Homework 2

Istituzioni di Algebra

Due date: November 18, 2024

1 Proving stuff

Definition. Let A be a ring. Elements $x_1, \dots, x_r \in A$ form a **regular sequence** if the following hold:

1. for every $i = 1, \dots, r$, the image of x_i in $A/(x_1, \dots, x_{i-1})$ is not a zero-divisor (for $i = 1$, this means that x_1 is not a zero-divisor in A);
2. $(x_1, \dots, x_r) \neq A$.

Exercise P1. Let A be a Noetherian local ring with maximal ideal P. Let $x_1, \dots, x_r$ be a minimal system of generators of P (this means that $P = (x_1, \dots, x_r)$, but no proper subset of $\{x_1, \dots, x_r\}$ generates P). Show that A is regular if and only if $x_1, \dots, x_r$ form a regular sequence.

Exercise P2. Let A be a Noetherian ring of Krull dimension $r \geq 2$. Show that for every i with $0 < i < r$ there exist infinitely many prime ideals in A of height i.

Hint. Start with the case where A is a domain, $r = 2$ and $i = 1$.

Exercise P3. Let R be a ring and $\mathfrak{m}$ be a maximal ideal of R. Suppose that R is $\mathfrak{m}$-adically complete. Prove the following variants of Hensel's lemma:

1. Let $f_1, \dots, f_n \in R[x_1, \dots, x_n]$. Suppose $(a_1, \dots, a_n) \in R^n$ satisfies
$$f_i(a_1, \dots, a_n) \equiv 0 \pmod{\mathfrak{m}} \quad \forall i = 1, \dots, n \quad \text{and} \quad \det\left(\frac{\partial f_i}{\partial x_j}(a_1, \dots, a_n)\right)_{i,j=1,\dots,n} \in R^\times.$$
Prove that there exists a unique $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_n) \in R^n$ such that $f_i(\widehat{a}) = 0$ for all $i = 1, \dots, n$ and $\widehat{a}_i \equiv a_i \pmod{\mathfrak{m}}$ for all $i = 1, \dots, n$.
2. Let $F, G, H \in R[x]$ be monic polynomials. Suppose that $F \equiv GH \pmod{\mathfrak{m}}$ and that $G \bmod \mathfrak{m}, H \bmod \mathfrak{m}$ are relatively prime as polynomials in $(R/\mathfrak{m})[x]$. Show that there exist monic polynomials $G', H' \in R[X]$ such that $F = G'H'$ and $G' \equiv G \pmod{\mathfrak{m}}, H' \equiv H \pmod{\mathfrak{m}}$.

Hint. You can prove part 2 directly, but it may be easier to deduce it from part 1 – after all, factoring a polynomial amounts to solving a system of equations...

Exercise P4. Come dangerously close to proving Bézout's theorem in the following way. Let k be a field and recall that for a homogeneous ideal I of $A = k[x_0, \dots, x_n]$ we have defined $h_I(d) := \dim_k (A/I)_d$.

1. Let $F \in A$ be a homogeneous polynomial of degree $d_F > 0$. Compute $h_I(d)$ for $I = (F)$ and d sufficiently large. For $n = 2$, compute $h_I(d)$ for all d.
2. Let F be as above and I be a homogeneous ideal of A with the following property: there exists d_0 such that for all homogeneous $G \in A$ of degree at least d_0, we have $FG \in I$ if and only if $G \in I$. Show that for all but finitely many $d \in \mathbb{N}$ we have
$$h_{I+(F)}(d) = h_I(d) - h_I(d - d_F).$$
3. Let now $F, G \in k[x_0, x_1, x_2]$ be non-associate irreducible homogeneous polynomials of degrees $d_F > 0, d_G > 0$. Compute $h_{(F,G)}(d)$ for $d \gg 0$.
4. (Bonus points) Can you explain why this is related to Bézout's theorem in the plane? What is missing to complete the proof of this theorem? How does the notion of length enter the picture?

2 Computing stuff

Exercise C1. Let k be a field of characteristic zero, $A := k[x,y,z]/(xy, xz, yz)$ and $B := k[x,y]/(xy(x-y))$. Show that the rings A and B are not isomorphic. Find elements $a \in A$ and $b \in B$ such that $A[1/a] \cong B[1/b] \cong k[x, 1/x]^3$.

Hint 1. Not surprisingly, drawing a picture can help!

Hint 2. Let k be a field, $n \geq 1$, and A a quotient of $k[x_1, \dots, x_n]$. Prove that, for every maximal ideal M of A, the A/M-vector space M/M^2 is generated by at most n elements.

Exercise C2. Consider the ring $A = \frac{\mathbb{Z}[X,Y]}{(Y^2 - X^5 - 2)}$. Denote by x, y the classes of X, Y in A. Consider the ideal of A given by $I = (5, x)$. Check that I is a maximal ideal and describe an isomorphism between $\mathrm{Gr}_I A = \bigoplus_{n \geq 0} I^n / I^{n+1}$ and a suitable polynomial ring with coefficients in a field.

Exercise C3. Let k be a field and $A = k[x_1, x_2, \dots, x_n, \dots]$ be the polynomial ring in infinitely many variables over k. Let $\mathfrak{m} = (x_1, x_2, \dots, x_n, \dots)$ be the maximal ideal generated by all the variables. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. We know that $\widehat{A}$ is complete with respect to the topology given by the ideals $M^n := \ker\left(\widehat{A} \to A/\mathfrak{m}^n\right)$, and that $\widehat{A}$ is local with maximal ideal M^1.

1. Find an element in $M^1 \setminus \mathfrak{m}\widehat{A}$, where as usual $\mathfrak{m}\widehat{A}$ denotes the ideal of $\widehat{A}$ generated by the image of $\mathfrak{m}$ in $\mathfrak{A}$.
2. Deduce that $\widehat{A}$, seen as an A-module, is not $\mathfrak{m}$-adically complete.

Exercise C4. Let p be a prime number. Consider the power series

$$\log_p(1+x) := \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n} \quad \text{and} \quad \exp_p(x) := \sum_{n \geq 0} \frac{x^n}{n!}.$$

1. Show that, for $p > 2$, the formal series $\log_p(1+x), \exp_p(x)$ given above converge for $x \in p\mathbb{Z}_p$. Show that they induce well-defined, continuous, mutually inverse bijections

   $$\log_p(x) : 1 + p\mathbb{Z}_p \rightleftharpoons p\mathbb{Z}_p : \exp_p$$

   which are also group homomorphisms between $(1 + p\mathbb{Z}_p, \cdot)$ and $(p\mathbb{Z}_p, +)$.

   Note. Here we have set $p\mathbb{Z}_p = (p)\mathbb{Z}_p = \{px : x \in \mathbb{Z}_p\}$ and $1 + p\mathbb{Z}_p = \{1 + px : x \in \mathbb{Z}_p\}$.
2. Take now $p = 2$, $x = 2$. Show that $\log_2(1+x)$ converges to a value $\alpha \in \mathbb{Z}_2$. Show that the series $\exp_2(\alpha)$ converges to a value $\beta \in \mathbb{Z}_2$. Compute β explicitly.

Exercise C5. Let p be a prime number and $k = \mathbb{F}_p(t)$ be the field of rational functions over $\mathbb{F}_p$. Let $A := k[x,y]/(x^p + y^2 - t)$ and $B := \overline{k}[x,y]/(x^p + y^2 - t)$. There is a natural inclusion $i : A \to B$. Find a maximal ideal $\mathfrak{n}$ of B with the following property: $B_{\mathfrak{n}}$ is not regular, but $A_{\mathfrak{m}}$ is regular, where $\mathfrak{m} = i^{-1}(\mathfrak{n})$.