apply classical min-max theory for C^1 functionals on Banach spaces (see e.g. [42, Theorem 3.2]) to conclude that ω_ε is achieved as the energy of a smooth critical couple $(u_\varepsilon, A_\varepsilon)$. In the following proposition, we show that $\omega_\varepsilon(\mathcal{M})$ is positive, so that the corresponding critical couples $(u_\varepsilon, A_\varepsilon)$ are nontrivial.

Proposition 5.7.8. *We have* $\omega_\varepsilon(\mathcal{M}) > 0$.

Proof. We argue by contradiction, though the proof could be made quantitative. Since we are proving only the positivity $\omega_\varepsilon(\mathcal{M}) > 0$ at this stage—making no reference to the dependence on ε—in what follows we take $\varepsilon = 1$ for convenience. Assume that we have a family $F \in \Gamma$ with $\max_{y\in\overline{D}} E(F(y)) < \delta$, with δ very small. Writing $F(y) = (u, A)$, this implies that

$$\|A - h(A)\|_{W^{1,2}} \le C\|dA\|_{L^2} < C\delta^{1/2}, \quad \|DA\|_{L^2} \le C(\delta^{1/2} + \|h(A)\|). \tag{5.7.6}$$

When $b_1(\mathcal{M}) \neq 0$, some additional work is required to deduce that the harmonic part $h(A)$ of A must also be small for all couples $(u, A) = F(y)$ in the family. In particular, we will need to employ the following lemma, showing that $h(A)$ lies close to the integral lattice $\Lambda \subset \mathcal{H}^1(\mathcal{M})$ when $E(u, A) < \delta$.

Lemma 5.7.9. *There exists* $C(\mathcal{M}) < \infty$ *such that if* $(u, A) \in X$ *satisfies* $E(u, A) < \delta$, *with* δ *small enough, then*

$$\operatorname{dist}(h(A), \Lambda) \le C\delta^{1/2}.$$

Proof. As in [102], it is convenient to define a box-type norm $|\cdot|_b$ on the space $\mathcal{H}^1(\mathcal{M})$ of harmonic one-forms as follows. Fix a collection $\gamma_1, \ldots, \gamma_{b_1(\mathcal{M})} \in C^\infty(S^1, \mathcal{M})$ of embedded loops generating $H_1(\mathcal{M}; \mathbb{Q})$ and, for $h \in \mathcal{H}^1(\mathcal{M})$, set

$$|h|_b := \max_{1\le i\le b_1(\mathcal{M})} \Big| \int_{\gamma_i} h \Big|. \tag{5.7.7}$$

Since $\mathcal{H}^1(\mathcal{M})$ is finite-dimensional, this is of course equivalent to any other norm on $\mathcal{H}^1(\mathcal{M})$. Assuming for simplicity that $\mathcal{M}$ is orientable, we may fix a collection of diffeomorphisms $\Phi_i : B_1^{m-1}(0) \times S^1 \to T(\gamma_i)$ onto tubular neighborhoods $T(\gamma_i)$ of γ_i, such that $\Phi_i(0, \theta) = \gamma_i(\theta)$. For every $t \in B_1^{m-1}$, set $\gamma_i^t(\theta) := \Phi_i(t, \theta)$.

Suppose now that $(u, A) \in X$ satisfies the energy bound

$$E(u, A) = \int_{\mathcal{M}} (|du - iuA|^2 + |dA|^2 + W(u)) < \delta. \tag{5.7.8}$$

As a consequence of the curvature bound $\|dA\|_{L^2} \le \delta^{1/2}$ and the definition of X, it follows that

$$\|A - h(A)\|_{L^2}^2 \le C\delta$$

as well. As in the proof of Proposition 5.7.6, applying a gauge transformation $\phi \cdot (u, A)$ by an appropriate choice of harmonic map $\phi : \mathcal{M} \to S^1$, we may assume moreover that

$$|h(A)|_b = \operatorname{dist}_b(h(A), \Lambda) \le \pi,$$

which together with the energy bound (5.7.8) and the definition of X leads us to the estimate

$$\int_{\mathcal{M}} |A|^2 \leq C(\mathcal{M}). \tag{5.7.9}$$

(Note that making a harmonic change of gauge preserves not only the energy $E(u, A)$, but also the distance $\mathrm{dist}_b(h(A), \Lambda)$, so it indeed suffices to establish the desired estimate in this gauge.)

Combining these estimates with a simple Fubini argument, we see that there exists a nonempty set S of $t \in B_1^{m-1}$ for which

$$\int_{\gamma_i^t} (|du - iuA|^2 + |dA|^2 + W(u)) < C\delta, \tag{5.7.10}$$

$$\int_{\gamma_i^t} |A - h(A)|^2 < C\delta, \tag{5.7.11}$$

and

$$\int_{\gamma_i^t} |A|^2 \leq C. \tag{5.7.12}$$

Recalling the pointwise bound (5.7.2) for $W(u)$, observe next that

$$|d(1-|u|)^2| = 2(1-|u|)|d|u|| \leq CW(u) + |du - iuA|^2,$$

so that, along a curve γ_i^t satisfying (5.7.10), it follows that

$$\|(1-|u|)^2\|_{C^0} \leq C\|(1-|u|)^2\|_{W^{1,1}} \leq C\delta. \tag{5.7.13}$$

Now, choose $\delta < \delta_1(\mathcal{M})$ sufficiently small that (5.7.13) gives

$$\|1 - |u|\|_{C^0} \leq \eta < \frac{1}{2}$$

on γ_i^t, so that $\phi := u/|u|$ defines there an S^1-valued map $\phi : \gamma_i^t \to S^1$, whose degree is given by

$$2\pi \deg(\phi) = \int_{\gamma_i^t} |u|^{-2} \langle du, iu \rangle.$$

When (5.7.10)–(5.7.12) hold, we observe next that

$$\int_{\gamma_i^t} |u|^2 |A - |u|^{-2}\langle iu, du\rangle| = \int_{\gamma_i^t} |\langle iu, iuA - du\rangle| \leq C\delta^{1/2}.$$

Since $|u| \geq \frac{1}{2}$ on γ_i^t, it follows that

$$\left|2\pi \deg(\phi) - \int_{\gamma_i^t} A\right| \leq \int_{\gamma_i^t} |A - |u|^{-2}\langle iu, du\rangle| \leq C\delta^{1/2} \tag{5.7.14}$$

as well. Combining this with (5.7.11), we then deduce that

$$\left|2\pi \deg(\phi) - \int_{\gamma_i^t} h(A)\right| \leq C\delta^{1/2}. \tag{5.7.15}$$

On the other hand, we already made a gauge transformation so that

$$\Big|\int_{\gamma_i} h(A)\Big| = \Big|\int_{\gamma_i^t} h(A)\Big| \le \pi.$$

So, for δ chosen sufficiently small that $C\delta^{1/2} < \pi$, it follows that the degree $\deg(\phi) = 0$. In particular, we can now conclude that

$$|h(A)|_b = \max_i \Big|\int_{\gamma_i} h(A)\Big| \le C\delta^{1/2},$$

giving the desired estimate. $\square$

Remark 5.7.10. If $\mathcal{M}$ is not orientable, we have the weaker conclusion $\mathrm{dist}(h(A), \frac{1}{2}\Lambda) \le C\delta^{1/2}$ (still sufficient for the sequel): indeed, whenever γ_i reverses the orientation, we can still parametrize a double cover of $T(\gamma_i)$ in the same way, with γ_i^t homotopic to γ_i traveled twice; in this case, the bound (5.7.15) implies that $2\int_{\gamma_i} h(A) = \int_{\gamma_i^t} h(A)$ has distance to $2\pi\mathbb{Z}$ bounded by $C\delta^{1/2}$, from which the claim follows.

Returning to the proof of Proposition 5.7.8, suppose again that we have a family $\overline{D} \ni y \mapsto F(y) \in X$ in Γ with

$$\max_{y\in\overline{D}} E(F(y)) < \delta.$$

For $\delta < \delta_1(\mathcal{M})$ sufficiently small, it follows from the lemma that $\mathrm{dist}_b(h(A), \Lambda) < \pi$ for every couple $(u, A) = F(y)$ in the family. In particular, since the assignment $(u, A) \mapsto h(A)$ gives a continuous map $X \to \mathcal{H}^1(\mathcal{M})$, and since $h(A) = A = 0$ for $y \in \partial\overline{D}$, it follows that 0 is the nearest point in the lattice Λ to $h(A)$ for every $y \in \overline{D}$, and the estimate therefore becomes

$$\|h(A)\| \le C\delta^{1/2}.$$

In particular, combining this with (5.7.6), we see now that

$$\|A\|_{W^{1,2}} \le C\delta^{1/2} \tag{5.7.16}$$

for every couple $(u, A) = F(y)$ in the family.

Now, for $(u, A) = F(y)$, our structural assumption (G) on $W(u)$ gives

$$\|u\|_{L^p}^p \le C + E(u, A) \le C + \delta,$$

which together with the smallness

$$\|A\|_{L^{2^*}} \le C\|A\|_{W^{1,2}} \le C\delta^{1/2}$$

of A in L^{2^*} (recalling that $p > m$) gives

$$\int_{\mathcal{M}} |uA|^2 \le C\delta.$$

Combining this with the fact that $\int_{\mathcal{M}} |du - iuA|^2 \leq E(u,A) < \delta$ by assumption, we then deduce that

$$\int_{\mathcal{M}} |du|^2 \leq C\delta$$

as well.

Finally, by (5.7.2) and the Poincaré inequality, we have

$$\begin{aligned} 1 - \Big|\frac{1}{\mathrm{vol}(\mathcal{M})}\int_{\mathcal{M}} u\Big| &\leq C\int_{\mathcal{M}} |1-|u|| + C\int_{\mathcal{M}} \Big|u - \frac{1}{\mathrm{vol}(\mathcal{M})}\int_{\mathcal{M}} u\Big| \\ &\leq C\Big(\int_{\mathcal{M}} W(u)\Big)^{1/2} + C\Big(\int_{\mathcal{M}} |du|^2\Big)^{1/2} \\ &\leq C\delta^{1/2}. \end{aligned}$$

As a consequence, we find that $\int_{\mathcal{M}} u_y$ is nonzero for all $(u_y, A_y) = F(y)$ in the family. But then the averaging map

$$\overline{D} \to \mathbb{C}, \quad y \mapsto \frac{\int_{\mathcal{M}} u_y}{|\int_{\mathcal{M}} u_y|} \tag{5.7.17}$$

gives a retraction $\overline{D} \to \partial\overline{D}$, whose nonexistence is well known. This gives the desired contradiction. □

Having shown positivity $\omega_\varepsilon(\mathcal{M}) > 0$ of the min-max energies, we can now deduce the lower bound in (5.7.1) from the following simple fact.

Proposition 5.7.11. *There exist $c(\mathcal{M}) > 0$ and $\varepsilon_0(\mathcal{M}) > 0$ such that the following holds, for $\varepsilon \leq \varepsilon_0$. If (u,∇) is critical for the functional E_ε, then either $E_\varepsilon(u,\nabla) \geq c$ or $E_\varepsilon(u,\nabla) = 0$.*

Remark 5.7.12. For future reference, we make the obvious observation that the trivial case $E_\varepsilon(u,\nabla) = 0$ can only occur when the bundle L is trivial.

Proof. By Proposition 5.7.5, critical points are smooth up to change of gauge. We claim that, whenever $E_\varepsilon(u,\nabla) > 0$, u has to vanish at some point $x_0 \in \mathcal{M}$. Once we have this, assume e.g. $E_\varepsilon(u,\nabla) \leq 1$; Corollary 5.4.4 (with $\Lambda = 1$) gives a constant $\varepsilon_0 > 0$ such that $r^{2-m}E_\varepsilon(u,\nabla,B_r(x_0))$ has a lower bound independent of ε and r, for any radius $\varepsilon < r < \mathrm{inj}(\mathcal{M})$, provided that $\varepsilon \leq \varepsilon_0$.

We show the contrapositive, namely we assume that u is nowhere vanishing and show that the energy is zero. Note that L must be trivial and we can use the section $\frac{u}{|u|}$ to identify L isometrically with the trivial line bundle $\mathbb{C} \times \mathcal{M}$, equipped with the canonical Hermitian metric. Under this identification, $u : \mathcal{M} \to \mathbb{C}$ takes values into positive real numbers. Writing $\nabla = d - iA$ and observing that $\langle \nabla u, iu\rangle = -|u|^2 A$, (5.2.5) becomes

$$\varepsilon^2 d^* dA + |u|^2 A = 0.$$

Integrating against A we get $\int_{\mathcal{M}}(\varepsilon^2|dA|^2 + u^2|A|^2) = 0$, so $A = 0$ and ∇ is the trivial connection. At a minimum point y_0 for u, (5.3.4) gives

$$0 \leq \frac{1}{2}\Delta|u|^2 = |du|^2 - \frac{1}{2\varepsilon^2}(1 - |u|^2)|u|^2 = -\frac{1}{2\varepsilon^2}(1 - u^2)u^2,$$

which forces $u(y_0) \geq 1$ and thus $u = 1$ everywhere, giving $E_\varepsilon(u, \nabla) = 0$. $\square$

Finally, we turn to the uniform upper bound. In the next statement, $L \to \mathcal{M}$ is a Hermitian line bundle with a fixed Hermitian reference connection ∇_0. We identify any other Hermitian connection ∇ with the real one-form A such that $\nabla s = \nabla_0 s - is \otimes A$ for all sections s.

Proposition 5.7.13. *Given a smooth section $u : \mathcal{M} \to L$, we can find a smooth couple (u', A') such that*

$$\begin{aligned} E_\varepsilon(u', A') \leq C\varepsilon^{-2} \operatorname{vol}\left(\left\{|u| \leq \frac{1}{2}\right\}\right) + C(1 + \varepsilon^2\|\nabla_0 u\|_{L^\infty}^2) \int_{\{|u|\leq\frac{1}{2}\}} |\nabla_0 u|^2 \\ + C\varepsilon^2 \int_{\mathcal{M}} |\omega_0|^2 \end{aligned} \tag{5.7.18}$$

for a universal constant C.

Proof. On $\{u \neq 0\}$ we let

$$w := \frac{u}{|u|}, \quad iw \otimes A := \nabla_0 w.$$

Note that the compatibility of ∇_0 with the Hermitian metric on L forces $\langle \nabla_0 w, w\rangle = 0$, so that A is a real one-form.

We fix a smooth function $\rho : [0, \infty] \to [0, 1]$ with

$$\rho(t) = 0 \text{ for } t \leq \frac{1}{4}, \quad \rho(t) = 1 \text{ for } t \geq \frac{1}{2}$$

and we set

$$(u', A') := \rho(|u|)(w, A),$$

where the right-hand side is meant to be zero on $\{u = 0\}$.

Writing $F_{\nabla_0} = -i\omega_0$, observe that $(\nabla_0 - iA)w = 0$, hence

$$|dA + \omega_0| = |F_A| = 0 \quad \text{on } \{u \neq 0\}.$$

In particular, $e_\varepsilon(u', A') = 0$ on $\{|u| > \frac{1}{2}\}$.

From the estimates $|d|u|| \leq |\nabla_0 u|$ and $|A| = |\nabla_0 w| \leq 2|u|^{-1}|\nabla_0 u|$, it follows that also

$$\begin{aligned} |\nabla_0 u'| &\leq C|\nabla_0 u|, \\ |A'| &\leq C|\nabla_0 u|, \\ |dA'| &\leq |\rho'(|u|)d|u| \wedge A| + |\omega_0| \leq C|\nabla_0 u||d|u|| + |\omega_0|, \end{aligned}$$

and the statement follows immediately. $\square$

Proof of (5.7.1). The method used in [102, Section 3] gives a continuous map $H : \overline{D} \to W^{1,2} \cap C^0(\mathcal{M}, \mathbb{C})$ such that $H(y) \equiv y$ for $y \in \partial D$ and

$$\begin{aligned} \|dH(y)\|_{L^\infty} &\leq C\varepsilon^{-1}, \\ \int_{\{|H(y)|\leq\frac{3}{4}\}} |dH(y)|^2 &\leq C, \\ \mathrm{vol}\left(\left\{|H(y)| \leq \frac{3}{4}\right\}\right) &\leq C\varepsilon^2 \end{aligned} \tag{5.7.19}$$

for all $y \in \overline{D}$—the full Dirichlet energy having a worse bound $\int_{\mathcal{M}} |dH(y)|^2 \leq C \log \varepsilon^{-1}$, which is the natural one in the setting of Ginzburg–Landau. By approximation, we can assume that H takes values in $C^\infty(\mathcal{M}, \mathbb{C})$, continuously in y, and still satisfies the same uniform bounds (5.7.19) (possibly increasing C and replacing $\frac{3}{4}$ with $\frac{1}{2}$).

To each section $H(y)$ of the trivial line bundle, Proposition 5.7.13 assigns in a continuous way an element $F(y) \in X$. From the way $F(y)$ is constructed, it is clear that $F \in \Gamma$. Finally, combining (5.7.18) with (5.7.19) gives

$$\omega_\varepsilon(\mathcal{M}) \leq \max_{y\in\overline{D}} E_\varepsilon(F(y)) \leq C. \qquad \square$$

Minimizers for nontrivial line bundles

Suppose now that L is a nontrivial line bundle, equipped with a Hermitian metric. Fix a smooth Hermitian connection ∇_0 and identify any other Hermitian connection ∇ with the real one-form A such that

$$\nabla = \nabla_0 - iA.$$

We can define $\widehat{X}$ and X as in the previous subsection. With this notation, observe that the curvature of ∇ is given by

$$F_\nabla = F_{\nabla_0} - idA.$$

Hence, writing $F_{\nabla_0} = -i\omega_0$, we have

$$E_\varepsilon(u, \nabla) = \int_{\mathcal{M}} |\nabla_0 u - iu \otimes A|^2 + \varepsilon^{-2} \int_{\mathcal{M}} W(u) + \varepsilon^2 \int_{\mathcal{M}} |\omega_0 + dA|^2.$$

Definition 5.7.14. For a fixed $m < p < \infty$, we define $\widehat{X}$ to be the Banach space of couples (u, A), where $u : \mathcal{M} \to L$ is an L^p section and $A \in \Omega^1(\mathcal{M}, \mathbb{R})$, both of class $W^{1,2}$, with the norm

$$\|(u, A)\| := \|u\|_{L^p} + \|\nabla_0 u\|_{L^2} + \|A\|_{L^2} + \|DA\|_{L^2}.$$

We let $X := \{(u, A) \in \widehat{X} : d^*A = 0\}$.

The analogous statements to Remark 5.7.4 and Propositions 5.7.5 and 5.7.6 hold, with identical proofs (replacing du and uA with $\nabla_0 u$ and $u \otimes A$, respectively).

Arguing as in the proof of Proposition 5.7.6, it is easy to see that a minimizing sequence for E_ε in X converges weakly—up to change of gauge—to a global minimizer $(u_\varepsilon, A_\varepsilon)$. We now show that the energy of these minimizers enjoys uniform upper and lower bounds as $\varepsilon \to 0$.

Proof of (5.7.1). The lower bound in (5.7.1) follows directly from Proposition 5.7.11 and Remark 5.7.12. In order to obtain the upper bound, pick a smooth section $s : \mathcal{M} \to L$ transverse to the zero section (see, e.g., [63, Theorem IV.2.1]) and let $N := \{s = 0\}$, which is a smooth embedded $(m-2)$-submanifold of $\mathcal{M}$. Proposition 5.7.13 applied to $\varepsilon^{-1}s$ gives a couple $(u'_\varepsilon, A'_\varepsilon)$ with

$$E_\varepsilon(u'_\varepsilon, A'_\varepsilon) \le C\varepsilon^{-2} \operatorname{vol}\left(\left\{|\varepsilon^{-1}s| \le \frac{1}{2}\right\}\right) + C\varepsilon^2 \int_{\mathcal{M}} |\omega_0|^2.$$

By transversality of s, the set $\{|s| \le \frac{\varepsilon}{2}\}$ is contained in a $C(s)\varepsilon$-neighborhood of N, whose volume is bounded by $C(s)\varepsilon^2$. We infer that

$$E_\varepsilon(u_\varepsilon, A_\varepsilon) \le E_\varepsilon(u'_\varepsilon, A'_\varepsilon) \le C\varepsilon^{-2} \operatorname{vol}\left(\left\{|s| \le \frac{\varepsilon}{2}\right\}\right) + C \le C. \qquad \square$$

Remark 5.7.15. When $\mathcal{M}$ is oriented, N can be oriented in such a way that $[N] \in H_{m-2}(\mathcal{M}, \mathbb{R})$ is Poincaré dual to the Euler class $e(L) \in H^2(\mathcal{M}, \mathbb{R})$ of the line bundle, which equals the first Chern class $c_1(L)$. The fact that the energy of our competitors concentrates along N suggests that, given a sequence of global minimizers $(u_\varepsilon, A_\varepsilon)$, up to subsequences the corresponding energy concentration varifold is induced by an integral mass-minimizing current whose homology class is Poincaré dual to $c_1(L)$. Theorem 5.6.10 provides the natural candidate Γ, which also satisfies $|\Gamma| \le \mu$.

Appendix

In this short appendix, we describe the essential ingredients needed to establish local regularity in the Coulomb gauge for finite-energy critical points (u, A) of the $(\varepsilon = 1)$ abelian Higgs energy $E(u, A)$, collecting some estimates which will be of use elsewhere in the chapter.

Consider the manifold with boundary $(\overline{\Omega}^m, g)$ given by a smooth, contractible domain $\Omega^m \subset\subset \mathbb{R}^m$ equipped with a C^2 metric g, and let $L \cong \mathbb{C} \times \Omega$ be the trivial line bundle over Ω, with the standard Hermitian structure. With respect to the metric g, we then define the Yang–Mills–Higgs energy

$$E(u, A) := \int_\Omega e(u, A) = \int_\Omega |du - iu \otimes A|^2 + |dA|^2 + W(u)$$

as in the preceding section. By (the first part of) Proposition 5.7.5, it is easy to see that a pair (u, A) in $W^{1,2}$ with

$$|u| \le 1 \tag{A.1}$$

is a critical point for E (with respect to smooth perturbations supported in Ω) if and only if the equations

$$d^*dA = \langle du - iu \otimes A, iu \rangle, \tag{A.2}$$

$$\Delta u = 2\langle idu, A\rangle + |A|^2 u - \frac{1}{2}(1-|u|^2)u - i(d^*A)u \tag{A.3}$$

are satisfied distributionally in Ω, where all geometric quantities and operators are defined with respect to the metric g.

Now, given a pair (u, A) in $W^{1,2}$ satisfying (A.2)–(A.3) and

$$E(u, A) \leq \Lambda < \infty, \tag{A.4}$$

we can select a *local Coulomb gauge* adapted to Ω as follows. Denote by $\theta \in W^{2,2}(\Omega, \mathbb{R})$ the unique solution of the Neumann problem

$$\Delta\theta = d^*A \text{ in } \Omega; \quad \frac{\partial\theta}{\partial\nu} = -A(\nu) \text{ on } \partial\Omega \tag{A.5}$$

with zero mean $\int_\Omega \theta = 0$. Then the gauge-transformed pair

$$(\widetilde{u}, \widetilde{A}) := (e^{i\theta}u, A + d\theta)$$

lies in $W^{1,2}$ and continues to satisfy (A.2)–(A.3), with

$$E(\widetilde{u}, \widetilde{A}) = E(u, A) \leq \Lambda,$$

but now with the additional constraints

$$d^*\widetilde{A} = 0 \text{ on } \Omega; \quad \widetilde{A}(\nu) = 0 \text{ on } \partial\Omega. \tag{A.6}$$

For the remainder of the section, we will assume that the pair (u, A) is already in the Coulomb gauge on Ω, so that A satisfies (A.6). Note that (A.2)–(A.3) then become

$$\Delta u = 2\langle idu, A\rangle + |A|^2 u - \frac{1}{2}(1-|u|^2)u, \tag{A.7}$$

$$\Delta_H A = \langle du - iu \otimes A, iu\rangle. \tag{A.8}$$

We now establish the local regularity for critical points (u, A) in the Coulomb gauge, giving in particular local estimates for (u, A) in $W^{2,q}$ norms.

Proposition A.1. *Let (u, A) solve (A.2)–(A.3) in the Coulomb gauge (A.6) on (Ω, g), with $|u| \leq 1$. If*

$$E(u, A; \Omega) \leq \Lambda \tag{A.9}$$

and

$$\|g\|_{C^2} + \|g^{-1}\|_{C^2} \leq \Lambda, \tag{A.10}$$

then for every compactly supported subdomain $\Omega' \subset\subset \Omega$ and $q \in (1, \infty)$ there exists $C_q(\Lambda, \Omega, \Omega') < \infty$ such that

$$\|u\|_{W^{2,q}(\Omega')} + \|A\|_{W^{2,q}(\Omega')} \leq C_q. \tag{A.11}$$

Proof. To begin, note that (A.8) and standard Bochner–Weitzenböck identities give the (weak) subequation

$$\begin{aligned} \Delta\frac{1}{2}|A|^2 &= -\langle \Delta_H A, A\rangle + |DA|^2 + \mathrm{Ric}(A,A) \\ &\geq -|du - iu\otimes A||A| + |DA|^2 - C(\Lambda)|A|^2 \end{aligned} \tag{A.12}$$

for $|A|^2$. On the other hand, as in Section 5.3, we also obtain from (A.3) the relation

$$\Delta\frac{1}{2}|u|^2 = |du - iu\otimes A|^2 - \frac{1}{2}(1-|u|^2)|u|^2. \tag{A.13}$$

Recalling that $|u|\leq 1$ and using Young's inequality, we can combine (A.12)–(A.13) to find an estimate of the form

$$\frac{1}{2}\Delta(|A|^2+|u|^2) \geq \alpha(|DA|^2+|du|^2) - C(\alpha,\Lambda)|A|^2 - C(\Lambda), \tag{A.14}$$

for any $0<\alpha<1$.

By standard estimates for one-forms A satisfying (A.6) (see, e.g., [58, Theorem 4.8]), we have the global L^2 bound

$$\|A\|_{W^{1,2}(\Omega)} \leq C(\Lambda,\Omega)\|dA\|_{L^2(\Omega)} \leq C(\Lambda,\Omega),$$

hence $|u|, |A|$ are both bounded in $W^{1,2}$ in terms of Λ (and Ω).

Note that (A.8) gives a local $W^{2,2}$ bound on A, by standard elliptic regularity. This, together with Sobolev embedding and (A.7), gives

$$\|u\|_{W^{2,p}(\Omega_0)} + \|A\|_{W^{2,2}(\Omega_0)} + \||A|^p\|_{W^{1,2}(\Omega_0)} \leq C(\Lambda,\Omega,\Omega_0) \tag{A.15}$$

for all $\Omega_0 \subset\subset \Omega$ and some $1<p<2$, depending only on m. We need the following observation, stated and proved separately for the sake of clarity.

Lemma A.2. *Defining $f\in W^{1,2}(\Omega)$ by*

$$f := (1+|A|^2+|u|^2)^{1/2},$$

we have the subequation

$$\Delta f^p \geq -C(p,\Lambda)f^p \tag{A.16}$$

and, for all $\Omega_0\subset\subset\Omega$,

$$\|f^p\|_{W^{1,2}(\Omega_0)} \leq C(\Lambda,\Omega,\Omega_0).$$

Proof. Since $u\in L^\infty\cap W^{1,2}\cap W^{2,p}_{loc}$ and $A\in W^{2,2}_{loc}$, a standard approximation argument shows that $|u|^2, |A|^2 \in W^{2,1}_{loc}$, so that (A.14) holds pointwise a.e.

Likewise, we have $f\in W^{2,1}_{loc}$ and the chain rule applies, giving

$$\Delta f = f^{-1}(|DA|^2 + |du|^2 - \langle A, D^*DA\rangle + \langle u,\Delta u\rangle) - f^{-1}|df|^2$$

pointwise. The first term equals $f^{-1}\Delta\frac{1}{2}f^2$, so recalling (A.14) we obtain

$$\Delta f \geq \alpha f^{-1}(|DA|^2 + |du|^2) - C(\alpha,\Lambda)f - f^{-1}|df|^2.$$

Also, since $f \in W^{1,2} \cap W^{2,p}_{loc}$, we have the pointwise inequalities

$$\begin{aligned}\Delta f^p &= p(p-1)f^{p-2}|df|^2 + pf^{p-1}\Delta f\\ &\geq p\alpha f^{p-2}(|DA|^2 + |du|^2) - C(\alpha,\Lambda)f^p + p(p-2)f^{p-2}|df|^2\\ &\geq p(\alpha + p - 2)f^{p-2}|df|^2 - C(\alpha,\Lambda)f^p.\end{aligned}$$

Choosing $\alpha := 2 - p$, inequality (A.16) follows. The second claim is an easy consequence of (A.15) and the fact that $|u| \leq 1$. □

Returning to the proof of Proposition A.1, we can now apply Moser iteration to (A.16), obtaining in particular that

$$\|A\|_{L^\infty(\Omega_1)} \leq C(\Lambda,\Omega,\Omega_1) \tag{A.17}$$

for any $\Omega_1 \subset\subset \Omega$.

Now, fixing some intermediate domain $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$ between Ω' and Ω, (A.7) together with the $L^\infty(\Omega_1)$ estimate for A give pointwise bounds of the form

$$|\Delta u| \leq C(\Lambda,\Omega,\Omega_1)(|du| + 1) \quad \text{in } \Omega_1. \tag{A.18}$$

And since

$$|du| \leq |du - iu \otimes A| + |A| \leq e(u,A) + C$$

in Ω_1, we obtain from the energy bound $E(u,A) \leq \Lambda$ and (A.18) the simple estimate

$$\|\Delta u\|_{L^2(\Omega_1)} \leq C(\Lambda,\Omega,\Omega_1),$$

and consequently

$$\|u\|_{W^{2,2}(\Omega_2)} \leq C$$

for any $\Omega' \subset\subset \Omega_2 \subset\subset \Omega_1$. Returning to the pointwise bound (A.18), we can now employ a simple iteration argument—combining L^q regularity theory with the Sobolev embedding $W^{2,r} \hookrightarrow W^{1,\frac{rn}{m-r}}$—over successive domains between Ω' and Ω, to arrive at the desired $W^{2,q}$ estimates for u.

Returning finally to (A.8), it therefore follows from the preceding estimates that

$$\|A\|_{L^\infty(\Omega'')} + \|\Delta_H A\|_{L^\infty(\Omega'')} \leq C(\Lambda,\Omega,\Omega'')$$

for some intermediate domain $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. In particular, this gives us upper bounds for $\|\Delta A\|_{L^q(\Omega'')}$ for every $q \in (1,\infty)$, and L^q regularity theory therefore gives us the desired estimates for A in $W^{2,q}(\Omega')$. □

Finally, we remark that higher regularity of u and A in the Coulomb gauge follows in a standard way—e.g., via Schauder theory—from the $W^{2,q}$ estimates obtained in the preceding proposition.

Remark A.3. With local regularity established, note that it is easy to find a globally smooth couple $(\widetilde{u}, \widetilde{\nabla})$ gauge equivalent to any critical pair (u, ∇) for E_ε on $L \to \mathcal{M}$. Indeed, for any critical pair (u, ∇) with $u \in W^{1,2} \cap L^\infty$ and $\nabla = \nabla_0 - iA$ (where ∇_0 is a smooth reference connection and $A \in W^{1,2}$), it follows from the local regularity results above that the gauge-invariant objects $|u|^2$ and $dA = F_\nabla - F_{\nabla_0}$ are smooth globally. Making a change of gauge $(u, \nabla) \to (\widetilde{u}, \widetilde{\nabla} = \nabla_0 - i\widetilde{A})$ such that

$$d\widetilde{A} = dA \text{ and } d^*\widetilde{A} = 0,$$

it follows from the smoothness of dA that the new connection $\widetilde{\nabla} = \nabla_0 - i\widetilde{A}$ is smooth. And since $\widetilde{u}$ satisfies

$$\widetilde{\nabla}^*\widetilde{\nabla}\widetilde{u} = \frac{1}{2\varepsilon^2}(1 - |u|^2)\widetilde{u}$$

where both $\widetilde{\nabla}$ and $|u|^2$ are smooth, standard results for linear elliptic equations imply that $\widetilde{u} \in \Gamma(L)$ is a smooth section as well.

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Curriculum Vitae

Personal information

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Address: ETH Zürich, HG FO 27.9. Rämistrasse 101, 8092 Zürich (Switzerland)

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Research interests

Geometric analysis and (elliptic) PDEs, with particular emphasis on the variational construction and regularity theory of minimal submanifolds; geometric measure theory; sub-Riemannian geometry.

Education

2016 – present **PhD**
Zürich Graduate School in Mathematics, ETH Zürich
Advisor: Prof. Tristan Rivière (ETH Zürich)

2014 – 2016 **Master Degree in Mathematics**
University of Pisa (Italy), 110/110 cum laude
Dissertation topic: *New regularity results for sub-Riemannian geodesics*
Advisors: Prof. Luigi Ambrosio (SNS), Prof. Davide Vittone (University of Padua)

2011 – 2014 **Bachelor Degree in Mathematics**
University of Pisa (Italy), 110/110 cum laude
Dissertation topic: *The kissing number of spheres in Euclidean spaces*
Advisor: Prof. Giovanni Alberti (University of Pisa)

2011 – 2016 **Diploma**
Scuola Normale Superiore (SNS)

Publications, preprints and surveys

- **The viscosity method for min-max free boundary minimal surfaces**. *arXiv preprint* 2007.06004, 2020.
- **Codimension two min-max minimal submanifolds from PDEs** (survey). *Oberwolfach Reports: Partial Differential Equations (workshop 1930)*, 2019.
- with D. Stern: **Minimal submanifolds from the abelian Higgs model**. *arXiv preprint* 1905.13726, 2019.
- **Parametrized stationary varifolds and the multiplicity one conjecture** (survey). *Oberwolfach Reports: Calculus of Variations (workshop 1831)*, 2018.
- with T. Rivière: **A proof of the multiplicity one conjecture for min-max minimal**

surfaces in arbitrary codimension. Accepted in *Duke Math J.*

- with T. Rivière: **The regularity of parametrized integer stationary varifolds in two dimensions**. Accepted in *Comm. Pure Appl. Math.*
- with F. Da Lio: **Free boundary minimal surfaces: a nonlocal approach**. Accepted in *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).*
- with R. Monti and D. Vittone: **On tangent cones to length minimizers in Carnot-Carathéodory spaces**. *SIAM J. Control Optim.* 56 (2018), no. 5, 3351–3369.
- with R. Monti and D. Vittone: **Existence of tangent lines to Carnot–Carathéodory geodesics**. *Calc. Var. PDE* 57 (2018), art. 75.
- **New regularity results for sub-Riemannian geodesics**. Master thesis, available online at the *ETD repository* of the University of Pisa, 2016.

Invited talks

June 2020	*Geometric analysis and calibrated geometries* (canceled), Zürich (Switzerland)
October 2019	*Analysis seminar at Queen Mary University*, London (United Kingdom)
July 2019	*Partial Differential Equations*, Oberwolfach (Germany)
June 2019	*Workshop on Geometric Measure Theory*, Alba di Canazei (Italy)
March 2019	*Variational approaches to PDE's*, Rome (Italy)
December 2018	*Workshop in Geometric Analysis*, Paris (France)
July 2018	*Calculus of Variations*, Oberwolfach (Germany)
June 2018	*Geometric Measure Theory in Verona*, Verona (Italy)
April 2018	*Analysis seminar at University of Padua*, Padua (Italy)
November 2017	*Analysis seminar at ETH Zürich*, Zürich (Switzerland)

Other conferences attended

October 2019	*PDEs and Geometric Measure Theory*, Zürich (Switzerland)
June 2019	*Geometric Analysis and General Relativity. A conference in honour of Gerhard Huisken's 60th birthday*, Zürich (Switzerland)
May 2018	*Geometric Analysis*, Edinburgh (United Kingdom)
June 2017	*Nonlinear analysis Conference*, Zürich (Switzerland)
June 2017	*23rd Rolf Nevanlinna Colloquium*, Zürich (Switzerland)
June 2017	*Advances in Geometric Analysis*, Zürich (Switzerland)
January 2016	*XXVI Convegno Nazionale di Calcolo delle Variazioni*, Levico Terme (Italy)

Teaching

Spring 2019	Teaching assistant for Differential Geometry II
Fall 2018	Teaching assistant for Fourier Analysis in Function Space Theory
Spring 2018	Teaching assistant for Functional Analysis II
Fall 2017	Teaching assistant for Functional Analysis I

Spring 2017	Teaching assistant for Products and Nonlinearities in Function Space Theory
Fall 2016	Teaching assistant for Functional Analysis I

Other informal seminars

November 2019	Uhlenbeck compactness and applications to $SU(2)$ instantons
November 2019	Inverse mean curvature flow: uniqueness of weak solutions and short time existence
April 2018	Gunther's proof of the isometric embedding theorem
Spring 2018	Lectures on minimal surfaces: existence of infinitely many minimal hypersurfaces in positive Ricci curvature, Gromov's width, Weyl's law for minimal hypersurfaces
Spring 2017	Lectures on the real Hardy space
November 2015	Immersions of S^2 with prescribed mean curvature
September 2015	The Cheeger–Gromoll soul theorem
July 2015	Oseledec's multiplicative ergodic theorem
April 2015	Convex integration techniques and counterexamples to Korn's inequality
February 2015	Malgrange–Ehrenpreis theorem and Paley–Wiener theorems
October 2014	The spectral theorem for bounded and unbounded self-adjoint operators
September 2014	A polynomial version of Van der Waerden's theorem
May 2014	The central limit theorem and the monotonicity of entropy

Languages

Italian	**Native**
English	**Fluent**
French	**Intermediate**
German	**Basic**

Honors and awards

July 2011	Silver medal at the International Mathematical Olympiad, held in Amsterdam, Netherlands
May 2011	Bronze medal at the Balkan Mathematical Olympiad, held in Iassy, Romania
May 2011	Gold medal at the Italian Mathematical Olympiad, held in Cesenatico, Italy
May 2010	Gold medal at the Italian Mathematical Olympiad, held in Cesenatico, Italy