Homework 1

Istituzioni di Algebra

Due date: October 28, 2024

1 Proving stuff

Let R be a (commutative, unitary) ring. Recall that a (commutative, unitary) ring A is said to be *finitely generated as an R-algebra* if there exists an integer $n \geq 0$ and a surjective homomorphism $\varphi : R[x_1, \ldots, x_n] \to A$.

Exercise P1.

1. Let K be a field of characteristic 0. Show that K is not finitely generated as a $\mathbb{Z}$-algebra.
2. Let A be a finitely generated $\mathbb{Z}$-algebra and let M be a maximal ideal of A. Prove that A/M is a finite field.

Exercise P2. Recall that the *Jacobson radical* $J(A)$ of a ring A is the intersection of all the maximal ideals of A.

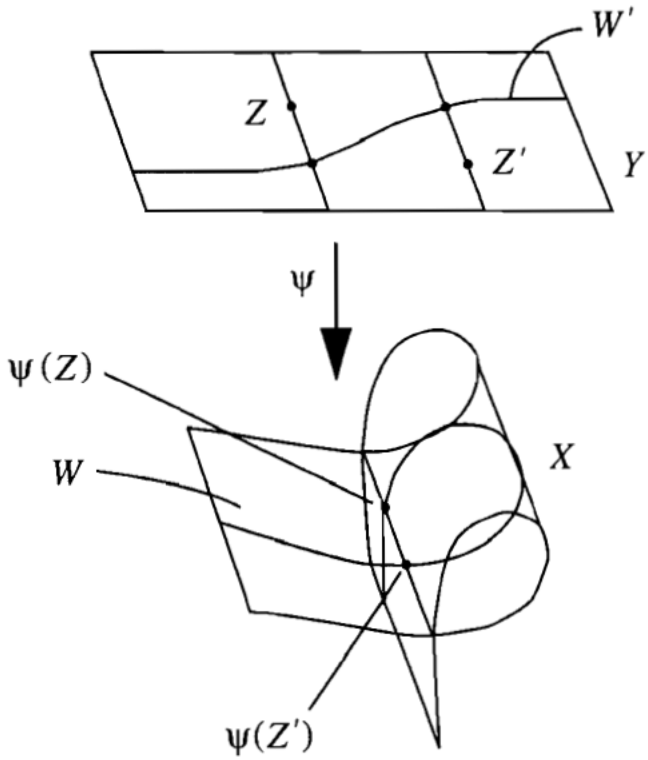
1. Let $A \subseteq B$ be an integral extension of rings. Show that $J(B) \cap A = J(A)$.
2. In the situation of the previous question, suppose furthermore that B is an integral domain. Show that $J(B) = 0$ if and only if $J(A) = 0$.

Exercise P3. Let k be a field and A be a finitely generated k-algebra.

1. Let I be a radical ideal of A and fix an element $f \in A \setminus I$. Let $(A/I)_f$ be the localisation of (A/I) at the multiplicative set $\{f^n : n \in \mathbb{N}\}$. Show that $(A/I)_f$ is not the zero ring.
2. With the same notation, let $\mathfrak{m}$ be a maximal ideal of $(A/I)_f$ and $\mathfrak{n}$ be the contraction of $\mathfrak{m}$ along the natural map $A \to (A/I) \to (A/I)_f$. Show that $\mathfrak{n}$ is a maximal ideal of A.
3. Deduce that every radical ideal of A is the intersection of a (possibly infinite) family of maximal ideals of A.

Exercise P4. In this exercise, you will construct an example where the conclusion of the going down theorem fails for an integral extension of domains $A \subseteq B$ where A is not integrally closed. Let $A = \mathbb{C}[X, Y, Z]/(Y^2 - X^3 - X^2)$. Denote by x, y, z the classes of X, Y, Z in A.

1. Show that A is not integrally closed and that its integral closure B is isomorphic to the polynomial ring $\mathbb{C}[t, z]$. Let $i : A \hookrightarrow B$ be the natural inclusion. Identifying B with $\mathbb{C}[t, z]$, determine explicitly the images $i(x)$ and $i(y)$.
2. Let $\mathfrak{p}_1 = (x, y, z - 1)$ and $\mathfrak{p}_2 = (x - z^2 + 1, y - z(z^2 - 1))$ be ideals of A. Check that $\mathfrak{p}_1 \supsetneq \mathfrak{p}_2$ is a strictly decreasing chain of prime ideals in A.
3. Find a prime $\mathfrak{q}_1$ of B lying over $\mathfrak{p}_1$ for which there is no prime ideal $\mathfrak{q}_2 \subsetneq \mathfrak{q}_1$ that lies over $\mathfrak{p}_2$.
4. Stare at the picture below until you're convinced that it *is* (essentially) this counterexample. (Bonus points if you give a brief explanation of how the picture relates to the rest of the problem. Your answer should involve a curve in the (t, z)-plane.)



2 Computing stuff

Exercise C1. Let
$$A = \frac{\mathbb{Z}[x,y]}{(y^2-2,(x-y)(x-2y-1)(x-2y+1))}.$$
Describe the irreducible components of Spec A and its connected components (in particular, you should say how many irreducible components and how many connected components there are). Determine the Krull dimension of A. Prove that every irreducible component of Spec A is homeomorphic to Spec B as a topological space, where B is a ring to be determined.

Exercise C2. It is known that the ideal $I := (x_1x_4 - x_2x_3, x_1x_3 - x_2^2, x_2x_4 - x_3^2)$ of the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$ is prime. Let $A = \frac{\mathbb{C}[x_1,x_2,x_3,x_4]}{I}$. Compute the Krull dimension of A.

(Bonus points if you prove that I is indeed a prime ideal.)

Exercise C3. Let
$$A := \{f \in \mathbb{C}[x,y] : f(0,0) = f(1,0) = f(2,0)\}.$$

1. Show that A is a domain. Describe the integral closure B of A in its fraction field.
2. Consider the map $i^\sharp : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to the natural inclusion $i : A \hookrightarrow B$. Show that $i^\sharp$ is surjective.
3. Prove that there exists a unique maximal ideal $\mathfrak{m}$ of A such that $(i^\sharp)^{-1}(\mathfrak{m})$ consists of more than one point. Determine, for each prime $\mathfrak{p} \in \operatorname{Spec} A$, the cardinality of the fibre $(i^\sharp)^{-1}(\mathfrak{p})$.

   Hint. It may be useful to work locally, restricting to suitable open subsets of Spec A.

Exercise C4. Let $A = \mathbb{C}[X,Y]/(Y^2 + Y - X^3 + X^2)$ and $\mathfrak{m}$ be the maximal ideal of A given by (x, y), where we denote by x, y the classes of X, Y in A.

1. Show that $A_\mathfrak{m}$ is a DVR.
2. Let v be the valuation on Frac(A) corresponding to the valuation ring $A_\mathfrak{m}$. Compute $v(f)$, where $f = (x+1)y + x^2$.